

Original Research Article

**MORE ON SOME PROPERTIES AND INEQUALITIES FOR A TWO-PARAMETER
GENERALIZATION OF THE INCOMPLETE EXPONENTIAL INTEGRAL
FUNCTION**

Abstract. Motivated by the p -analogue of the exponential integral function [5], we introduce a Two-parameter generalization of the Incomplete Exponential Integral function. We also establish some properties satisfied by the function and then generalized inequalities established by Sroysang in [1] and [2]. We employ the Holder's inequality for integrals and the Young's inequality for scalars.

Keywords: Two-parameter Generalization of the incomplete exponential integral function, Holder's inequality and Young's inequality for scalars.

1. INTRODUCTION

The classical exponential integral function is defined by Schloemich in [3] as

$$(1) \quad E_n(x) = \int_1^\infty t^{-n} e^{-tx} dt \quad x > 0, \quad n \in \mathbb{N}.$$

For any $1 < a < b$ and $n \in \mathbb{N}_0$, the incomplete exponential integral function ${}_a^b E_n(x)$ is defined by [1] as

$$(2) \quad {}_a^b E_n(x) = \int_a^b t^{-n} e^{-xt} dt$$

for all $x > 0$.

In [1] it was proved that the incomplete exponential integral function is nonincreasing and then gave the inequality as follows,

$$(3) \quad {}_a^b E_{m+n} \left(\frac{x}{u} + \frac{y}{v} \right) \leq \left({}_a^b E_{um}(x) \right)^{\frac{1}{u}} \left({}_a^b E_{vn}(y) \right)^{\frac{1}{v}}$$

where $1 < a < b$, $x, y > 0$, $u > 1 = \frac{1}{u} + \frac{1}{v}$, and $m + n, um, vn \in \mathbb{N}_0$,

Also

$$(4) \quad {}_a^b E_n(xy) \leq \left({}_a^b E_n(ux) \right)^{\frac{1}{u}} \left({}_a^b E_n(vy) \right)^{\frac{1}{v}}$$

where $1 < a < b, x, y > 1, n \in \mathbb{N}_0, u > 1, \frac{1}{u} + \frac{1}{v} = 1$, and $x + y \leq xy$.

Furthermore,

$$(5) \quad {}_a^b E_n(xy) \geq \left({}_a^b E_n(ux)\right)^{\frac{1}{u}} \left({}_a^b E_n(vy)\right)^{\frac{1}{v}}$$

where $1 < a < b, x > 0, 0 < y < 1, n \in \mathbb{N}_0, 0 < p < 1 = \frac{1}{u} + \frac{1}{v} = 1$ and $x + y \geq xy$.

And finally,

$$(6) \quad {}_a^b E_n(xy) \geq \left({}_a^b E_n\left(\frac{rx^u}{u}\right)\right)^{\frac{1}{r}} \left({}_a^b E_n\left(\frac{sy^v}{v}\right)\right)^{\frac{1}{s}}$$

where $1 < a < b, x, y > 1, n \in \mathbb{N}_0, u > 1, 0 < r < 1$ and $\frac{1}{u} + \frac{1}{v} = 1 = \frac{1}{r} + \frac{1}{s}$.

And in [2] generalizations of inequalities (3),(4), (5) and (6) were established as follows,

For $x_i > 0, n_i \geq 0$, and $u_i > 1$, be such that $u_i n_i \in \mathbb{N}_0$, for all $i \in \mathbb{N}_m$. Assume that $1 < a < b, \sum_{i=1}^m u_i = 1$, and $\sum_{i=1}^m n_i \in \mathbb{N}_0$.

$$(7) \quad {}_a^b E_{\sum_{i=1}^m n_i} \left(\sum_{i=1}^m \frac{x_i}{u_i}\right) \leq \prod_{i=1}^m \left({}_a^b E_{u_i n_i}(x)\right)^{\frac{1}{u_i}}$$

is valid. Which is a generalization of (3).

For $n \in \mathbb{N}_0, x_i$ and $u_i > 1$ for all $i \in \mathbb{N}_m$. Assume that $1 < a < b, \sum_{i=1}^\infty u_i = 1$ and $\sum_{i=1}^m x_i \leq \prod_{i=1}^m x_i$.

