Orthonormal bases on $L^2(\mathbb{R}^+)$

Abstract

We derive the explicit form of eigenvectors of selfadjoint extension H_ξ, parametrized by $\xi \in \langle 0, \pi)$, of differential expression $H = -\frac{d^2}{dx^2} + \frac{x^2}{4}$ together with the spectrum $\sigma(H_\xi)$ on the space $L^2(\mathbb{R}^+)$. For each ξ the set of eigenvectors form an orthonormal basis of $L^2(\mathbb{R}^+)$.

1 Introduction

The basic examples of quantum mechanics is a quantization the harmonic oscillator. A selfadjoint Hamiltonian H_D of the one–dimensional linear harmonic oscillator is generated by the differential expression

$$H = -\frac{d^2}{dx^2} + \frac{x^2}{4} \tag{1}$$

with appropriate definition domain D. It is known that the operator H_D has a pure point spectrum and its eigenfunctions form the orthonormal basis in $L^2(\mathbb{R})$, and H_D is a unique selfadjoint operator generated by H on $L^2(\mathbb{R})$.

The situation is quite different on $L^2(\mathbb{R}^+)$, there is one–parametric set of selfadjoint operators H_ξ, $\xi \in \langle 0, \pi)$ with corresponding definition domains D_ξ and with the same differential expression (1) ([1],p. 137). All these selfadjoint operators are selfadjoint extensions of the closed symmetric operator $\hat{H}$ with the domain $\hat{D} = \bigcap_{\xi \in \langle 0,\pi)} D_\xi$. Following the theorem ([2] p. 246) all these extension have the same essential spectrum. As in the case of the operator $H_{\xi=0}$, where it applies $\sigma_{ess}(H_{\xi=0}) = \emptyset$, it applies for all operators H_ξ, $\xi \in \langle 0, \pi)$. In other words, for any $\xi \in \langle 0, \pi)$ there exist an orthonormal basis formed by eigenvectors of H_ξ. The objective of this paper is to derive explicit form of the orthonormal basis and express $\sigma(H_\xi)$.

2 Parabolic cylinder functions

The parabolic cylinder fuctions [3] (9.240, 9.210)

$$D_\nu(x) = e^{-\frac{x^2}{4}}[\frac{\sqrt{\pi}\,2^{\frac{\nu}{2}}}{\Gamma(\frac{1-\nu}{2})}{}_1\Phi_1(-\frac{\nu}{2},\frac{1}{2},\frac{x^2}{2}) - \frac{\sqrt{\pi}\,2^{\frac{\nu+1}{2}}}{\Gamma(\frac{-\nu}{2})}\,x\,{}_1\Phi_1(\frac{1-\nu}{2},\frac{3}{2},\frac{x^2}{2})] \tag{2}$$

are the solutions of the Weber differential equation [3] (9.255)

$$(\frac{d^2}{dx^2} - \frac{x^2}{4} + \nu + \frac{1}{2})D_\nu(x) = 0.$$

Values $\nu \in \{0, 1, 2, \dots\} \equiv \mathbb{N}_0$ need special attention, because of

$$\frac{1}{\Gamma(\frac{-\nu}{2})} = 0, \ \ \Gamma(\frac{1-\nu}{2}) = \infty \ \ \text{for } \nu = 1, 3, 5, \dots$$

and

$$\frac{1}{\Gamma(\frac{-\nu}{2})} = \infty, \ \ \Gamma(\frac{1-\nu}{2}) = 0 \ \ \text{for } \nu = 0, 2, 4, \dots.$$

Definition (2) then gives $D_\nu(x) = h_\nu(x)$ known hermitian functions [3] (9.253).

The following relations holds for PCFs [3](7.711, 8.370, 8.372):

$$\int_0^\infty |D_\nu(x)|^2 dx = \frac{1}{c(\nu)^2}, \ \ c(\nu) = \sqrt{\sqrt{\frac{2}{\pi}}\frac{\Gamma(-\nu)}{\beta(-\nu)}}, \ \ \beta(-\nu) = \sum_{k=0}^{\infty}\frac{(-1)^k}{-\nu+k} \tag{3}$$

(note that $c(\nu)\ D_\nu$ is normalized), and

$$\int_0^\infty D_\nu(x)D_\mu(x)dx = \frac{\pi\,2^{\frac{1}{2}(\nu+\mu+1)}}{\mu-\nu}[\frac{1}{\Gamma(\frac{1-\mu}{2})\Gamma(-\frac{\nu}{2})} - \frac{1}{\Gamma(\frac{1-\nu}{2})\Gamma(-\frac{\mu}{2})}]. \tag{4}$$

