Linear Hybrid Multistep Block Method for Direct Solution of Initial Value Problems of Third Order Ordinary Differential Equations

Abstract

In this article, we focus on linear hybrid multistep method for direct solution of initial value problems of third order ordinary differential equations without reduction to system of first-order ordinary differential equations. The derivation of the method involved using collocation and interpolation techniques with power series as basis function to produce a system of linear equations. The unknown parameters in the system of equations were obtained through the Gaussian elimination technique. The values of the determined parameters were then substituted and evaluated at different grid and off-grid points to produce the required continuous block method. The discrete scheme obtained from the method is self-starting with improved accuracy and a larger interval of absolute stability. Basic properties of the method were investigated. The results showed that the method is zero stable, consistent , convergent and of order seven. The performance of the method was tested by solving linear and nonlinear problems of general third order ordinary differential equations. The result were found to compare favourably with some existing methods in literature.

Keywords: Linear multistep method, hybrid points, convergence, interpolation, collocation and block method.

AMS Subect Classification: 65J05, 65L06

1 Introduction

This article considers the numerical solution of initial value problems (ivps) of general third order ordinary differential equations is of the form:

$$y''' = f(x, y, y', y''), y(0) = y_0, y' = y'_0, y'' = y''_0 \tag{1}$$

In the past, third order ordinary differential equations (1) were traditionally solved by reducing them to first order initial value problem of ordinary differential equations and then solving the resulting equations using any of the available numerical methods. This numerical approach and its attendant drawbacks had been extensively referenced by many authors. (See Awoyemi 1988,1999, Awoyemi and Kayode 2005, Kayode *et al.* 2014, Duromola 2016). As a result of this, alternative solutions for these problems had been considered by several authors. Different methods of solving higher-order differential equations without reduction to a set of first order ordinary differential equations have been produced. These methods include those of Awoyemi and Kayode 2008, Olabode 2019, Kayode and Adeyeye 2011, Areo and Adeniyi 2013, Kayode and Obarhua 2013, Obarhua and Kayode 2016. Duromola and Momoh (2019) developed a hybrid numerical method with block extension for direct solution of initial value problems of third-order ordinary differential equations consider method of the form:

$$y_{\frac{j}{2}}(x) = \sum_{j=1}^{k-1} \alpha_{\frac{j}{2}}(x) y_{n+\frac{j}{2}} + h^3 \left(\sum_{j=0}^{k} \beta_{\frac{j}{2}}(x) f_{n+\frac{j}{2}} \right), \tag{2}$$

The method has order of accuracy of order five. The focus of the present work is to provide method with improved order of accuracy.

2 Derivation of the Block Method

This paper focused on direct solution of initial value problems of general third order ordinary differential equations of the type:

$$y''' = f\left(x, y, y'y''\right),\ y(0) = y_0,\ y'(0) = y_0',\ y''(0) = y_0'',\ f \in \Re^n[a,b], y \in \Re, x \in [a,b] \tag{3}$$

using power series approximation solution of the form

$$y(x) = \sum_{j=0}^{(c+i)-1} a_j x^j, \tag{4}$$

where $a_j, j = 0, 1, 2, \cdots, k$ are the coefficients to be determined.

The third derivative of equation (5) is obtained as:

$$y'''(x) = \sum_{j=3}^{k} j(j-1)(j-2) a_j x^{j-3} \tag{5}$$

Combining (4) and (6) yield the differential system:

$$f(x, y(x), y'(x), y''(x)) = \sum_{j=3}^{k} j(j-1)(j-2) a_j x^{j-3} \tag{6}$$

Interpolate equation (5) at $x = x_{n+j}, j = \frac{3}{2}, 2, \frac{5}{2}$ and collocate equation (6) at $x = x_{n+j}, j = o, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$ to have a set of linear equation of

$$AX = B \tag{7}$$

where

$$X = \begin{bmatrix} x_{11} & x_{12} & . & . & . & x_{1n} \\ x_{21} & x_{22} & . & . & . & . \\ x_{31} & x_{32} & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ x_{n1} & x_{n2} & . & . & . & x_{nn} \end{bmatrix}, A = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{bmatrix}, B = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix} \tag{8}$$

Solving (7) for A and substituting the value of $a_j s$ in (5) yields the following continuous schemes after some simplifications:

$$y(t) = \sum_{j=0}^{9} \alpha_{\frac{j}{2}}(t) y_{n+\frac{j}{2}} + h^3 \sum_{j=0}^{9} \beta_{\frac{j}{2}}(t) f_{n+\frac{j}{2}}(t) \tag{9}$$

Using the transformation in Awoyemi and Kayode (2008),

$$t = \left(\frac{x - x_{n+k-1}}{h} \right) \tag{10}$$

$$\frac{dt}{dx} = \frac{1}{h}$$

where k denotes step number. We proceed by evaluating at different values of t

$\alpha_{\frac{3}{2}} = 10t$

$\alpha_2 = (-4t^2 + 16t - 15)$

$\alpha_{\frac{5}{2}} = 6t$

$$\beta_0 = \frac{1}{7257600} h^3 \left(1280\,t^9 - 20160\,t^8 + 134400\,t^7 - 493920\,t^6 + 1091328\,t^5 - 1481760\,t^4 + 1209600\,t^3 - 551055\,t^2 + 115147\,t - 3930\right)$$

$$\beta_{\frac{1}{2}} = -\frac{1}{1209600} h^3 \left(1280\,t^9 - 19200\,t^8 + 119040\,t^7 - 389760\,t^6 + 701568\,t^5 - 604800\,t^4 + 442860\,t^2 - 327713\,t + 77970\right)$$

$$\beta_1 = \frac{1}{483840} h^3 \left(1280\,t^9 - 18240\,t^8 + 105216\,t^7 - 309792\,t^6 + 471744\,t^5 - 302400\,t^4 - 53049\,t^2 + 221359\,t - 114810\right)$$

