SOME PROPERTIES AND INEQUALITIES FOR A TWO-PARAMETER GENERALIZATION OF THE INCOMPLETE EXPONENTIAL INTEGRAL FUNCTION

Abstract. Motivated by the p-analogue of the exponential integral function [5], we introduce a two-parameter generalization of the Incomplete Exponential Integral function. By using the classical Holders and Youngs inequalities, among other analytical techniques, we establish some new inequalities involving the generalized function.

Keywords: Two-parameter Generalization of the incomplete exponential integral function, Holder's inequality and Young's inequality for scalars.

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1. Introduction

The classical exponential integral function is defined by Schloemich in [3] as

$$E_n(x) = \int_1^\infty t^{-n} e^{-tx} \ \ dt \qquad x > 0, \qquad n \in \mathbb{N}. \tag{1}$$

For any $1 < a < b$ and $n \in \mathbb{N}_0$, the incomplete exponential integral function ${}_a^bE_n(x)$ is defined by [1] as

$$ {}_a^bE_n(x) = \int_a^b t^{-n} e^{-xt} \ \ dt \tag{2}$$

for all $x > 0$.

In [1] it was proved that the incomplete exponential integral function is nonincreasing and then gave the inequality as follows,

$$ {}_a^bE_{m+n}\left(\frac{x}{u}+\frac{y}{v}\right) \leq \left({}_a^bE_{um}(x)\right)^{\frac{1}{u}} \left({}_a^bE_{vn}(y)\right)^{\frac{1}{v}} \tag{3}$$

where $1<a<b$, $x,y>0$, $u>1=\frac{1}{u}+\frac{1}{v}$, and $m+n, um, vn \in \mathbb{N}_0$,
Also

$$ {}_a^bE_n(xy) \leq \left({}_a^bE_n(ux)\right)^{\frac{1}{u}} \left({}_a^bE_n(vy)\right)^{\frac{1}{v}} \tag{4}$$

where $1<a<b$, $x,y>1$, $n\in\mathbb{N}_0$, $u>1$, $\frac{1}{u}+\frac{1}{v}=1$, and $x+y\leq xy$.
Furthermore,

$$ {}_a^bE_n(xy) \geq \left({}_a^bE_n(ux)\right)^{\frac{1}{u}} \left({}_a^bE_n(vy)\right)^{\frac{1}{v}} \tag{5}$$

where $1<a<b$, $x>0$, $0<y<1$, $n\in\mathbb{N}_0$, $0<p<1=\frac{1}{u}+\frac{1}{v}=1$ and $x+y\geq xy$.
And finally,

$$ {}_a^bE_n(xy) \geq \left({}_a^bE_n\left(\frac{rx^u}{u}\right)\right)^{\frac{1}{r}} \left({}_a^bE_n\left(\frac{sy^v}{v}\right)\right)^{\frac{1}{s}} \tag{6}$$

where $1<a<b$, $x,y>1$, $n\in\mathbb{N}_0$, $u>1$, $0<r<1$ and $\frac{1}{u}+\frac{1}{v}=1=\frac{1}{r}+\frac{1}{s}$.
And in [2] generalizations of inequalities (3),(4), (5) and (6) were established as follows,
For $x_i>0$, $n_i\geq 0$, and $u_i>1$, be such that $u_in_i\in\mathbb{N}_0$, for all $i\in\mathbb{N}_m$. Assume that $1<a<b$, $\sum_{i=1}^m u_i=1$, and $\sum_{i=1}^m n_i\in\mathbb{N}_0$.

$$ {}_a^bE_{\sum_{i=1}^m n_i}\left(\sum_{i=1}^m \frac{x_i}{u_i}\right) \leq \prod_{i=1}^m \left({}_a^bE_{u_i}n_i(x)\right)^{\frac{1}{u_i}} \tag{7}$$

is valid. Which is a generalization of (3).
For $n\in\mathbb{N}_0$, x_i and $u_i>1$ for all $i\in\mathbb{N}_m$. Assume that $1<a<b$, $\sum_{i=1}^\infty u_i=1$ and $\sum_{i=1}^m x_i\leq\prod_{i=1}^m x_i$.
Then

$$ {}_a^bE_n\left(\prod_{i=1}^m x_i\right) \leq \prod_{i=1}^m \left({}_a^bE_n(u_ix_i)\right)^{\frac{1}{u_i}} \tag{8}$$

