A Comparative Study of Secondary School Mathematics Teaching Materials from the Perspective of Reasoning--Taking "Pythagorean Theorem" as an Example

ABSTRACT

Education has always been a hot topic of discussion. From the results of junior high school mathematics assessments in recent years, there are differences in mathematical abilities between Eastern Europe and Europe and America. Since reasoning is crucial to understanding mathematics,we compare the "Pythagorean Theorem" in textbooks from China, the United States, and Singapore, examining the theorem's different types of reasoning and how it relates to previously acquired knowledge. The reasoning categories proposed by Shi Ningzhong, Harel, and Sowder serve as the fundamental basis for categorizing the varieties of reasoning in this research. The introduction of proofs for the hook and Pythagorean theorem is investigated for each edition of the textbook, and concerning the deductive reasoning ability, the People's Education Press edition(PEP) and Zhejiang Education edition(ZJE) editions are in the first tier, followed by Singaporean textbooks and American textbooks. In terms of the correlation between the Pythagorean theorem and what has been learned, the ZJE involves the most points of correlation. Finally, we discuss the implications of these results for teachers' explanations of the Pythagorean theorem: teachers can briefly introduce the geometric deductive proof process of the Pythagorean theorem to stimulate students' interest in learning mathematics.

Keywords: Pythagorean theorem; types of reasoning; deductive reasoning.

1 INTRODUCTION

Education is the foundation of social modernization construction and a key representation of a comprehensive national strength. Education plays a crucial role in facing challenges.[1]The globe is looking towards education reform as a result of the advancement of science and technology. According to the most recent 2019 Trends in International Mathematics and Science Study (TIMSS) data, East Asian nations and regions are in the lead, with Singapore, Chinese Taipei, South Korea, Japan, and Hong Kong, China having the top five math scores for eighth-graders.According to the Program for International Student Assessment (PISA) test results, children from China, Singapore, and Japan performed better on the math portion of the exam, and China came out on top in all three abilities—reading, math, and science—in the 2018 PISA test. When seen as a whole, East Asian students perform better academically in Mathematics than pupils in developed nations like the United States and Europe, and the distribution of mathematics test scores around the world exhibits a polarized trend with significant variances. There are a number of causes for this occurrence. Some scholars believe that the quality of school resources has a significant impact on students as a whole, while others believe that different textbooks and teaching content lead to differences among students.Textbooks are viewed as a potential implementation course that significantly impacts the educational process.[2]Therefore, we examine the impact of various teaching materials on students' academic performance.

Research on education reform has focused on school curricula as a platform for implementing education. The Compulsory Education Curriculum Program and Curriculum Standards (2022 Edition)[1], which represents a thorough reform of the application of compulsory education curriculum standards since 2011, was published by China's Ministry of Education in 2022. It places a strong emphasis on the necessity of enhancing core literacy and being oriented toward core literacy in junior high school. Education is the most important thing in forming individuality, and on this basis, we hope to improve and cultivate students' core mathematical literacy through education.[3]**Error! Reference source not found.**Logical thinking is a component of mathematics' core literacy. The development of mathematics itself is achieved by reasoning; therefore, it is essential to strengthen pupils' capacity for reasoning in middle school.Fan L believes that these discrepancies unequivocally show that there is no international research community consensus regarding the function and significance of proof in the mathematics curriculum and the manner in which proof is taught in school mathematics.[4] We believe it is obvious that further reliable study in this area is essential before a genuine consensus can be formed. As a result, more thorough research in this area is required.

The Pythagorean Theorem, one of the most famous theorems in mathematics, is one of the important theorems in geometry theorems in junior high school and is an important content in developing students' reasoning ability. It reveals the quantitative relationship between the three sides of a right triangle, establishes a bond between geometry and algebra, and has a wide range of applications in the study of graphics and real life.