$$\beta_{\frac{3}{2}} = -\frac{1}{362880} h^3 \left(1280\,t^9 - 17280\,t^8 + 92928\,t^7 - 249984\,t^6 + 341376\,t^5 - 201600\,t^4 + 135726\,t^2 - 290177\,t + 211530\right)$$

$$\beta_2 = \frac{1}{483840}\,h^3\left(1280\,t^9 - 16320\,t^8 + 82176\,t^7 - 206304\,t^6 + 266112\,t^5 - 151200\,t^4 - 26787\,t^2 + 196771\,t - 175290\right)$$

$$\beta_{\frac{5}{2}} = -\frac{1}{1209600}\,h^3\left(1280\,t^9 - 15360\,t^8 + 72960\,t^7 - 174720\,t^6 + 217728\,t^5 - 120960\,t^4 + 26160\,t^2 - 9473\,t + 2370\right)$$

$$\beta_3 = \frac{1}{7257600}\,h^3\left(1280\,t^9 - 14400\,t^8 + 65280\,t^7 - 151200\,t^6 + 184128\,t^5 - 100800\,t^4 + 19635\,t^2 - 953\,t - 3930\right)$$

Evaluating (9) at $t = 0, \frac{1}{2}, 1, 3$ yield the discrete three step formulas.

$$y_n - 10\,y_{n+\frac{3}{2}} + 15\,y_{n+2} - 6\,y_{n+\frac{5}{2}} = \frac{h^3}{241920}\Big(-131\,f_n - 15594\,f_{n+\frac{1}{2}} - 57405\,f_{n+1} - 141020\,f_{n+\frac{3}{2}} - 87645\,f_{n+2} - 474\,f_{n+\frac{5}{2}} - 131\,f_{n+3}\Big) \quad (11)$$

$$y_{n+1/2} - 3\,y_{n+\frac{5}{2}} + 8\,y_{n+2} - 6\,y_{n+\frac{3}{2}} = \frac{h^3}{241920}\Big(61\,f_n - 744\,f_{n+\frac{1}{2}} - 12819\,f_{n+1} - 63464\,f_{n+\frac{3}{2}} - 43689\,f_{n+2} - 240\,f_{n+\frac{5}{2}} - 65\,f_{n+3}\Big) \quad (12)$$

$$y_{n+1} - y_{n+\frac{5}{2}} + 3\,y_{n+2} - 3\,y_{n+\frac{3}{2}} = \frac{h^3}{241920}\Big(31\,f_n - 249\,f_{n+\frac{1}{2}} + 654\,f_{n+1} - 15866\,f_{n+\frac{3}{2}} - 14781\,f_{n+2} + 3\,f_{n+\frac{5}{2}} - 32\,f_{n+3}\Big) \quad (13)$$

$$y_{n+3} - 3\,y_{n+\frac{5}{2}} + 3\,y_{n+2} - y_{n+\frac{3}{2}} = \frac{h^3}{241920}\Big(32\,f_n - 255\,f_{n+\frac{1}{2}} + 921\,f_{n+1} - 1774\,f_{n+\frac{3}{2}} + 16986\,f_{n+2} + 14109\,f_{n+\frac{5}{2}} + 221\,f_{n+3}\Big) \quad (14)$$

In line with Kayode *et al.* (2014), the normalized form of the general block method is given by

$$Ay_i = Ey_n + h^{N-p}df(y_n) + h^{N-p}BF(y_n) \quad (15)$$

By combining the equations, the first and the second derivatives of the schemes are solved using Maple 17 application and the value of $y_{n+\frac{1}{2}}, ..., y_{n+3}$, $y'_{n+\frac{1}{2}}, ..., y'_{n+3}$, $y''_{n+\frac{1}{2}}, ..., y''_{n+3}$ are obtained as follows:

$$h^a \sum_{s=0}^{q} \phi_{m,s} y^e_{n+s} = h^e \sum_{s=0}^{q} \nabla_{m,s} y^p_n + h^{p-e}\Big(\sum_{s=0}^{q} \nabla_{m,s} f_n + \sum_{s=0}^{q} \delta_{m,s} f_{n+s}\Big) \quad (16)$$

where a is the power of the derivation of the continuous method and p is the order of the problem to solve. Equation (16) is solved for $s = 0, \frac{1}{2}, 1, ...3$ to obtain the following proposed third-step hybrid multistep method:

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2}\,{y'}_n h + \frac{1}{8}\,{y''}_n h^2 + \frac{h^3}{29030400}\Big(343801\,f_n + 506604\,f_{n+\frac{1}{2}} - 494715\,f_{n+1} + 414160\,f_{n+\frac{3}{2}} - 226605\,f_{n+2} + 71364\,f_{n+\frac{5}{2}} - 9809\,f_{n+3}\Big)$$

$$y_{n+1} = y_n + {y'}_n h + \frac{1}{2}\,{y''}_n h^2 + \frac{h^3}{226800}\Big(13774\,f_n + 35976\,f_{n+\frac{1}{2}} - 24465\,f_{n+1} + 20800\,f_{n+\frac{3}{2}} - 11370\,f_{n+2} + 3576\,f_{n+\frac{5}{2}} - 491\,f_{n+3}\Big)$$

$$y_{n+\frac{3}{2}} = y_n + \frac{3}{2}\,{y'}_n h + \frac{9}{8}\,{y''}_n h^2 + \frac{h^3}{358400}\Big(+52893\,f_n + 172692\,f_{n+\frac{1}{2}} - 72495\,f_{n+1} + 80640\,f_{n+\frac{3}{2}} - 44145\,f_{n+2} + 13932\,f_{n+\frac{5}{2}} - 1917\,f_{n+3}\Big)$$