is valid. Which is also generalization of (4).
For $n\in\mathbb{N}_0$, $0<x_i<1$ and $0<u_i<1$ for all $i\in\mathbb{N}_m$. Assume that $1<a<b$, $\sum_{i=1}^\infty u_i=1$ and $\sum_{i=1}^m x_i\geq\prod_{i=1}^m x_i$.
Then

$$ {}_a^bE_n\left(\prod_{i=1}^m x_i\right) \geq \prod_{i=1}^m \left({}_a^bE_n(u_ix_i)\right)^{\frac{1}{u_i}} \tag{9}$$

is valid. Which is also generalization of (5).
Inequality (6) is generalized as follows,
for $n\in\mathbb{N}_0$, $x_i>1$, $u_i>1$ and $0<r_i<1$ for all $i\in\mathbb{N}_m$. Assume that $1<a<b$ and $\sum_{i=1}^m u_i=$

$1 = \sum_{i=1}^{m} r_i$,

$$ {}_a^bE_n\left(\prod_{i=1}^{m} x_i\right) \geq \prod_{i=1}^{m}\left({}_a^bE_n\left(\frac{r_i x_i^{u_i}}{u_i}\right)\right)^{\frac{1}{r_i}}. \tag{10} $$

The focus of this paper is on the incomplete exponential integral function defined in [4] as

$$ E_n(a,x) = \int_x^{\infty} t^{-n}e^{-at} \quad dt \qquad x \geq 0, \quad a > 0, \quad n \in \mathbb{N}_0 \tag{11} $$

Clearly, $E_n(a,1) = E_n(a)$.

The p-analogue of the exponential integral function, $E_{n,p}(x)$ is defined for $x > 0$, $p > 1$ and $n \in \mathbb{N}_0$ by [5]

$$ E_{n,p}(x) = \int_1^p t^{-n}A_p^{-xt}dt, \tag{12} $$

and the i-th derivative of (12) is given by [6]

$$ E_{n,p}^{(i)}(x) = \left(\ln A_p^{-1}\right)^i \int_1^p t^{i-n}A_p^{-xt}dt, \tag{13} $$

where, $E_{n,p}(x) \longrightarrow E_n(x)$ as $p \longrightarrow \infty$, $A_p = (1+\frac{1}{p})^p$ and $E_{n,p}^{(i)}(x) \longrightarrow E_n^{(i)}(x)$ as $p \longrightarrow \infty$.

The function emerges in the investigation of radiative exchanges in a two-dimensional planar medium [7].

This special function has been investigated in diverse ways (see [8], [9], [10], [11], [12], [13] and the related references therein).

The objective of this paper is to introduce a two-parameter generalization of the incomplete exponential integral function of (12) and to establish some properties of the function. In this paper, we will generalize inequalities (3), (4), (5), (6), (7), (8), (9) and (10).

2. Preliminaries

We begin with the following well known results(see for instance [14], [15], [16] or [17]).

Lemma 2.1. *(Holder's Inequality) Let $\eta, \mu > 1$ and $\frac{1}{\eta} + \frac{1}{\mu} = 1$. If $f(t)$ and $g(t)$ are continuous real-valued functions on $[a,b]$, then inequality*

$$ \int_a^b |f(t)g(t)|dt \leq \left(\int_a^b |f(t)|^{\eta}dt\right)^{\frac{1}{\eta}}\left(\int_a^b |g(t)|^{\mu}dt\right)^{\frac{1}{\mu}}, \tag{14} $$

holds. With equality when $|g(t)| = c|f(t)|^{\eta-1}$. If $\eta = \mu = 2$, the inequality becomes Schwarz's Inequality.

Lemma 2.2. *(Young's Inequality) Let $a, b > 0$, $\eta, \mu > 1$, and $\frac{1}{\eta} + \frac{1}{\mu} = 1$. Then inequality*

$$ab \leq \frac{a^{\eta}}{\eta} + \frac{b^{\mu}}{\mu}, \tag{15}$$

holds.

