Original Research Article

SOME CHARACTERIZATIONS OF WHOLE EDGE DOMINATION IN BIPOLAR FUZZY GRAPHS

Abstract. In this paper, Some bounds and theorems on whole edge domination number in bipolar fuzzy graph are established. Further, some results on whole edge domination number in bipolar fuzzy graph are investigated with the support of some examples. The concepts of perfect whole edge domination in bipolar fuzzy graph, complete perfect whole edge domination in bipolar fuzzy graph, semi-perfect whole edge domination in bipolar fuzzy graph are discussed and investigated some of their properties and also some results on perfect whole edge domination number in bipolar fuzzy graph are contributed with the support of some examples.

Key words and phrases. Bipolar fuzzy graph (BFG), strong edge, Domination number, Edge domination number, Whole edge domination number.

1. Introduction

Graph theory has now become a new language that deals with all scientific and even literary sciences. It can give another face to prove most scientific problems through its simple tools. One of the most important topics that graph theory deals with is the topic of domination because it has very wide applications in most fields. The concept of edge domination in graphs was introduced by Mitchell and Hedetniemi [4].

Zadeh [7] introduced the notion of fuzzy subset of a set as a method of presenting uncertainty in 1965. In recent years, some kinds of domination in fuzzy graphs have been studied. Most of those belong to the vertex domination of fuzzy graphs [6]. The above observation motivates the researcher in the study of edge domination in fuzzy graphs.

In 1994, Zhang [8] [9] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets. The basic idea of fuzzy graph was introduced by mathematician Kauffman in the year 1973. After two years, Rosenfeld [5] introduced the concept of fuzzy graphs. Further, In [1] Akram extended fuzzy graph into bipolar fuzzy graph. In [3], Karunambigai et al. defined the domination, the domination number in bipolar fuzzy graphs.

Bipolar fuzzy sets are an extension of fuzzy sets whose membership degree range is [-1,1]. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree (0,1] of an element indicates that the element somewhat satisfies the property, and the membership degree [-1,0) of an element indicates that the element somewhat satisfies the implicit counter-property. In [2], the domination will be calculated by means of the set of edges.

In this paper, many bounds and properties of whole edge domination in bipolar fuzzy graph have been determined. Moreover, for certain graphs, this number has been introduced. Finally, the effect of deletion, addition, and contraction of an edge on the domination of this definition has been calculated.

2. Preliminaries

Definition 2.1. A fuzzy subset μ on a set X is a map $\mu : X \to [0,1]$. A map $v : X \times X \to [0, 1]$ is called a fuzzy relation on X if $v(x,y) \leq min(\mu(x), \mu(y))$ for all $x, y \in X$.

Definition 2.2. Let X be a non empty set. A bipolar fuzzy set M in X is an object having the form $B = \{(x, \mu_B^+, \mu_B^-)/x \in X\}$ where, $\mu_B^+ : X \to [0,1]$ and $\mu_B^- : X \to [-1,0]$ are mappings.

Definition 2.3. A Bipolar fuzzy graph (BFG) is of the form $G = (V, E)$ where

1.: $V = v_1, v_2, v_3, ...v_n$ such that $\mu_1^+ : X \to [0,1]$ and $\mu_1^- : X \to [-1,0]$.

2.: $\varepsilon \subset V \times V$ where $\mu_2^+ : V \times V \to [0,1]$ and $\mu_2^- : V \times V \to [-1,0]$ such that $\mu_{2ij}^+ = \mu_2^+(v_i, v_j) \leq min(\mu_1^+(v_i), \mu_1^+(v_j))$ and
$\mu_{2ij}^- = \mu_2^-(v_i, v_j) \geq max(\mu_1^-(v_i), \mu_1^-(v_j))$ for all $(v_i, v_j) \in E$.

Definition 2.4. Let u be a vertex in a BFG $G = (V, E)$ then $N(u) = \{v : v \in V\}$ and (u, v) is a strong edge in G is called neighbourhood of u in G.

Definition 2.5. A set $D \subseteq E$ is said to be an edge dominating set if every edge in $E - D$ is adjacent to some edge in D. The edge domination number of G is the cardinality of a smallest edge dominating set of G and is denoted by γ.

Definition 2.6. In a graph $G = (V, E)$, a proper subset $D \subseteq E$ is called whole edge dominating set ($WEDS$), if every edge in D is adjacent to all edges in $E - D$.