Based on the aforementioned factors, we use the "Pythagorean Theorem" as an example and choose four textbooks for a comparative study: the People's Education Press edition, the Zhejiang Education edition, the Singapore textbook, and the American textbook. Then we explored the differences in inference types introduced by the concept of Pythagorean theorem.Through the comparison of various textbooks, the advantages and disadvantages of the textbooks are clarified, to provide reference opinions for the introduction of inferential theorems in the compilation of mathematics textbooks in China. At the same time, it can enable teachers to better grasp the teaching materials and provide directions for specific teaching.

2Literature review

2.1 Research status of inference models

J Lithner[5] divides mathematical reasoning into imitative and creative mathematically founded reasoning. Imitative reasoning, i.e., memorizing solutions, is divided into two main types of reasoning: memorized reasoning (MR) and algorithmic reasoning (AR). Memorized reasoning , is one by recalling the process of solving a problem; the other type of memorized reasoning involves making judgments about conclusions through prior experience. Algorithmic reasoning is the selection of a solution algorithm, and the impact of the rest of the reasoning process on the results can be negligible. All conceptually difficult parts are handled by algorithms, which may limit students' learning effectiveness. Creative mathematically founded reasoning (CMR)needs to meet the three conditions of novelty, rationality, and mathematical foundation. This reasoning is rational and logical.

Bergqvistand Ewa[6] examined what types of reasoning models college students need to use for exams. They divided them into four categories: memorized reasoning, algorithmic reasoning, local creative reasoning which is reasoning in which all but one of the steps are familiar to the student, and global creative reasoning which is reasoning that builds on the intrinsic nature of mathematics and in which the student may be required to use creative thinking processes. Research has shown that the first two types of reasoning weaken students' understanding of math concepts.

Harel and Sowder[7]offer a thorough analysis of the proof. The development of pupils' capacity to "reason deductively" is an aim shared by mathematics curricula around the globe. They divide reasoning into three categories: inference from external certainty (first), which mainly refers to theories proposed by authoritative figures such as mathematicians and some basic propositions that do not need to be proven; empirical reasoning (second), reasoning or perceiving through special examples; and deductive reasoning (third), which is divided into transformational and axiomatic reasoning.

The transmissibility of the logical reasoning process is split into two categories in Shi Ningzhong's study of mathematical logic. The first is deductive reasoning, which is a type of relational transmissive reasoning, or reasoning from the general to the specific, in which the set of satisfied relations or objects of study addressed remains constant. In the second type, inductive reasoning, in relational transitive reasoning, the range of satisfaction relationships or research objects discussed is from small to large, that is, from special to general reasoning. Induction and analogy are the two subcategories of inductive reasoning.Induction is based on the fact that a set comprises many things with the same properties and so brings holistic conclusions. The analogy is based on the fact that two sets have the same attributes and that one of sets A can infer the property of P, which means that the set B can also infer the property of P.

3 FRAMEWORKS OF INQUIRY

Reasoning is an activity of thinking that starts from a previously known statement of truth and deduces a new statement.[8]Developing students' mathematical reasoning skills is a goal of many curricula and an important part of the culture of the mathematics education research community.[9]The introduction of the concept of the Pythagorean theorem requires teachers to guide students to infer and obtain the Pythagorean theorem from their already learned knowledge. In this reasoning process, different types of reasoning are associated with different knowledge. The concept of a research framework is the core of each research field.[10] Therefore, we compare and analyze the presentation process of the Pythagorean theorem in different textbooks from two dimensions: the inference types introduced by the Pythagorean theorem and the correlation between inference types and learned knowledge.

3.1 TYPES OF REASONING FOR CONSTRUCTING PROOFS OF THE PYTHAGOREAN THEOREM

The "Mathematics Curriculum Standards for Compulsory Education (2022 Edition)" (hereinafter referred to as the "Curriculum Standards (2022)") puts forward the following requirements for middle school reasoning ability: for some simple problems, the ability to infer general conclusions through special results; Understand the structure and connections of propositions, explore and express the process of argumentation; Understand the rigor of mathematics and develop a habit of logical expression and communication. The Curriculum Standards (2020) also elaborates on the connotations of inductive reasoning and deductive reasoning.