3. Definition of a Two-Parameter Generalization of the incomplete exponential integral

Definition 3.1. Let $x > 0$, $v \geq 1$, $p > 1$, and $n \in \mathbb{N}_0$. Then, the function is defined as

$$E_{n,p}(x, v) = \int_v^p t^{-n} A_p^{-xt} dt, \tag{16}$$

where, $E_{n,p}(x, 1) = E_{n,p}(x)$, $E_{n,p}(x, v) \longrightarrow E_n(x, v)$ as $p \longrightarrow \infty$, $E_{1,p}(1, v) \longrightarrow \Gamma(0, v)$ as $p \longrightarrow \infty$ and $E_{n,p}(x, v) = x^{n-1} E_{n,p}(xv, v)$.

3.1. Some Properties and Inequalities of $E_{n,p}(x, v)$.

Lemma 3.2. *The recursive relation*

$$\ln A_p^x E_{n,p}(x, v) = v^{-n} A_p^{-vx} - p^{-n} A_p^{-px} - n E_{n+1,p}(x, v), \tag{17}$$

holds for $n \in \mathbb{N}_0$, $v \geq 1$.

Proof. Using (16) and by means of integration by parts

$$\begin{aligned} E_{n,p}(x,v) &= \int_v^p t^{-n} A_p^{-xt} dt \\ &= \left[-\frac{t^{-n} A_p^{-xt}}{\ln A_p^x} \right]_v^p - \frac{n}{\ln A_p^x} \int_v^p t^{-(n+1)} A_p^{-xt} dt \\ &= \frac{1}{\ln A_p^x} \left[v^{-n} A_p^{-vx} - p^{-n} A_p^{-px} - n E_{n+1,p}(x,v) \right], \end{aligned}$$

which concludes the proof.

Theorem 3.3. *Let $p > 1$ and $m, n \in \mathbb{N}_0$ such that $\eta m, \mu n \in \mathbb{N}_0$.*
Then, the inequality

$$E_{n,p}\left(\prod_{i=1}^m x_i, v \right) \leq \prod_{i=1}^m \left(E_{n,p}(\eta_i x_i, v) \right)^{\frac{1}{\eta_i}}, \tag{18}$$

holds for $x, y > 0$, $v \geq 1$, $\eta > 1$ and $\frac{1}{\eta} + \frac{1}{\mu} = 1$, *Proof.* Using (16) and Hölder's inequality for integrals, we have

$$\begin{aligned}
E_{n,p}\left(\prod_{i=1}^{m} x_i, v\right) &\leq E_{n,p}\left(\sum_{i=1}^{m} x_i, v\right) \\
&= \int_v^p t^{-n} A_p^{-t\left(\sum_{i=1}^{m} x_i, v\right)} dt \\
&= \int_v^p \left(\prod_{i=1}^{m} t^{-\frac{n}{\eta_i}} A_p^{-(x_i,v)t}\right) dt \\
&\leq \prod_{i=1}^{m} \left(\int_v^p t^{-n} A_p^{-\eta_i (x_i,v)t} dt\right)^{\frac{1}{\eta_i}} \\
&= \prod_{i=1}^{m} \left(E_{n,p}(\eta_i x_i, v)\right)^{\frac{1}{\eta}{i}}.
\end{aligned}$$

Which is a generalization of (3) and (7)

Theorem 3.4. *Let $p > 1$ and $m,n \in \mathbb{N}_0$ such that $\eta m, \mu n \in \mathbb{N}_0$. Then, the inequality*

$$E_{\sum_{i=1}^{m} n_i,p}\left(\sum_{i=1}^{m} \frac{x_i, v}{\eta_i}\right) \leq \prod_{i=1}^{m} \left(E_{\eta_i,n_i,p}(x,v)\right)^{\frac{1}{\eta}{i}}, \tag{19}$$

holds for $x,y > 0$, $v \geq 1$, $\eta > 1$ and $\frac{1}{\eta} + \frac{1}{\mu} = 1$,

Proof. Using (16) and Hölder's inequality for integrals, we have

$$\begin{aligned}
E_{\sum_{i=1}^{m} n_i,p}\left(\sum_{i=1}^{m} \frac{x_i, v}{\eta_i}\right) &= \int_v^p t^{-\sum_{i=1}^{m} n_i} A_p^{-t\left(\sum_{i=1}^{m} \frac{x_i,v}{\eta_i}\right)} dt \\
&= \int_v^p \left(\prod_{i=1}^{m} t^{-n_i} A_p^{-\left(\frac{x_i,v}{\eta_i}\right)t}\right) dt \\
&\leq \prod_{i=1}^{m} \left(\int_v^p t^{-\eta_i,n_i} A_p^{-(x_i,v)t} dt\right)^{\frac{1}{\eta_i}} \\
&= \prod_{i=1}^{m} \left(E_{\eta_i,n_i,p}(x,v)\right)^{\frac{1}{\eta}{i}}.
\end{aligned}$$