$$y_{n+2} = y_n + 2\,{y'}_n h + 2\,{y''}_n h^2 + \frac{h^3}{14175}\Big(3863\,f_n + 13992\,f_{n+\frac{1}{2}} - 3390\,f_{n+1} + 6800\,f_{n+\frac{3}{2}} - 3255\,f_{n+2} + 1032\,f_{n+\frac{5}{2}} - 142\,f_{n+3}\Big)$$

$$y_{n+\frac{5}{2}} = y_n + \frac{5}{2}\,{y'}_n h + \frac{25}{8}\,{y''}_n h^2 + \frac{h^3}{1161216}\Big(505625\,f_n + 1945500\,f_{n+\frac{1}{2}} - 256875\,f_{n+1} + 1070000\,f_{n+\frac{3}{2}} - 358125\,f_{n+2} + 136500\,f_{n+\frac{5}{2}} - 18625\,f_{n+3}\Big)$$

$$y_{n+3} = y_n + 3\,{y'}_n h + \frac{9}{2}\,{y''}_n h^2 + \frac{h^3}{2800}\Big(1782\,f_n + 7128\,f_{n+\frac{1}{2}} - 405\,f_{n+1} + 4320\,f_{n+\frac{3}{2}} - 810\,f_{n+2} + 648\,f_{n+\frac{5}{2}} - 63\,f_{n+3}\Big) \quad (17)$$

$${y'}_{n+\frac{1}{2}} = {y'}_n + \frac{1}{2}\,{y''}_n h + \frac{h^2}{483840}\Big(28549\,f_n + 57750\,f_{n+\frac{1}{2}} - 51453\,f_{n+1} + 42484\,f_{n+\frac{3}{2}} - 23109\,f_{n+2} + 7254\,f_{n+\frac{5}{2}} - 995\,f_{n+3}\Big)$$

$${y'}_{n+1} = {y'}_n + {y''}_n h + \frac{h^2}{7560}\Big(1027\,f_n + 3492\,f_{n+\frac{1}{2}} - 1680\,f_{n+1} + 1576\,f_{n+\frac{3}{2}} - 873\,f_{n+2} + 276\,f_{n+\frac{5}{2}} - 38\,f_{n+3}\Big)$$

$$y'_{n+\frac{3}{2}} = y'_n + h^2 + \frac{3}{2}\, y''_n h + \frac{h^2}{17920}\, \Big(3795\, f_n + 14850\, f_{n+\frac{1}{2}} - 2403\, f_{n+1} + 6300\, f_{n+\frac{3}{2}} - 3267\, f_{n+2} + 1026\, f_{n+\frac{5}{2}} - 141\, f_{n+3}\Big)$$

$$y'_{n+2} = y'_n + 2\, y''_n h + \frac{h^2}{945}\, \Big(272\, f_n + 1128\, f_{n+\frac{1}{2}} - 18\, f_{n+1} + 656\, f_{n+\frac{3}{2}} - 210\, f_{n+2} + 72\, f_{n+\frac{5}{2}} - 10\, f_{n+3}\Big)$$

$$y'_{n+\frac{5}{2}} = y'_n + 5/2\, y''_n h + \frac{h^2}{96768}\, \Big(35225\, f_n + 150750\, f_{n+\frac{1}{2}} + 9375\, f_{n+1} + 102500\, f_{n+\frac{3}{2}} - 5625\, f_{n+2} + 11550\, f_{n+\frac{5}{2}} - 1375\, f_{n+3}\Big)$$

$$y'_{n+3} = y'_n + 3\, y''_n h + \frac{h^2}{280}\, \Big(123\, f_n + 540\, f_{n+\frac{1}{2}} + 54\, f_{n+1} + 408\, f_{n+\frac{3}{2}} + 27\, f_{n+2} + 108\, f_{n+\frac{5}{2}}\Big) \quad (18)$$

$$y''_{n+\frac{1}{2}} = y''_n + \frac{h}{120960}\, \Big(19087\, f_n + 65112\, f_{n+\frac{1}{2}} - 46461\, f_{n+1} + 37504\, f_{n+\frac{3}{2}} - 20211\, f_{n+2} + 6312\, f_{n+\frac{5}{2}} - 863\, f_{n+3}\Big)$$

$$y''_{n+1} = y''_n + \frac{h}{7560}\, \Big(1139\, f_n + 5640\, f_{n+\frac{1}{2}} + 33\, f_{n+1} + 1328\, f_{n+\frac{3}{2}} - 807\, f_{n+2} + 264\, f_{n+\frac{5}{2}} - 37\, f_{n+3}\Big)$$

$$y''_{n+\frac{3}{2}} = y''_n + \frac{h}{4480}\, \Big(685\, f_n + 3240\, f_{n+\frac{1}{2}} + 1161\, f_{n+1} + 2176\, f_{n+\frac{3}{2}} - 729\, f_{n+2} + 216\, f_{n+\frac{5}{2}} - 29\, f_{n+3}\Big)$$

$$y''_{n+2} = y''_n + \frac{h}{945}\, \Big(143\, f_n + 696\, f_{n+\frac{1}{2}} + 192\, f_{n+1} + 752\, f_{n+\frac{3}{2}} + 87\, f_{n+2} + 24\, f_{n+\frac{5}{2}} - 4\, f_{n+3}\Big)$$

$$y''_{n+\frac{5}{2}} = y''_n + \frac{h}{24192}\, \Big(3715\, f_n + 17400\, f_{n+\frac{1}{2}} + 6375\, f_{n+1} + 16000\, f_{n+\frac{3}{2}} + 11625\, f_{n+2} + 5640\, f_{n+\frac{5}{2}} - 275\, f_{n+3}\Big)$$

$$y''_{n+3} = y''_n + \frac{h}{280}\Big(41\, f_n + 216\, f_{n+\frac{1}{2}} + 27\, f_{n+1} + 272\, f_{n+\frac{3}{2}} + 27\, f_{n+2} + 216\, f_{n+\frac{5}{2}} + 41\, f_{n+3}\Big) \quad (19)$$

3 Analysis of the method

This section presented the analysis of the basic properties of the proposed third-step hybrid multistep method.