3 Two Lemmas and two Theorems

It is known that the differential expression (1)

$$H = -\frac{d^2}{dx^2} + \frac{x^2}{4}$$

with definition domain

$$D_\xi\,(H) := \{f \in \tilde{\mathcal{D}}, f(0)\cos\xi - f'(0)\sin\xi = 0\}, \tag{5}$$

is a selfadjoint operator on $L^2(R^+)$ for all $\xi \in \langle 0, \pi)$, and $\tilde{\mathcal{D}} = \{f\,f' \in a.c.(0,\infty) : f, H\,f, \in L^2(R^+)\}$ [1] (p. 127, p. 137)

So, if D_ν will belong to $\mathcal{D}_\xi(H)$ for some ν then D_ν will be an eigenvector of the considered selfadjoint operator with eigenvalues $\nu + 1/2$. Eq. (3) guarantees that D_ν lies in $L^2(\mathbb{R}^+)$).

The last condition generates the relationship

$$D_\nu(0)\cos\xi - D'_\nu(0)\sin\xi = 0. \tag{6}$$

Although, values $D_\nu(0)$ and $D'_\nu(0)$ can be calculated using definition (2), we have to distinguish two cases:

1. when $\nu \notin \mathbb{N}_0$ we obtain

$$\eta\Gamma(-\frac{\nu}{2}) - \Gamma(\frac{1-\nu}{2}) = 0,\ \eta = \frac{1}{\sqrt{2}}\cot\xi. \tag{7}$$

2. when $\nu \in \mathbb{N}_0$ we obtain

$$h_\nu(0)\cos\xi - h'_\nu(0)\sin\xi = 0. \tag{8}$$

   If ν is odd, then $h_\nu(0) = 0,\ h'_\nu(0) = 1$, and Eq. (8) is fulfilled only if $\xi = 0$. If ν is even, then $h_\nu(0) = 1,\ h'_\nu(0) = 0$, and Eq. (8) is fulfilled only if $\xi = \frac{\pi}{2}$. In both cases, condition (8) is fulfilled by the set of Hermitian functions $\{h_0, h_2, h_4, \ldots\}$ and $\{h_1, h_3, h_5, \ldots\}$, respectively. It is known that both sets form orthonormal bases in $L^2(\mathbb{R}^+)$.

Eq.(7) has to be solved for ν.

First we prove two lemmas.

Lemma 1:

1. If $\nu \in (2M-1, 2M), M = 1, 2, \ldots$ or $\nu < 0$, then $\beta(-\nu) \geq 0$,
2. If $\nu \in (2M-2, 2M-1), M = 1, 2, \ldots$, then $\beta(-\nu) < 0$.

Proof:

1. Using the relationship

$$\beta(-\nu) = \sum_{k=0}^{\infty} \frac{1}{(-\nu+2k)(-\nu+2k+1)},$$

[3](8.372), it is possible to show by elementary calculation that $(-\nu + 2k)(-\nu + 2k + 1) > 0$ for all $k = 0, 1, \ldots$ if $\nu \in (2M-1, 2M), M = 0, 1, \ldots$, or $\nu < 0$.

2. In this case we rewrite the sum $\beta(-\nu)$ in the following form:

$$\beta(-\nu) = -\frac{1}{\nu} + \sum_{k=0}^{\infty} \frac{1}{-\nu+k+1} + \frac{1}{-\nu+k+2} = -\frac{1}{\nu} - \sum_{k=0}^{\infty} \frac{1}{(-\nu+k+1)(-\nu+k+2)}.$$

For considered values of $\nu \in (2M-2, 2M-1), M = 1, 2, \ldots$ the products $(-\nu+k+1)(-\nu+k+2)$ are positive, and therefore all denominators of the members in the previous sum are positive and so $\beta \leq 0$ (note that $-\frac{1}{\nu} < 0$). □

Remark: Comparing functions $\Gamma(-\nu)$ and $\beta(-\nu)$ we have the relationship

$$\operatorname{sgn}(\Gamma(-\nu)) = \operatorname{sgn}(\beta(-\nu)),\ \nu \in \mathbb{R}.$$

It shows that normalization factor $c(\nu)$ (Eq.3)) is correctly defined.

Lemma 2:

Function $y(\nu) := \frac{\Gamma(\frac{1-\nu}{2})}{\Gamma(-\frac{\nu}{2})}$ has the following properties:

1. There are asymptotes for $\nu_{\text{as}} \in \{2n+1 | n \in \mathbb{N}_0\}$ and $\lim_{\nu \to \nu_{\text{as}}^+} y(\nu) = \infty$, $\lim_{\nu \to \nu_{\text{as}}^-} y(\nu) = -\infty$. Further $\lim_{\nu \to -\infty} y(\nu) = +\infty$.
2. The set $\{2n | n \in \mathbb{N}_0\}$ consists of all zero points of y.
3. In the intervals $(-\infty, 1), (M, M+1), M = 0, 1, \ldots, y$ is continuous decreasing function.