Which is also generalization of (4) and (8)

Theorem 3.5. *Let $p > 1$ and $m, n \in \mathbb{N}_0$ such that $\eta m, \mu n \in \mathbb{N}_0$.*
Then, the inequality

$$E_{n,p}\left(\prod_{i=1}^{m} x_i, v\right) \geq \prod_{i=1}^{m} \left(E_{n,p}(\eta_i x_i, v)\right)^{\frac{1}{\eta}_i}, \tag{20}$$

holds for $x, y > 0$, $v \geq 1$, $\eta > 1$ and $\frac{1}{\eta} + \frac{1}{\mu} = 1$,

Proof. Using (16) and Hölder's inequality for integrals, we have

$$\begin{aligned}
E_{n,p}\left(\prod_{i=1}^{m} x_i, v\right) &\geq E_{n,p}\left(\sum_{i=1}^{m} x_i, v\right) \\
&= \int_{v}^{p} t^{-n} A_p^{-t\left(\sum_{i=1}^{m} x_i, v\right)} dt \\
&= \int_{v}^{p} \left(\prod_{i=1}^{m} t^{-\frac{n}{\eta_i}} A_p^{-(x_i, v)t}\right) dt \\
&\geq \prod_{i=1}^{m} \left(\int_{v}^{p} t^{-n} A_p^{-\eta_i (x_i, v)t} dt\right)^{\frac{1}{\eta_i}} \\
&= \prod_{i=1}^{m} \left(E_{n,p}(\eta_i x_i, v)\right)^{\frac{1}{\eta}_i}.
\end{aligned}$$

Which is also generalization of (5) and (9)

Theorem 3.6. *Let $p > 1$ and $m, n \in \mathbb{N}_0$ such that $\eta m, \mu n \in \mathbb{N}_0$.*
Then, the inequality

$$E_{n,p}\left(\prod_{i=1}^{m} x_i, v\right) \geq \prod_{i=1}^{m} \left(E_{n,p}\left(r_i\left(\frac{x_i^{\eta_i}, v}{\eta_i}\right)\right)\right)^{\frac{1}{r_i}}, \tag{21}$$

holds for $x, y > 0$, $v \geq 1$, $\eta > 1$, $\frac{1}{\eta} + \frac{1}{\mu} = 1$ and $\prod_{i=1}^{m} x_i, v \leq \sum_{i=1}^{m} \frac{x_i^{\eta_i}, v}{\eta_i}$

Proof. Using (16) and Hölder's inequality for integrals, we have

$$\begin{aligned}
E_{n,p}\left(\prod_{i=1}^{m} x_i, v\right) &\geq E_{n,p}\left(\sum_{i=1}^{m} \frac{x_i^{\eta_i}, v}{\eta_i}\right) \\
&= \int_v^p t^{-n} A_p^{-t\left(\sum_{i=1}^{m} \frac{x_i^{\eta_i}, v}{\eta_i}\right)} dt \\
&= \int_v^p \left(\prod_{i=1}^{m} t^{-\frac{n}{r_i}} A_p^{-\left(\frac{x_i^{\eta_i}, v}{\eta_i}\right)t}\right) dt \\
&\geq \prod_{i=1}^{m} \left(\int_v^p t^{-n} A_p^{-r_i\left(\frac{x_i^{\eta_i}, v}{\eta_i}\right)t} dt\right)^{\frac{1}{r_i}} \\
&= \prod_{i=1}^{m} \left(E_{n,p}\left(r_i\left(\frac{x_i^{\eta_i}, v}{\eta_i}\right)\right)\right)^{\frac{1}{r_i}}.
\end{aligned}$$

Which is also a generalization of (6) and (10)

4. CONCLUSION

we introduced a two-parameter generalization of the Incomplete Exponential Integral function. By using the classical Holders and Youngs inequalities, among other analytical techniques, we established some new inequalities involving the generalized function. Furthermore, we generalized inequalities (3), (4), (5), (6), (7), (8), (9) and (10).

Conflict of Interests

The authors declare that there is no conflict of interests.

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