We consulted materials, combined with the requirements for cultivating middle school students' mathematical reasoning ability and the specific content of the Pythagorean theorem, and referred to the reasoning model framework proposed by Shi Ningzhong, Harel, and Sowder.

The three types of reasoning proposed by Harel and Sowder, among which "empirical reasoning" is basically equivalent to the inductive reasoning proposed by Shi Ningzhong, and "deductive reasoning" is equivalent to the deductive reasoning proposed by Shi Ningzhong.

Inspired by the existing studies, wedivided the types of reasoning into the following groups, depending on the degree of reliance on numbers and shapes: reasoning about the fundamental theorem, reasoning that relies on the area of a special case, based on general model area inference, reasoning based on combining numbers and shapes, and geometric deductive reasoning. There are very few examples of "reasoning about the fundamental theorem" type, and the majority of them fall under the category of "reasoning that relies on the area of a special case" type. We investigated all versions of the textbook for examples of "geometric deductive reasoning" type for the Pythagorean theorem reasoning but was unable to discover any.

The " reasoning about the fundamental theorem " refers to the theorem being given directly without the need for the student to prove it through reasoning, similar to the "inference from external certainty " proposed by Harel and Sowder; " reasoning that relies on the area of a special case " denotes that the theorem is obtained by relying on the area of a special right triangle or a right triangle with a specified side length, Similar to the "empirical reasoning" proposed by Harel and Sowder;"based on general model area inference" refers to the exploration of general right triangles and the derivation of the Pythagorean theorem based on the relationship between the areas of the shapes; "reasoning based on combining numbers and shapes" refers to the use of algebraic form and graphical reasoning to derive the Pythagorean theorem, and "geometric deductive reasoning" refers to the use of rigorous geometric reasoning.The last three are similar to the "deductive reasoning" proposed by Harel and Sowder.

Table 1 Links between different types of reasoning

Shi Ningzhong	Harel and Sowder	the author
	inference from external certainty	reasoning about the fundamental theorem
inference by induction	empirical reasoning	reasoning that relies on the area of a special case
deductive inference	deductive inference	based on general model area inference
		reasoning based on combining numbers and shapes
		geometric deductive reasoning

3.2 CONNECTIONS BETWEEN REASONING ABOUT THE PYTHAGOREAN THEOREM AND WHAT HAS ALREADY BEEN LEARNED

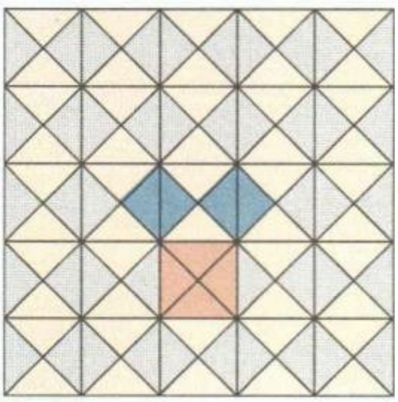
In the process of inferring Pythagorean theorem, different types of inference have different degrees of relevance to the knowledge already learned. Generally speaking, the higher the dependence on geometric area, the less learned knowledge associated with the Pythagorean theorem may be; The higher the degree of dependence on deductive reasoning, the more learned knowledge may be associated with the Pythagorean theorem.

4RESULTS

4.1 TYPES OF REASONING FOR CONSTRUCTING PROOFS OF THE PYTHAGOREAN THEOREM

1. Reasoning models for the PEP version's derivation of the Pythagorean theorem. Two different methods of reasoning—“reasoning about the Fundamental Theorem” and “Based on general model area inference”—are employed to derive the Pythagorean Theorem.