3.1 Order of the method

If $y(x)$ is continuously differentiable, the according to Lamber (1973),the linear operator associated with formular in (22) is defined by:

$$\alpha_{\frac{s}{2}}[y(x):h] = y(x_n + \frac{s}{2}h) - (\sum_{a=0}^{9} \alpha_a(\frac{s}{2})y^{(a)}(x)h^a - h^3 \sum_{a=0}^{9} \beta_a(\frac{s}{2})y^{(3)}(x_n + \frac{s}{2}h), s = 1,2,3) \quad (20)$$

Taking $y(x)$ as the valid solution of (1), the Taylor series expansion about the poinf x after using (35) give the formular for the local truncation error written as :

$$\alpha_{\frac{s}{2}}[y(x):h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + ... + C_{p+2} h^{p+2} y^{(p+2)}(x) + C_{p+3} h^{p+3} y^{(p+3)}(x) \quad (21)$$

The term C_{p+3} is called the error constant and implies that the local truncation error is given by

$$\alpha_{\frac{s}{2}}[y(x):h] = C_{p+3} h^{p+3} y^{(p+3)}(x) + Oh^{p+4} \quad (22)$$

Since $C_0 = C_1 = C_2 = ... = C_{p+2} = 0, C_{p+3} \neq 0$,refer to Badmus and Yahaya (2014); then formular in (22) have order $p = 7$ with error constant given by $C_{p+3} = \frac{4001}{1857945600}$

3.2 Zero Stability of the Block

According to Fatunla (1991), a block is said to be zero stable if the root of the characteristic polynomial satisfy $|z| \leq 1$ and the root of $|z| = 1$ has multiplicity not exceeding the oder of the differential equation. Moreover, as $h^n \to 0$, $p(z) = z^{y-n}(\lambda - 1)$, where n is the order of the differential equation, for the block (22) ,$r = 24, n = 3, \rho(z) = \lambda^{21}(\lambda - 1)^3 = 0$

This implies $\lambda = 0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1$

Therefore, the block method is zero stable.

3.3 Consistency of the method

From Eq.(14), the first and the second characteristic polynomial of the method are given by:

$\rho(r) = r^3 - 3r^{\frac{5}{2}} + 3r^2 - r^{\frac{3}{2}}$

$\sigma(r) = \frac{32}{241920} - \frac{255}{241920}r^{\frac{1}{2}} + \frac{921}{241920}r^1 - \frac{1774}{241920}r^{\frac{3}{2}} + \frac{16986}{241920}r^2 + \frac{14109}{241920}r^{\frac{5}{2}} + \frac{211}{241920}r^3$

This implies that the method presented in this paper is consistent since it satisfy the following conditions:

(i). The order of the method is $p = 7 \geq 1$, which is obvious

(ii). For the method, $\alpha_1 = 1$,$\alpha_2 = -3$,$\alpha_3 = 3$,$\alpha_4 = -1$. Hence $\sum_{j=0}^{4} a_j = 0$

(iii). If $p(r) = r^3 - 3r^{\frac{5}{2}} + 3r^2 - r^{\frac{3}{2}}$ and

$\rho'(r) = 3r^2 - \frac{15}{2}r^{\frac{3}{2}} + 6r - \frac{3}{2}r^{\frac{1}{2}}$

$\rho'''(r) = 6 - \frac{45}{8}r^{\frac{-1}{2}} + 0 + \frac{3}{8}r^{\frac{-3}{2}}$

$\therefore \rho'''(1) = \frac{3}{4} = 3!\sigma(1)$.

The conditions are satisfied , therefore the method is consistent.

3.4 Convergence of the method

According to Henrici (1962), the necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable. Thus, it has been successfully shown from the above conditions, it could be seen that the method is convergent.

3.5 Region of absolute stability of the method

We consider the stability polynomial written in general form

$$\pi(r, \bar{h}) = \rho(r) - \bar{h}\sigma(r) = 0 \qquad (23)$$

$\bar{h} = h^2\lambda$ and $\lambda = \frac{\delta(f)}{\sigma(y)}$ is assumed constant. The stability polynomial of the formular (14) becomes:

$$y_{n+3} - 3\,y_{n+\frac{5}{2}} + 3\,y_{n+2} - y_{n+\frac{3}{2}} - \frac{h^3}{241920}\Big(32\,f_n - 255\,f_{n+\frac{1}{2}} + 921\,f_{n+1} - 1774\,f_{n+\frac{3}{2}} + 16986\,f_{n+2} + 14109\,f_{n+\frac{5}{2}} + 221\,f_{n+3}\Big) = 0 \qquad (24)$$

where

$\rho(r) = r^3 - 3r^{\frac{5}{2}} + 3r^2 - r^{\frac{3}{2}}$ and

$\sigma(r) = \frac{32}{241920} - \frac{255}{241920}r^{\frac{1}{2}} + \frac{921}{241920}r - \frac{1774}{241920}r^{\frac{3}{2}} + \frac{16986}{241920}r^2 + \frac{14109}{241920}r^{\frac{5}{2}} + \frac{221}{241920}r^3$

$$\bar{h} = \frac{\rho(r)}{\sigma(r)} \qquad (25)$$

By inserting the value of $\rho(r)$ and $\sigma(r)$ into equation (25), we obtain boundary locus equation for the method as:

$$\bar{h}(r) = \frac{r^3 - 3r^{\frac{5}{2}} + 3r^2 - r^{\frac{3}{2}}}{\frac{32}{241920} - \frac{255}{241920}r^{\frac{1}{2}} + \frac{921}{241920}r - \frac{1774}{241920}r^{\frac{3}{2}} + \frac{16986}{241920}r^2 + \frac{14109}{241920}r^{\frac{5}{2}} + \frac{221}{241920}r^3} \tag{26}$$