Definition 2.7. In a graph $G = (V, E)$, If X is a $WEDS$, then D is called minimal $WEDS$, if it has no proper $WEDS$.

Definition 2.8. A minimal $WEDS$ has smallest cardinality is called whole edge domination number denoted by $\gamma_{whe}(G)$.

Definition 2.9. The number of edges (the cardinality of E) is called the size of a bipolar fuzzy graph (BFG) and is denoted by
$S(G) = \sum_{v_i, v_j \in E} \left(\dfrac{1 + \mu_2^+(v_i, v_j) + \mu_2^-(v_i, v_j)}{2} \right)$, for all $(v_i, vj) \in E$.

Definition 2.10. An edge (u, v) is said to be strong edge in BFG, $G = (V, E)$ if $\mu_2^+(u,v) \geq (\mu_2^+)^\infty(u,v)$ and $\mu_2^-(u,v) \leq (\mu_2^-)^\infty(u,v)$ where

$$(\mu_2^+)^\infty(u,v) = max\{(\mu_2^+)^k(u,v)/k = 1, 2, 3, ...n\} \text{ and}$$
$$(\mu_2^-)^\infty(u,v) = min\{(\mu_2^-)^k(u,v)/k = 1, 2, 3, ...n\}.$$

Definition 2.11. Let $G = (V, E)$ be an bipolar fuzzy graph. Let e_i and e_j be two edges of G. We say that e_i dominates e_j, if e_i is a strong arc in G and adjacent to e_j.

Definition 2.12. A bipolar fuzzy graph $G = (A, B)$ is called strong if
$\mu_2^+(xy) = min\{\mu_1^+(x), \mu_1^+(y)\}$ and $\mu_2^-(xy) = min\{\mu_1^-(x), \mu_1^-(y)\}$ for all $xy \in E$.

Definition 2.13. The complement of a strong bipolar fuzzy graph $G = (A, B)$ of $G^* = (V, E)$ is a strong bipolar fuzzy graph $\bar{G} = (\bar{A}, \bar{B})$ on $\bar{G}$, where $\bar{A} = (\bar{\mu_A^+}, \bar{\mu_A^-})$ and $\bar{B} = (\bar{\mu_B^+}, \bar{\mu_B^-})$ are defined by

- $\bar{V} = V$
- $\bar{\mu_A^+} = \mu_A^+$ and $\bar{\mu_A^-} = \mu_A^-$, for all $x \in V$
-

$$\bar{\mu_B^+}(xy) = \begin{cases} 0 & if \mu_B^+(xy) > 0 \\ min(\mu_A^+(x), \mu_A^+(y)) & if \mu_B^+(xy) = 0 \end{cases}$$

$$\bar{\mu_B}(xy) = \begin{cases} 0 & if \mu_B^-(xy) < 0 \\ min(\mu_A^-(x), \mu_A^-(y)) & if \mu_B^-(xy) = 0 \end{cases}$$

Definition 2.14. A strong bipolar fuzzy graph G is called self complementary if $\bar{\bar{G}} = G$.

3. Whole Edge Domination in Bipolar Fuzzy Graph

Some dimension of whole edge dominating set in bipolar fuzzy graph are mentioned.

Definition 3.1. In a BFG $G = (V, E)$, a proper subset $D \subseteq E$ is called whole edge dominating set ($WEDS$), if every edge in D is strong to all edges in $E - D$.

Definition 3.2. In a BFG $G = (V, E)$, If D is a $WEDS$, then D is called minimal $WEDS$, if it has no proper $WEDS$.

Definition 3.3. A minimal $WEDS$ in BFG has smallest cardinality is called whole edge domination number in BFG denoted by $\gamma_{whe}(G)$.

Definition 3.4. Let $G = (V, E)$ be a bipolar fuzzy graph and $e \in E$, when we delete an edge e from G then the edges of G are partition into two sets

$$E_-^0 = \{e \in E, \gamma_{whe}(G - e) = \gamma(G)\},$$
$$E_-^- = \{e \in E, \gamma_{whe}(G - e) < \gamma(G)\}.$$

Example 3.5. Consider a BFG, (Figure 1:),

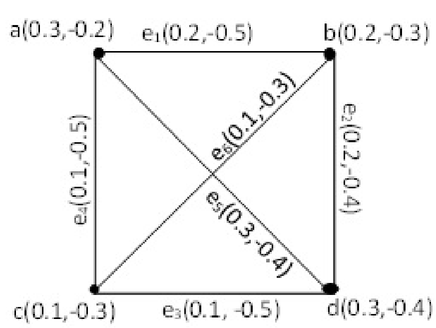

Figure 1 Bipolar fuzzy graph (BFG) no 1

In this example, If we remove the edge e_3, then $\gamma_{whe}(G-e) < \gamma(G)$.