Then

$$(8) \quad {}_a^b E_n \left(\prod_{i=1}^m x_i\right) \leq \prod_{i=1}^m \left({}_a^b E_n(u_i x_i)\right)^{\frac{1}{u_i}}$$

is valid. Which is also generalization of (4).

For $n \in \mathbb{N}_0, 0 < x_i < 1$ and $0 < u_i < 1$ for all $i \in \mathbb{N}_m$. Assume that $1 < a < b, \sum_{i=1}^\infty u_i = 1$ and

$\sum_{i=1}^m x_i \geq \prod_{i=1}^m x_i$.

Then

$$(9) \quad {}_a^b E_n \left(\prod_{i=1}^m x_i\right) \geq \prod_{i=1}^m \left({}_a^b E_n(u_i x_i)\right)^{\frac{1}{u_i}}$$

is valid. Which is also generalization of (5).

Inequality (6) is generalized as follows,

for $n \in \mathbb{N}_0, x_i > 1, u_i > 1$ and $0 < r_i < 1$ for all $i \in \mathbb{N}_m$. Assume that $1 < a < b$ and $\sum_{i=1}^m u_i =$

$1 = \sum_{i=1}^m r_i$,

$$(10) \quad {}_a^b E_n \left(\prod_{i=1}^m x_i\right) \geq \prod_{i=1}^m \left({}_a^b E_n\left(\frac{r_i x_i^{u_i}}{u_i}\right)\right)^{\frac{1}{r_i}}.$$

The focus of this paper is on the incomplete exponential integral function defined in [4] as

$$(11) \quad E_n(a, x) = \int_x^\infty t^{-n} e^{-at} dt \quad x \geq 0, \quad a > 0, \quad n \in \mathbb{N}_0$$

Clearly, $E_n(a, 1) = E_n(a)$.

The p -analogue of the exponential integral function, $E_{n,p}(x)$ is defined for $x > 0$, $p > 1$ and $n \in \mathbb{N}_0$ by [5]

$$(12) \quad E_{n,p}(x) = \int_1^p t^{-n} A_p^{-xt} dt,$$

and the i -th derivative of (12) is given by [6]

$$(13) \quad E_{n,p}^{(i)}(x) = (\ln A_p^{-1})^i \int_1^p t^{i-n} A_p^{-xt} dt,$$

where, $E_{n,p}(x) \rightarrow E_n(x)$ as $p \rightarrow \infty$, $A_p = (1 + \frac{1}{p})^p$ and $E_{n,p}^{(i)}(x) \rightarrow E_n^{(i)}(x)$ as $p \rightarrow \infty$.

The function emerges in the investigation of radiative exchanges in a two-dimensional planar medium [7].

This special function has been investigated in diverse ways (see [8], [9], [10] and the related references therein).

The objective of this paper is to introduce a Two-parameter generalization of the incomplete exponential integral function of (12) and to establish some properties of the function. In this paper, we will generalize inequalities (3), (4), (5), (6), (7), (8), (9) and (10).

2. PRELIMINARIES

We begin with the following well known results(see for instance [11], [12], [13] or [14]).

Lemma 2.1. (Holder's Inequality) Let $\eta, \mu > 1$ and $\frac{1}{\eta} + \frac{1}{\mu} = 1$. If $f(t)$ and $g(t)$ are continuous real-valued functions on $[a, b]$, then inequality

$$(14) \quad \int_a^b |f(t)g(t)| dt \leq \left(\int_a^b |f(t)|^\eta dt \right)^{\frac{1}{\eta}} \left(\int_a^b |g(t)|^\mu dt \right)^{\frac{1}{\mu}},$$

holds. With equality when $|g(t)| = c|f(t)|^{\eta-1}$. If $\eta = \mu = 2$, the inequality becomes Schwarz's Inequality.

Lemma 2.2. (Young's Inequality) Let $a, b > 0$, $\eta, \mu > 1$, and $\frac{1}{\eta} + \frac{1}{\mu} = 1$. Then inequality

$$(15) \quad ab \leq \frac{a^\eta}{\eta} + \frac{b^\mu}{\mu},$$

holds.