Introduced from the actual example of floor tiles (Fig. 1), starting from the area relationship of three squares, the relationship between the three sides of the isosceles triangle can be explored, and it can be found that the area of the two squares with the right-angled side of the isosceles triangle as the side length and the area of the square with the hypotenuse of the isosceles triangle as the side length can obtain the relationship between the three sides of the isosceles triangle: the square of the hypotenuse is equal to the sum of the squares of the two right-angled sides.After that, the right triangle is explored from the special case, and for the given two right triangles (Fig. 2), the area of the square with the three sides of the right triangle as the side lengths are calculated, thus obtaining the Pythagorean theorem: if the lengths of the two right-angled sides of the right triangle are. The length of the hypotenuse is c , then $a^2+b^2=c^2$. This is the reasoning that relies on the area of a special case in the PEP.



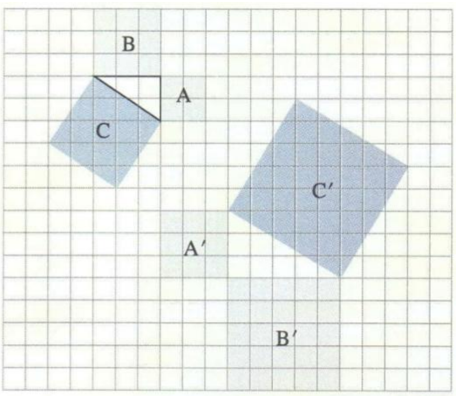


Figure 1floor tiles**Figure 2** area of the square with the three sides of the right triangle

Thethe People's Education Press edition of the proof of the Pythagorean theorem gives the ancient Chinese Zhao Shuang's method of proof. Using the "cut and fill method", such as in Figure 3, (1) in the side length of a, b the square together, their area is a^2+b^2 , and then this figure is broken down into four congruent right triangles and a square, (1) in the two right triangles rotated to the position of (2), you will get a c as the side length of the square, the area is c^2 . Since the figures in Figures 3(1) and (3) both consist of four congruent right triangles and a small square they have equal areas, we get $a^2+b^2=c^2$. This is "based on general model area inference" in the PEP.

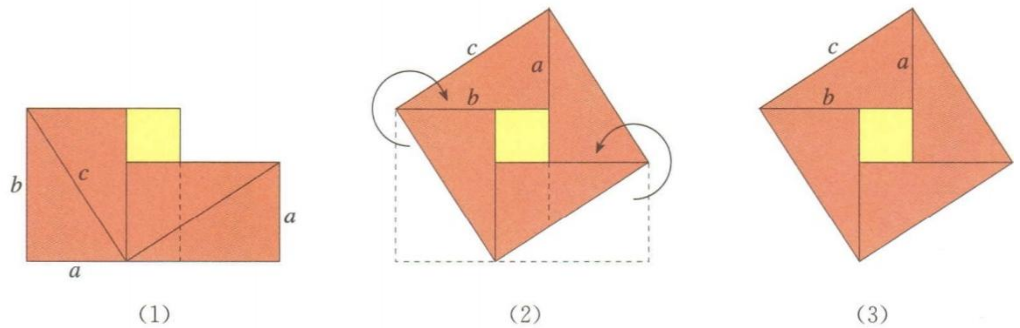
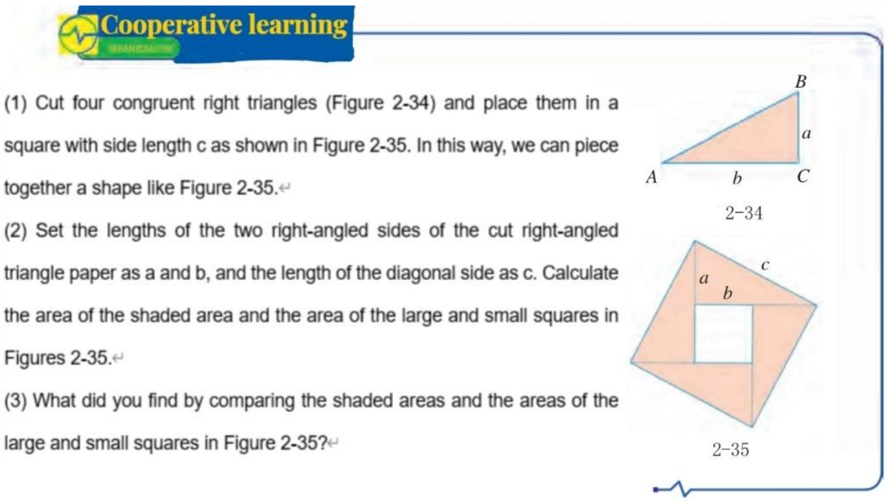