Proof:

1. The first assertion is a direct consequence of explicit form [3](8.314) of function Γ.

For the remaining assertions it is sufficient to prove that the sequence $\{y(-2n) | n \in \mathbb{N}_0\}$ is growing and $\lim_{n\to\infty} y(-2n) = +\infty$. As $y(-2n)$ can be expressed as

$$y(-2n) = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)} = \frac{\sqrt{\pi}(2n-1)!!}{2^n(n-1)!}$$

[3](8.339), the assertion can be easily verified.

2. Γ – function has no zero points. Therefore $y(\nu) = 0$ only if $|\Gamma(-\frac{\nu}{2})| = \infty$, i. e. $\nu = 2n$.

3. For $y'(\nu)$ we obtain

$$\frac{dy(\nu)}{d\nu} = \frac{1}{2}\frac{\Gamma(\frac{1-\nu}{2})}{\Gamma(-\frac{\nu}{2})}\,[\psi(-\frac{\nu}{2}) - \psi(\frac{1-\nu}{2})],\ \ \psi(\mu) = \frac{d}{d\mu}\lg\Gamma(\mu).$$

Using the relationships

$$\psi(-\frac{\nu}{2}) - \psi(\frac{1-\nu}{2}) = -2\,\beta(-\nu),\ \text{ and } \Gamma(-\nu) = \frac{2^{-\nu-1}}{\sqrt{\pi}}\Gamma(\frac{1-\nu}{2})\,\Gamma(-\frac{\nu}{2}),$$

[3](8.370,8.335) we obtain

$$\frac{dy(\nu)}{d\nu} = -2^{\nu}\frac{\sqrt{\pi}}{\Gamma(-\frac{\nu}{2})^2}\Gamma(-\nu)\,\beta(-\nu).$$

As $\Gamma(-\nu)\,\beta(-\nu) > 0$ (see Remark) the proof is completed. □

The consequence of this Lemma is a Theorem

Theorem 1:

For any $\eta \in \mathbb{R}$ and any $M \in \mathbb{N} = \{1, 2, \dots\}$ there is just one solution $\nu_\eta^{(M)}$ of Eq. (7) in the interval I_M, where

$$I_1 = (-\infty, 1),\ I_M = (2M-1, 2M+1),\ M = 2, 3, \dots$$

No further solution of Eq. (7) exists.

Table 1: Example of first 11 values of ν_η

$\nu_{-2.18}$	$\nu_{-0.51}$	ν_0	$\nu_{0.23}$	$\nu_{0.51}$	$\nu_{0.97}$	$\nu_{2.18}$
0.77051	0.399912	0	-0.311391	-0.875066	-2.33401	-9.95
2.66471	2.26065	2.	1.86885	1.71369	1.5141	1.26337
4.59639	4.20523	4.	3.90249	3.78578	3.62177	3.36297
6.54652	6.1743	6.	5.91892	5.82117	5.67849	5.42659
8.50776	8.15402	8.	7.92911	7.84326	7.715227	7.47292
10.4764	10.1394	10.	9.93622	9.85874	9.74156	9.50897
12.4503	12.1283	12.	11.9415	11.8704	11.7617	11.5382
14.4281	14.1195	14.	13.9457	13.8795	13.7777	13.5626
16.409	16.1123	16.	15.9491	15.887	15.7908	15.5834
18.3922	18.1062	18.	17.9519	17.8932	17.8019	17.6014
20.3773	20.101	20.	19.9543	19.8985	19.8113	19.6172

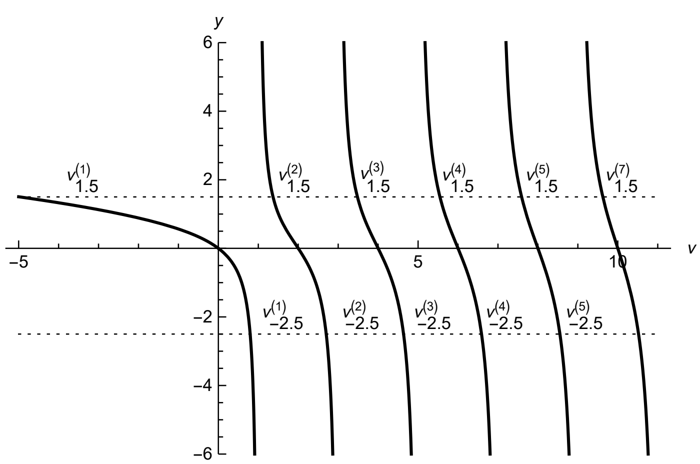

Figure 1: Function $y(\nu) = \frac{\Gamma\left(\frac{1-\nu}{2}\right)}{\Gamma\left(-\frac{\nu}{2}\right)}$