Evaluate the imaginary part to zero, then we have

$$\bar{h}(\theta) = \frac{241920(10626 - 18414cos\frac{\theta}{2} + 11880cos\theta - 5555cos\frac{3\theta}{2} + 1782cos2\theta - 351cos\frac{5\theta}{2} + 32cos3\theta)}{491698284 + 4243036cos\frac{\theta}{2} - 10299024cos\theta + 16428274cos\frac{3\theta}{2} - 5701404cos2\theta - 112710cos\frac{5\theta}{2} + 14144cos3\theta}$$

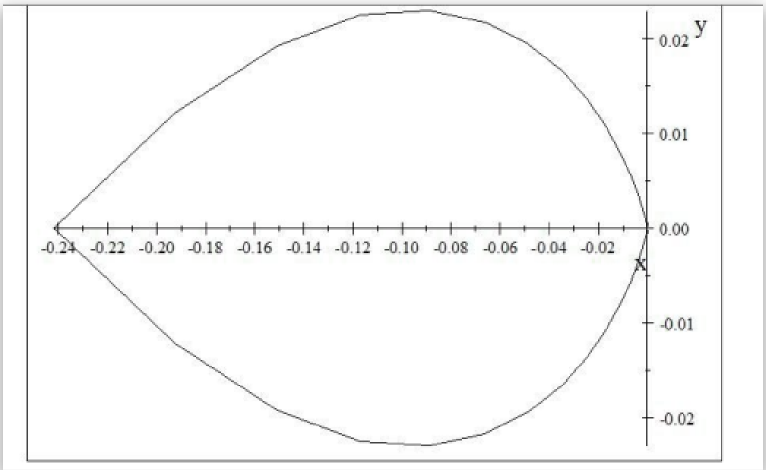

Figure 1: Region of Absolute Stability of the proposed method

4 Numerical application

The method was utilized to solve some specific initial value problems of third order ODE to verify its accuracy, workability, and applicability. The following notations are used to represent the current findings:

XVC: Value of the independent variable where a numerical value is taken

ERC: Exact result at XVC

NRC: Numerical result of XVC

ERR: E rror in proposed method at XVC

The table showing $Erc = |y_{ex} - y_c|$, where y_{ex} is the exact solution, y_c is the computed result and A_c absolute error defined as $Erc = |y_c - y_{ex}|$ was also reported.

4.1 Problem 1

$y''' = -y,$

$y(0) = 1, y'(0) = -1, y''(0) = 1, h = \frac{1}{10}$

Exact solution: $y(x) = e^{-x}$

Source: Kuboye (2015)

Table 1: Comparison of result obtained with proposed method and that of Kuboye (2015) on Problem 1, $h = 0.1$

XVR	ERC	NRC	ERR p=7,k =3	ERR in Kuboye (2015) p= 8
0.10	0.90483741803595957316	0.90483741803595837255	1.20061E-15	6.055156E-13
0.20	0.81873075307798185867	0.81873075307797625912	5.59955E-15	2.13841E-12
0.30	0.74081822068171786607	0.74081822068170467209	1.319398E-14	7.39575E-12
0.40	0.67032004603563930074	0.67032004603561636696	2.293378E-14	2.15813E-12
0.50	0.60653065971263342360	0.67032004603561636696	3.512021E-14	1.484579E-11
0.60	0.54881163609402643263	0.54881163609397670806	4.972457E-14	1.098521E-11
0.70	0.49658530379140951470	0.49658530379134358118	6.593352E-14	3.142886E-11
0.80	0.44932896411722159143	0.44932896411713766565	8.392578E-14	2.309530E-11
0.90	0.40656965974059911188	0.40656965974049548523	1.0362665E-13	5.154149E-11
1.00	0.36787944117144232160	0.36787944117131795034	1.2437126E-13	8.200535E-11