Figure 3 cut and fill method

2. The methodology by which the Pythagorean theorem in the ZJE version was derived. In the ZJE version, the reasoning is used to derive the Pythagorean theorem by reasoning based on combining numbers and shapes.


Cooperative learning

(1) Cut four congruent right triangles (Figure 2-34) and place them in a square with side length c as shown in Figure 2-35. In this way, we can piece together a shape like Figure 2-35.

(2) Set the lengths of the two right-angled sides of the cut right-angled triangle paper as a and b, and the length of the diagonal side as c. Calculate the area of the shaded area and the area of the large and small squares in Figures 2-35.

(3) What did you find by comparing the shaded areas and the areas of the large and small squares in Figure 2-35?

Figure 4area of the large square

As shown in Figure 4, the area of the large square is calculated as c^2 , the area of the small square is $(b-a)^2$, the sum of the areas of four congruent right triangles is $2ab$, and with the help of algebraic operations $c^2-(b-a)^2=2ab$, using the knowledge of perfect square and combining like terms, the Pythagorean theorem is obtained $a^2 + b^2 = c^2$. This is "reasoning based on combining numbers and shapes" in the ZJE.

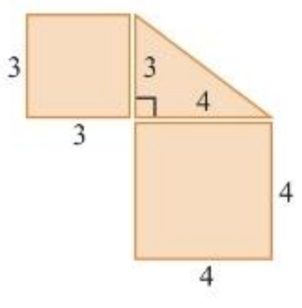
3. The American textbook's justification for how the Pythagorean theorem was derived. The Pythagorean theorem is derived from the American textbook using the fundamental theorem's logic, as seen in Figure 5. The Pythagorean Theorem is explicitly stated. Students do not need to provide evidence in order to obtain something equivalent to the axiom.

> **Pythagorean Theorem**
> the sum of the squares of the lengths of the legs in a right triangle is equal to the square of the length of the hypotenuse; $a^2 + b^2 = c^2$

Figure 5Pythagorean theorem

4. The Singapore textbook's justification for obtaining the Pythagorean Theorem. The Pythagorean Theorem is derived in the Singapore textbook using "reasoning that relies on the area of a special case".

The Singapore textbook's introduction to the Pythagorean theorem gives right triangles with fixed side lengths, as shown in Figure 6. Make the parallel sides of the hypotenuse in the larger square AB ,and across the center of the larger square C make EF perpendicular to AB , as shown in Figure 7. Along the AB , EF sides of the larger square will be cut into four and the smaller square a total of five, rearranged on the hypotenuse to form a new square, as shown in Figure 8.



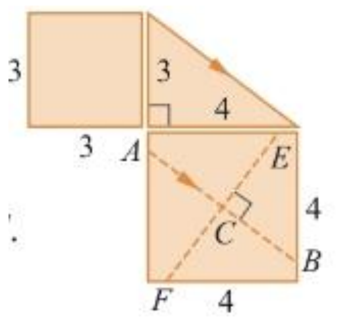


Figure 6 **Figure 7**

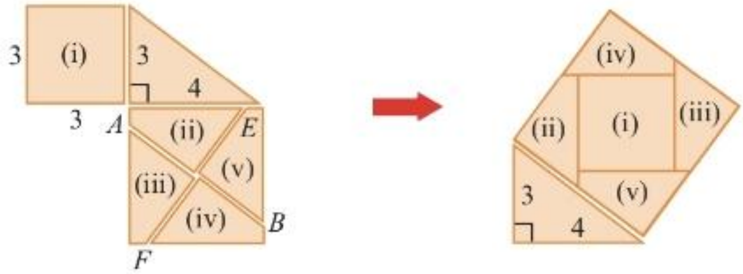


Figure 8

Figure 6-8 Singapore textbook's justification for obtaining the Pythagorean Theorem