Let Ω_ξ denote the set

$$\Omega_\xi = \{\nu^{(M)}_{\cot\xi},\ M = 1, 2, \ldots\},\ \xi \in \langle 0, \pi),\ \nu^{(M)}_{\cot\xi} = \nu^{(M)}_\eta,$$

(we understand $\Omega_0 = \{0, 2, 4, \ldots\}$),

and let denote further by $\mathcal{E}_\xi$ the set

$$\mathcal{E}_\xi = \{c(\nu)D_\nu |\, \nu \in \Omega_\xi\}.$$

The set $\mathcal{E}_\xi \subset \mathcal{D}_\xi$ contains all eigenvectors of the selfadjoint operator H_ξ, and the set $\{\nu + \frac{1}{2}, \nu \in \Omega_\xi\}$ contains all eigenvalues of H_ξ.

Note that orthogonality of two different eigenvectors can bee seen also from the Eq.(4). For different $\mu,\ \nu$ fulfilling the condition $\Gamma(\frac{1-\mu}{2})/\Gamma(-\frac{\mu}{2})) = \Gamma(\frac{1-\nu}{2})/\Gamma(-\frac{\nu}{2}) = \eta$, which is our case, Eq. (4) is the scalar product in $L^2(R^+)$ equal to zero. Moreover, the Eq. (3) guarantees that the eigenvectors are normalized.

We denote further by $\hat{H}$ the restriction of H_ξ to domain

$$\hat{D} = \{f \in \tilde{D},\ f(0) = f'(0) = 0\} \subset D_\xi(H) \subset L^2(\mathbb{R}^+).$$

Operator $\hat{H}$ is closed, symmetric with deficiency indices (1,1) [1] (prop. 4.8.6, p.129), and H_ξ is a selfadjoint extension of $\hat{H}$ for any $\xi \in \langle 0, \pi)$. Selfadjoint extensions $H_{\xi=0}$ and $H_{\xi=\frac{\pi}{2}}$ have pure point spectra, which is equivalent to the existence of orthonormal bases in $L^2(\mathbb{R}^+)$. The basis is in the case $D_{\xi=0}(H) = \{h_{2n+1}|n = 0, 1, 2, \ldots\}$, and it is in the case $D_{\xi=\frac{\pi}{2}}(H) = \{h_{2n+1}|n = 0, 1, 2, \ldots\}$. As we mentioned in the introduction, the same is true for all operators H_ξ with any parameter ξ.

Consequently, one can write theorem

Theorem 2

The set $\mathcal{E}_\xi$ consisting of eigenvectors of H_ξ is an orthonormal basis in $L^2(\mathbb{R}^+)$ for any $\xi \in \langle 0, \pi)$, and $\sigma(H_\xi) = \{\nu + \frac{1}{2}, \nu \in \Omega_\xi\}$.

4 Concluding remarks

The results we present can be translated to the case $L^2(\mathbb{R}^-)$. In this case an orthonormal basis in $L^2(\mathbb{R}^-)$ is

$$\tilde{\mathcal{E}_\xi} := \{\tilde{D}_\nu |\, \nu \in \Omega_\xi\}, \tilde{D}_\nu(x) := D_\nu(-x).$$

These two bases can be combined to the base in $L^2(\mathbb{R})$. As $L^2(\mathbb{R}) = L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^-)$. Then for any pair $(\xi, \sigma) \in \langle 0, \pi) \times \langle 0, \pi)$ the set $\mathcal{E}_\xi \oplus \tilde{\mathcal{E}_\sigma}$ is a basis in $L^2(\mathbb{R})$. Explicitly

$$\mathcal{E}_\xi \oplus \tilde{\mathcal{E}}_\sigma = \{(D_\nu, 0),\ \nu \in \Omega_\xi\} \cup \{(0, \tilde{D}_\nu),\ \nu \in \Omega_\sigma\}.$$

Of note, the known orthonormal basis $\{h_n, n = 0, 1, \ldots\}$ of $L^2(\mathbb{R})$ consisting of hermitian functions is not contained in this set. Functions h_n are eigenvectors of selfadjoint operator H_D with definition domain

$$D = \{f, f' \text{absolutely continuous}, f,\ Hf \in L^2(\mathbb{R})\},$$

and operator H_D is physically interpreted as Hamiltonian of quantum linear harmonic oscillator.

References

[1] J. Blank, P. Exner, M. Havlíček, Hilbert Space Operators in Quantum Physics, 2nd edition, Springer, 2008 .

[2] J.Weidmann, Linear Operators in Hilbert Space, Springer Verlag, New York 1980

[3] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, 4th edition, Fizmatfiz, Moskva 1963