3. DEFINITION OF A TWO-PARAMETER GENERALIZATION OF THE INCOMPLETE EXPONENTIAL INTEGRAL

Definition 3.1. Let $x > 0, v \geq 1, p > 1,$ and $n \in \mathbb{N}_0.$ Then, the function is defined as

$$(16) \quad E_{n,p}(x, v) = \int_v^p t^{-n} A_p^{-xt} dt,$$

where, $E_{n,p}(x, 1) = E_{n,p}(x), E_{n,p}(x, v) \rightarrow E_n(x, v)$ as $p \rightarrow \infty, E_{1,p}(1, v) \rightarrow \Gamma(0, v)$ as $p \rightarrow \infty$ and $E_{n,p}(x, v) = x^{n-1} E_{n,p}(xv, v).$

3.1. Some Properties and Inequalities of $E_{n,p}(x, v).$

Lemma 3.2. *The recursive relation*

$$(17) \quad \ln A_p^x E_{n,p}(x, v) = v^{-n} A_p^{-vx} - p^{-n} A_p^{-px} - n E_{n+1,p}(x, v),$$

holds for $n \in \mathbb{N}_0, v \geq 1.$

Proof. Using (16) and by means of integration by parts

$$\begin{aligned} E_{n,p}(x, v) &= \int_v^p t^{-n} A_p^{-xt} dt \\ &= \left[-\frac{t^{-n} A_p^{-xt}}{\ln A_p^x} \right]_v^p - \frac{n}{\ln A_p^x} \int_v^p t^{-(n+1)} A_p^{-xt} dt \\ &= \frac{1}{\ln A_p^x} [v^{-n} A_p^{-vx} - p^{-n} A_p^{-px} - n E_{n+1,p}(x, v)], \end{aligned}$$

which concludes the proof.

Theorem 3.3. Let $p > 1$ and $m, n \in \mathbb{N}_0$ such that $\eta m, \mu n \in \mathbb{N}_0.$

Then, the inequality

$$(18) \quad E_{n,p} \left(\prod_{i=1}^m x_i, v \right) \leq \prod_{i=1}^m (E_{n,p}(\eta_i x_i, v))^{\frac{1}{\eta_i}},$$

holds for $x, y > 0, v \geq 1, \eta > 1$ and $\frac{1}{\eta} + \frac{1}{\mu} = 1,$

Proof. Using (16) and Hölder’s inequality for integrals, we have

$$\begin{aligned}
 E_{n,p} \left(\prod_{i=1}^m x_i, \nu \right) &\leq E_{n,p} \left(\sum_{i=1}^m x_i, \nu \right) \\
 &= \int_{\nu}^p t^{-n} A_p^{-t(\sum_{i=1}^m x_i, \nu)} dt \\
 &= \int_{\nu}^p \left(\prod_{i=1}^m t^{-\frac{n}{\eta_i}} A_p^{-(x_i, \nu)t} \right) dt \\
 &\leq \prod_{i=1}^m \left(\int_{\nu}^p t^{-n} A_p^{-\eta_i(x_i, \nu)t} dt \right)^{\frac{1}{\eta_i}} \\
 &= \prod_{i=1}^m (E_{n,p}(\eta_i x_i, \nu))^{\frac{1}{\eta_i}}.
 \end{aligned}$$

Which is a generalization of (3) and (7)

Theorem 3.4. Let $p > 1$ and $m, n \in \mathbb{N}_0$ such that $\eta m, \mu n \in \mathbb{N}_0$.

Then, the inequality

$$(19) \quad E_{\sum_{i=1}^m n_i, p} \left(\sum_{i=1}^m \frac{x_i, \nu}{\eta_i} \right) \leq \prod_{i=1}^m (E_{\eta_i, n_i, p}(x, \nu))^{\frac{1}{\eta_i}},$$

holds for $x, y > 0, \nu \geq 1, \eta > 1$ and $\frac{1}{\eta} + \frac{1}{\mu} = 1$,