Thus, by equality of areas, as shown in Figure 9, we get that the sum of the squares of the two right angle edge is equal to square of hypotenuse, i.e. $a^2 + b^2 = c^2$. This is the reasoning in the Singapore textbook “reasoning that relies on the area of a special case”.

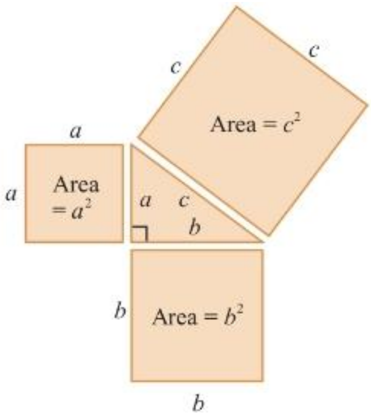


Figure 9 Sum of the squares of the two right angle

In summary, the types of reasoning in each version of the textbook are shown in Table 2.

Table 2 Types of reasoning in each version of the textbook

Textbook version	Type of reasoning
PEP	Reasoning that relies on the area of a special case, based on general model area inference
ZJE	reasoning based on combining numbers and shapes
United States of America	Reasoning about the fundamental theorem
Singaporean	Reasoning that relies on the area of a special case

4.2 CONNECTIONS BETWEEN REASONING ABOUT THE PYTHAGOREAN THEOREM AND WHAT HAS ALREADY BEEN LEARNED

Through the relevant learned knowledge involved in the reasoning process of Pythagorean theorem in various textbooks, Table 3 is obtained.

Table 3 Knowledge points associated with reasoning about the Pythagorean theorem in each version of the textbook

Textbook version	Associated Learned Knowledge Points
PEP	Calculation of square area, concept of congruent triangles, cut and fill method
ZJE	Calculating the area of squares and right triangles, concept of congruent triangles, perfect square equation, combining like terms
United States of America	square operation
Singaporean	Calculation of square area, concepts and graphing of parallelism and perpendicularity, cut and fill method

5Discussion

The Pythagorean Theorem has nearly 400 different proofs, and Euclid's Geometria Original uses geometric deduction to demonstrate it. Despite reading several textbook editions, no proofs via geometric deduction were discovered.In junior high school teaching, due to the limitations of teaching time and students' knowledge foundation, complex geometric deduction proofs cannot be written in textbooks. However, teachers can give students a quick introduction to them in order to develop their deductive thinking skills and to pique their interest in learning more about the Pythagorean theorem, and mathematics.

6 Conclusions

From the reasoning perspective of the introduction of the Pythagorean theorem in each version, the PEP involves “reasoning that relies on the area of a special case” and “based on general model area inference”. They fall under the categories of "empirical reasoning" and "deductive reasoning," respectively, according to Harel and Sowder[7]. The PEP of the Pythagorean Theorem is introduced in a way that moves from the specific to the general, first examining the relationship between the three sides of isosceles right triangles before moving on to right triangles in general. The form of reasoning chosen is " reasoning that relies on the area of a special case " and then "based on general model area inference", similar to Shining's "inductive reasoning," which goes from the particular to the general. The ZJE uses "reasoning based on combining numbers and shapes", which is the perfect combination of numbers and shapes. It also simplifies the proof process. It deduces the Pythagorean theorem from general triangles, and the combination of numbers and shapes simplifies the proof process. The introduction of the Pythagorean theorem in American textbooks directly provides a theorem, which is the"reasoning about the fundamental theorem". Similar to the People's Education Press edition, the Singapore textbook employs " reasoning that relies on the area of a special case " and employs the cut-and-fill technique. However, the cut-and-fill technique is distinct since it does not begin with a universal right-angled triangle but rather calls for a specific geometric base and a fixed right-angled triangle.