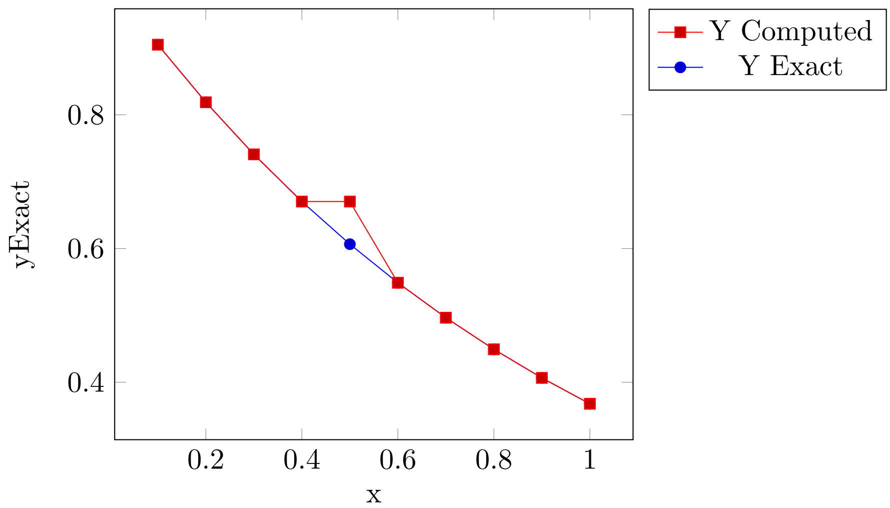

Figure 2: Solution curve of problem 1 as compared with the exact solution

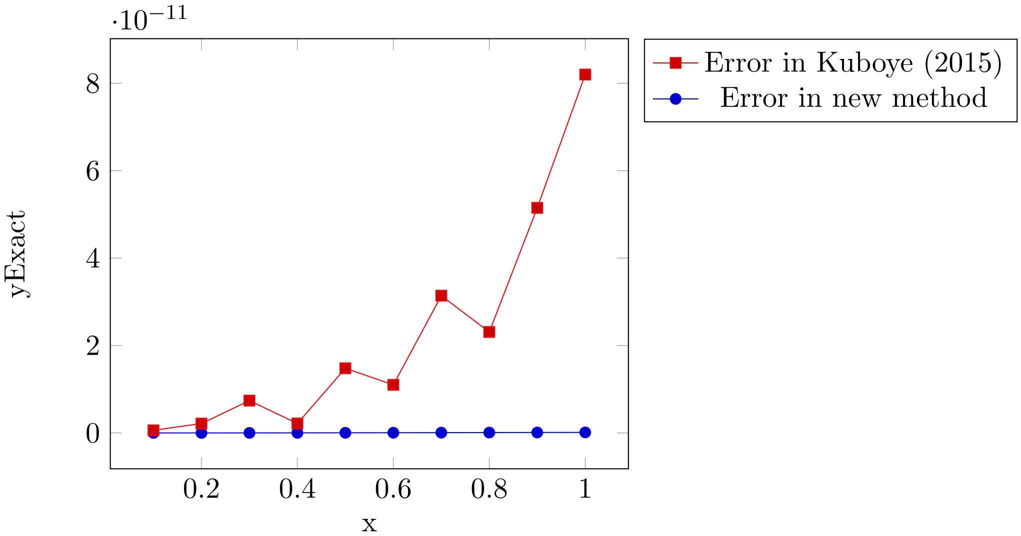

Figure 3: Comparison of absolute errors of the proposed method on problem 1 as compared with Kuboye (2015)

4.2 Problem 2

$y''' + 4y' - x = 0, h = 0.1$

$y(0) = 0, y'(0) = 0, y''(0) = 1, h = \frac{1}{10}$

Exact solution: $y(x) = \frac{3}{16}(1 - cos2x) + \frac{1}{18}x^2$

Source: Olumide (2021)

Table 2: Comparison of result obtained with proposed method and that of Olumide, $h = 0.1$

XVR	ERC	NRC	ERR in newly method, k=3 , p=7	Error in Olumide (2021), p=7
0.10	0.0049875166547671941600	0.0049875166545145588800	2.526352800E-13	1.6654E-08
0.20	0.019801063624459046980	0.019801063623291460450	1.167586530E-12	3.8095E-07
0.30	0.043999572204435319270	0.043999572201726883342	2.708435928E-12	1.5664E-06
0.40	0.076867491997406483580	0.076867491992833991841	4.572491739E-12	3.9865E-06
0.50	0.11744331764972380299	0.11744331764300377051	6.72003248E-12	7.9597E-06
0.60	0.16455792103562370419	0.16455792102655810685	9.06559734E-12	1.3680E-05
0.70	0.21688116070620482401	0.21688116069492027661	1.128454740E-11	2.1195E-05
0.80	0.27297491043149163616	0.27297491041831323226	1.317840390E-11	3.0396E-05
0.90	0.33135039275495382287	0.33135039274028196095	1.467186192E-11	4.10008E-05
1.00	0.39052753185258919756	0.39052753183702406999	1.556512757E-11	5.2605E-05

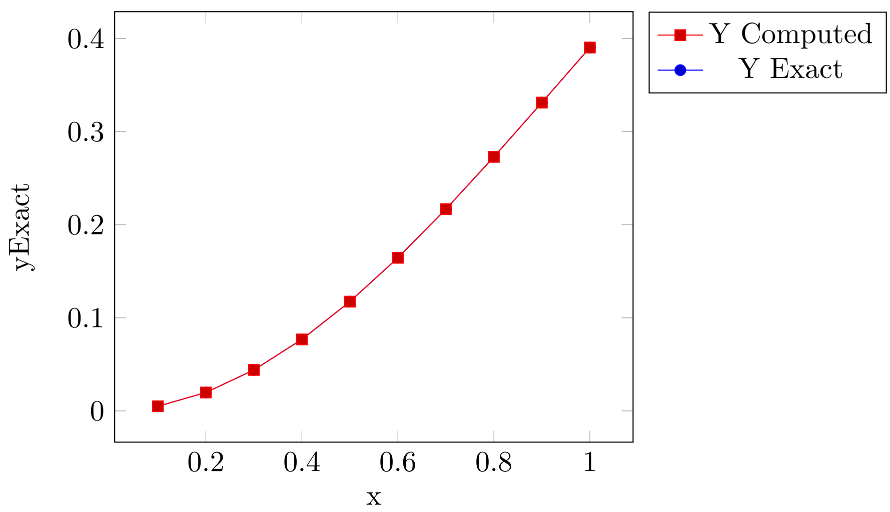

Figure 4: Solution curve of problem 2 as compared with the exact solution

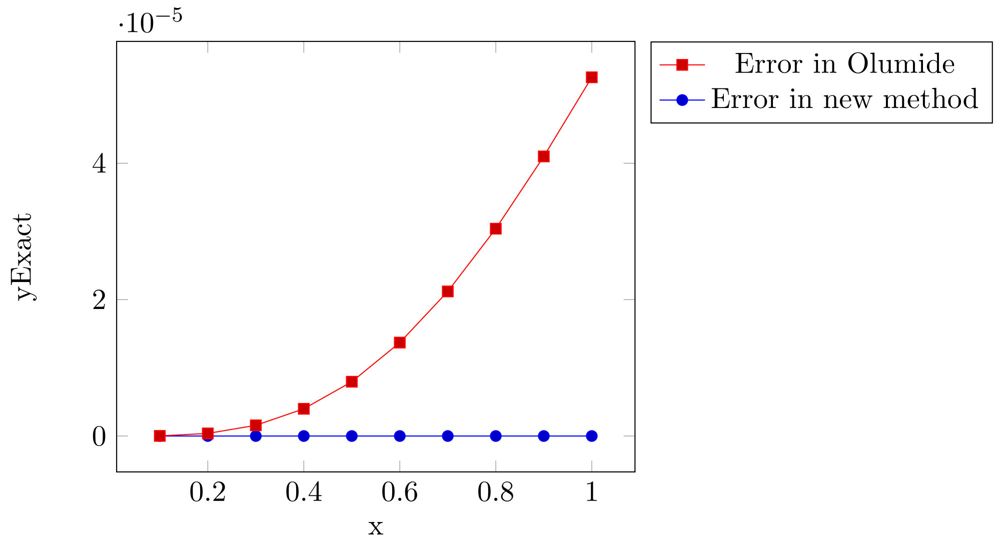

Figure 5: Comparison of absolute errors of the proposed method on problem 2 as compared with Olumide (2021)