Proof. Using (16) and Hölder’s inequality for integrals, we have

$$\begin{aligned}
 E_{\sum_{i=1}^m n_i, p} \left(\sum_{i=1}^m \frac{x_i, \nu}{\eta_i} \right) &= \int_{\nu}^p t^{-\sum_{i=1}^m n_i} A_p^{-t(\sum_{i=1}^m \frac{x_i, \nu}{\eta_i})} dt \\
 &= \int_{\nu}^p \left(\prod_{i=1}^m t^{-n_i} A_p^{-\left(\frac{x_i, \nu}{\eta_i}\right)t} \right) dt \\
 &\leq \prod_{i=1}^m \left(\int_{\nu}^p t^{-\eta_i n_i} A_p^{-(x_i, \nu)t} dt \right)^{\frac{1}{\eta_i}} \\
 &= \prod_{i=1}^m (E_{\eta_i, n_i, p}(x, \nu))^{\frac{1}{\eta_i}}.
 \end{aligned}$$

Which is also generalization of (4) and (8)

Theorem 3.5. Let $p > 1$ and $m, n \in \mathbb{N}_0$ such that $\eta m, \mu n \in \mathbb{N}_0$.

Then, the inequality

$$(20) \quad E_{n,p} \left(\prod_{i=1}^m x_i, v \right) \geq \prod_{i=1}^m (E_{n,p}(\eta_i x_i, v))^{\frac{1}{\eta_i}},$$

holds for $x, y > 0, v \geq 1, \eta > 1$ and $\frac{1}{\eta} + \frac{1}{\mu} = 1$,

Proof. Using (16) and Hölder’s inequality for integrals, we have

$$\begin{aligned} E_{n,p} \left(\prod_{i=1}^m x_i, v \right) &\geq E_{n,p} \left(\sum_{i=1}^m x_i, v \right) \\ &= \int_v^p t^{-n} A_p^{-t(\sum_{i=1}^m x_i, v)} dt \\ &= \int_v^p \left(\prod_{i=1}^m t^{-\frac{n}{\eta_i} A_p^{-(x_i, v)t}} \right) dt \\ &\geq \prod_{i=1}^m \left(\int_v^p t^{-n} A_p^{-\eta_i(x_i, v)t} dt \right)^{\frac{1}{\eta_i}} \\ &= \prod_{i=1}^m (E_{n,p}(\eta_i x_i, v))^{\frac{1}{\eta_i}}. \end{aligned}$$

Which is also generalization of (5) and (9)

Theorem 3.6. Let $p > 1$ and $m, n \in \mathbb{N}_0$ such that $\eta m, \mu n \in \mathbb{N}_0$.

Then, the inequality

$$(21) \quad E_{n,p} \left(\prod_{i=1}^m x_i, v \right) \geq \prod_{i=1}^m \left(E_{n,p} \left(r_i \left(\frac{x_i^{\eta_i}, v}{\eta_i} \right) \right) \right)^{\frac{1}{r_i}},$$

holds for $x, y > 0, v \geq 1, \eta > 1, \frac{1}{\eta} + \frac{1}{\mu} = 1$ and $\prod_{i=1}^m x_i, v \leq \sum_{i=1}^m \frac{x_i^{\eta_i}, v}{\eta_i}$

Proof. Using (16) and Hölder’s inequality for integrals, we have

$$\begin{aligned}
 E_{n,p} \left(\prod_{i=1}^m x_{i,v} \right) &\geq E_{n,p} \left(\sum_{i=1}^m \frac{x_i^{\eta_i, v}}{\eta_i} \right) \\
 &= \int_v^p t^{-n} A_p^{-t \left(\sum_{i=1}^m \frac{x_i^{\eta_i, v}}{\eta_i} \right)} dt \\
 &= \int_v^p \left(\prod_{i=1}^m t^{-\frac{n}{r_i} A_p^{-\left(\frac{x_i^{\eta_i, v}}{\eta_i} \right) t}} \right) dt \\
 &\geq \prod_{i=1}^m \left(\int_v^p t^{-n} A_p^{-r_i \left(\frac{x_i^{\eta_i, v}}{\eta_i} \right) t} dt \right)^{\frac{1}{r_i}} \\
 &= \prod_{i=1}^m \left(E_{n,p} \left(r_i \left(\frac{x_i^{\eta_i, v}}{\eta_i} \right) \right) \right)^{\frac{1}{r_i}}.
 \end{aligned}$$

Which is also a generalization of (6) and (10)

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