Regarding the relationship between the Pythagorean Theorem and prior knowledge, the Zhejiang Education edition involves five knowledge points, including "calculating the area of a square," "calculating the area of a right-angled triangle," "the concept of congruent triangles," "Perfect Flatness," "Combining Similar Terms," and other five knowledge points; the People's Education Press edition and Singapore textbook involve three to five associated knowledge points. There are not many linked knowledge points in American textbooks because it is a simple theorem reasoning that provides the theorem directly.

In order to establish new propositions or conclusions based on rules, one must first start with basic facts and propositions. The development of mathematical theorems depends heavily on logical reasoning. Even while curricular standards vary from country to country, they nonetheless have similar objectives for students' deductive reasoning skills. We discover that the Zhejiang and the People's Education Press edition textbooks are at the first tier, the Singapore textbook is at the second tier, and the American textbook is at the third tier, taking the development of students' deductive reasoning skills as a reference. Comparatively speaking, Asian mathematics textbooks place a greater emphasis on students' ability to use deductive reasoning than European and American textbooks .This also partially explains why Asian students have higher academic achievements in mathematics than those in Europe and America.

7REFERENCE

[1] Hera, R., and N. Sari. "SEMINAR NASIONAL MATEMATIKA DAN PENDIDIKAN MATEMATIKA UNY 2015 713 Literasi Matematika: Apa." Mengapa dan Bagaimana (2015): 713-720.

[2] Wijaya, A. "How Indonesian and Singaporean Students (might) Learn the Pythagorean Theorem: A Comparative Analysis of Textbooks." Journal of Physics: Conference Series. Vol. 1320. No. 1. IOP Publishing, 2019.

[3] Maslihah, Siti, S. B. Waluya, and Amin Suyitno. "The role of mathematical literacy to improve high order thinking skills." Journal of Physics: Conference Series. Vol. 1539. No. 1. IOP Publishing, 2020.Chi Hua Hong. "Analysis of the teaching value of the Pythagorean theorem in junior high school mathematics[J]. Mathematics Teaching Newsletter,2017(17):57-59.

[4] Fan, Lianghuo, et al. "A comparative study on the presentation of geometric proof in secondary mathematics textbooks in China, Indonesia, and Saudi Arabia." Research on mathematics textbooks and teachers’ resources: Advances and issues (2018): 53-65.

[5] Lithner, Johan. "A research framework for creative and imitative reasoning." Educational Studies in mathematics 67 (2008): 255-276. Narengele. Research on the level of inductive reasoning in mathematics of junior high school students[D]. Northeast Normal University, 2020. DOI: 10.27011/d.cnki.gdbsu.2019.000047.

[6] "Types of reasoning required in university exams in mathematics." The Journal of Mathematical Behavior 26.4 (2007): 348-370.Harel G, Sowder L. Towards comprehensive perspectives on the learning and teaching of proof [J]. Lester F. Second handbook of research on mathematics teaching and learning.,2007,2:805-842.

[7] Harel G, Sowder L. Towards comprehensive perspectives on the learning and teaching of proof [J]. Lester F. Second handbook of research on mathematics teaching and learning.,2007,2:805-842.

[8] Ayal, Carolina S., et al. "The Enhancement of Mathematical Reasoning Ability of Junior High School Students by Applying Mind Mapping Strategy." Journal of Education and Practice 7.25 (2016): 50-58.

[9] Jeannotte, Doris, and Carolyn Kieran. "A conceptual model of mathematical reasoning for

school mathematics." Educational Studies in mathematics 96 (2017): 1-16.

[10] Lester, Frank K. "On the theoretical, conceptual, and philosophical foundations for research in mathematics education." Zdm 37 (2005): 457-467.