4.3 Problem 3

$y''' = e^x, h = 0.1$

$y(0) = 3, y'(0) = 1, y''(0) = 5, h = \frac{1}{10}$

Exact solution: $y(x) = 2 + 2x^2 + e^x$

Source: Awoyemi (2021)

Table 3: Comparison of result obtained with proposed method and that of Awoyemi, $h = 0.1$

x-value	y-exact-solution	y-computed solution	Error in newly method,p=7	Error in Awoyemi (2015), p=8
0.10	3.1251709180756476248	3.1251709180756460575	1.5673E-15	2.531308E-14
0.02	3.3014027581601698339	3.3014027581601625265	7.3074E-15	1.612044E-13
0.03	3.5298588075760031040	3.5298588075759858842	1.72198E-14	4.023448E-13
0.40	3.8118246976412703178	3.8118246976412395954	3.07224E-14	7.536194E-13
0.50	4.1487212707001281468	4.1487212707000784350	4.97118E-14	1.212364E-12
0.60	4.5421188003905089749	4.5421188003904347878	7.41871E-14	1.780798E-12
0.70	4.9937527074704765216	4.9937527074703731589	1.033627E-13	2.456702E-12
0.80	5.5055409284924676046	5.5055409284923278058	1.397988E-13	2.212097E-11
0.90	6.0796031111569496638	6.0796031111567661698	1.834940E-13	5.231993E-11
1.00	6.7182818284590452354	6.7182818284588118470	2.333884E-13	8.86011E-11

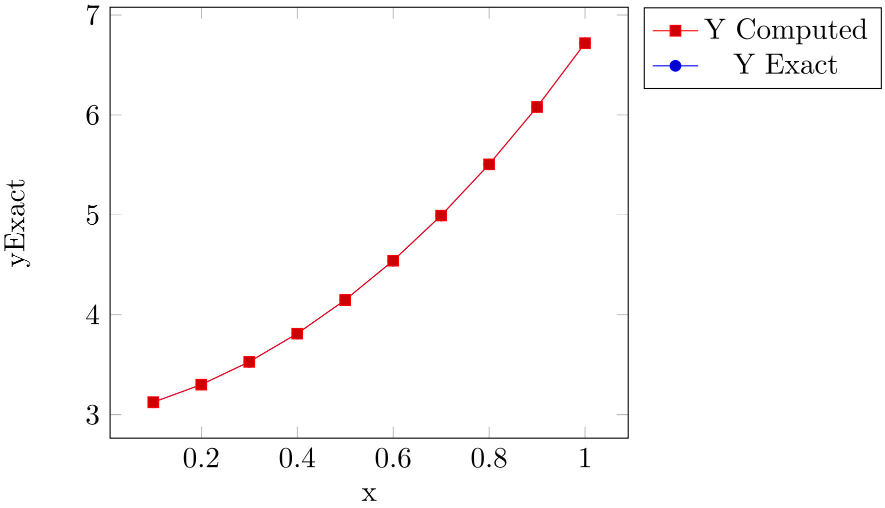

Figure 6: Solution curve of problem 3 as compared with the exact solution

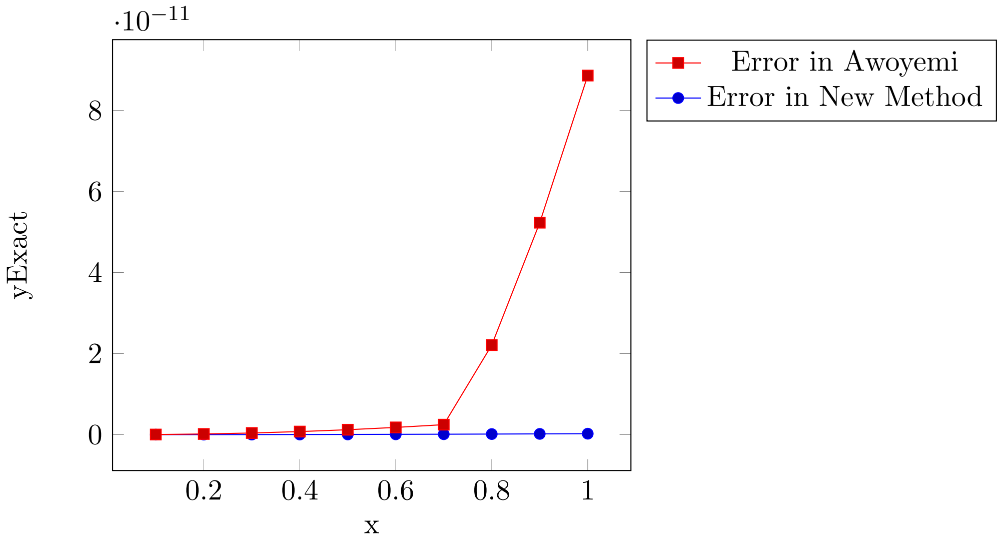

Figure 7: Comparison of absolute errors of the proposed method on problem 3 as compared with Awoyemi (2015)

4.4 Problem 4

$y''' - y'(2xy'' + y') = 0,$

$y(0) = 0, y'(0) = 0, y''(0) = \frac{1}{2}, h = \frac{1}{10}$

Exact solution: $y(x) = 1 + \frac{1}{2}log(\frac{2+x}{2-x})$

Source: Ezekiel, Ogunware and Adogbe (2016)

x	yapprox	yExact	Error in new method k=3,p=7	Error in Ezekiel,Adogbe ,Ogunware (2016,),p=9
0.1	1.050041729	1.050041729	1.59595E-16	1.93148E-08
0.2	1.100335348	1.100335348	2.91434E-16	5.60825E-07
0.3	1.151140436	1.151140436	7.49401E-16	3.75510E-06
0.4	1.202732554	1.202732554	1.44329E-15	1.34028E-05
0.5	1.255412812	1.255412812	2.9976E-15	3.25906E-05
0.6	1.309519604	1.309519604	6.32827E-15	5.81649E-05
0.7	1.365443754	1.365443754	1.28786E-14	7.15239E-05
0.8	1.42364893	1.42364893	2.63123E-14	2.56483E-05
0.9	1.484700279	1.484700279	5.66214E-14	1.70915E-04
1	1.549306144	1.549306144	1.27787E-13	6.70643E-04

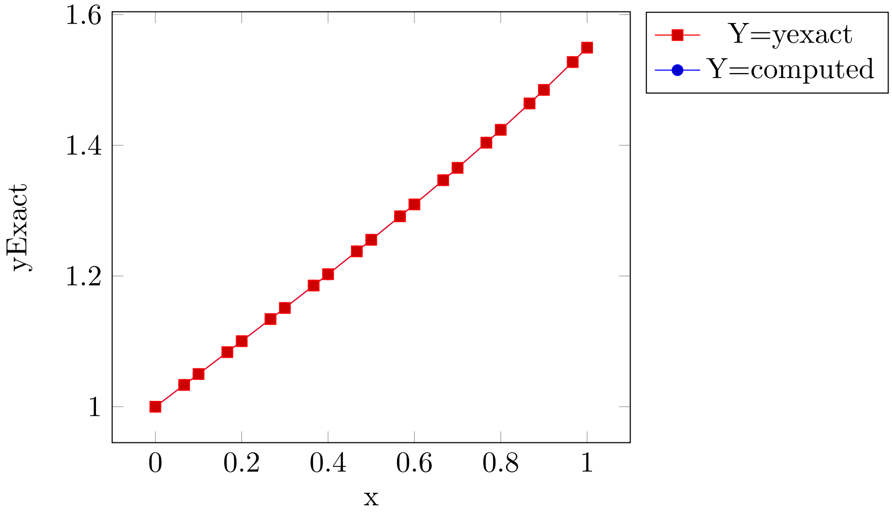

Figure 8: Solution curve of problem 4 as compared with the exact solution

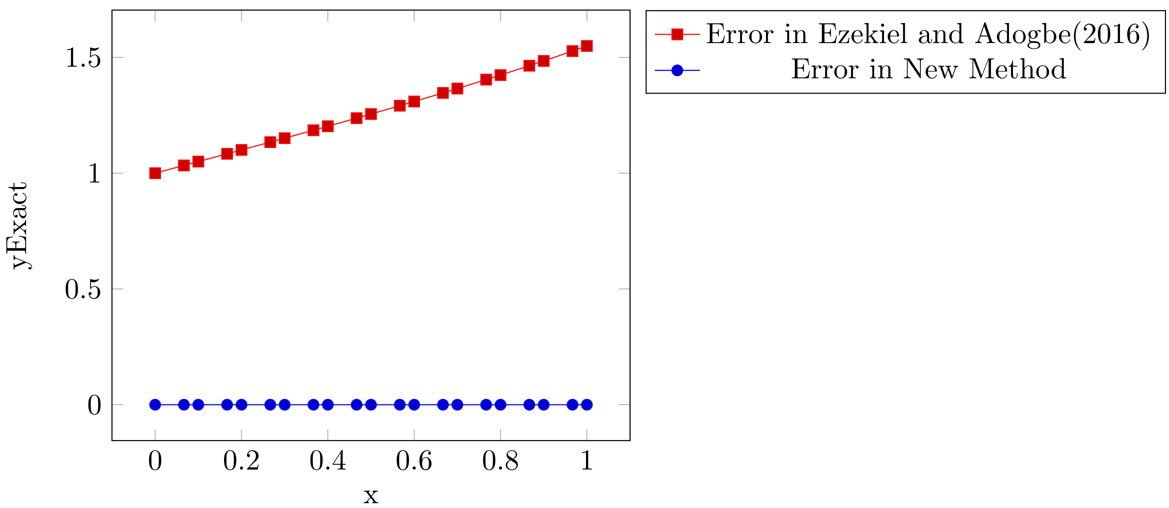

Figure 9: Comparison of absolute errors of the proposed method on problem 4 as compared with Ezekiel, Ogunware and Adogbe (2016)

5 Discussion

In this research work, we have applied the procedure of collocation and interpolation to develop linear hybrid multistep method for solving initial value problem of third order ordinary differential equations. In the table 1, table 2 and table 3. It is shown that our new method is more accurate than the method proposed by Kuboye (2015) , Olumide (2021) and Awoyemi (2021). It has been seen from figure 2 to figure 9 that our new method yield better result than the results presented by Kuboye (2015) , Olumide (2021), Ezekiel *et al.*(2016) and Awoyemi (2021) despite the high order of their method. Our new method of order seven are more efficiency and accurate than his method of order nine.

6 Conclusion

The 3-step methods has been developed for the direct solution of third order ordinary differential equations. The methods were obtained from the same continuous methods derived via interpolation and collocation procedure. The stability properties and region of the methods were discussed. The methods were applied in block form. Numerical results obtained using the obtained block methods showed that they are efficient and adequate for general third order problems of ordinary differential equations. In fact, when the new technique result were compared to the block method proposed by Kuboye et al (2015), Olabode et al (2009), Olumide et al (2021), and Awoyemi et al (2021), the new method was more accurate.

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