Definition 3.6. Let $G=(V,E)$ be a bipolar fuzzy graph and $e \in G$, when we add an edge e from G then the edges of G are partition into two sets

$$E_+^0 = \{e \in E, \gamma_{whe}(G+e) = \gamma(G)\},$$
$$E_+^- = \{e \in E, \gamma_{whe}(G+e) < \gamma(G)\}.$$

Example 3.7. Consider a BFG, (Figure 2:),

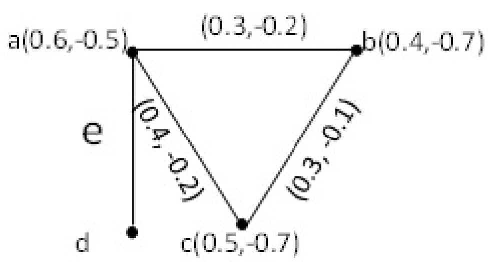

FIGURE 2 Bipolar fuzzy graph (BFG) no 2

Example 3.8. Consider a BFG, (Figure 3:),

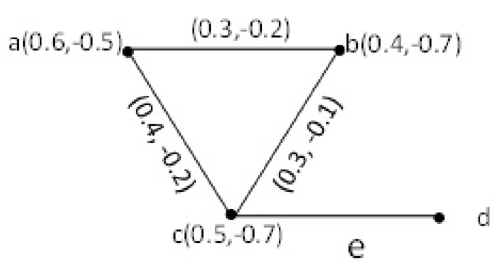

FIGURE 3 Bipolar fuzzy graph (BFG) no 3

In the above examples,
If we add the edge e (Figure :2), then $\gamma_{whe}(G+e) < \gamma(G)$.
If we add the edge e,(Figure :3), then $\gamma_{whe}(G+e) = \gamma(G)$.

Theorem 3.9. *If a bipolar fuzzy graph $G=(V,E)$ has a whole edge domination number $\gamma_{whe}(G)$, then E_+^0, E_+^- and E_+^+ are not empty sets.*

Proof. Two cases are appear as follows.
Case :(i)
If a graph G is a star and add an edge incident to the vertices then the BFG $(G+e)$

also has $\gamma_{whe}(G+e)$.

Case :(ii)

If there is no edge in D is adjacent to the addition an edge e, then two cases are appear as follows.

i) If we take $G = P_4$ and add the edge that is incident to the two pendants vertices of P_4, so the BFG $G+e = C_4$, therefore, the BFG $G+e$ also has $\gamma_{whe}(G+e)$.
ii) If we take $C(G) = C_4$ and add an edge incident to the vertices, so $G+e$ contains an edge adjacent to all edges. Then the BFG $G+e$ also has $\gamma_{whe}(G+e)$. □

Theorem 3.10. *If a bipolar fuzzy graph $G = (V, E)$ has a whole edge domination number γ_{whe}, then E_-^0, E_-^- and E_-^+ are not empty sets.*

Proof. As same manner in the previous theorem by deleting the edge. □

Corollary 3.11.

- *For the path bipolar fuzzy graph P_n with $n \geq 3$,*
  1. *If $3 \leq n \leq 4$ then the bipolar fuzzy graph has $\gamma_{whe}(P_n)$.*
  2. *If $n \geq 5$ then the bipolar fuzzy graph has no $\gamma_{whe}(P_n)$.*

- *For the cycle bipolar fuzzy graph C_n with $n \geq 3$,*
  1. *If $3 \leq n \leq 4$ then the bipolar fuzzy graph has $\gamma_{whe}(C_n)$.*
  2. *If $n \geq 5$ then the bipolar fuzzy graph has no $\gamma_{whe}(C_n)$.*

- *For the complete bipolar fuzzy graph K_n with $n \geq 3$,*
  1. *If $3 \leq n \leq 4$ then the bipolar fuzzy graph has $\gamma_{whe}(K_n)$.*
  2. *If $n \geq 5$ then the bipolar fuzzy graph has no $\gamma_{whe}(K_n)$.*

- *For a Wheel bipolar fuzzy graph W_n with $n \geq 3$,*
  1. *If $n = 3$, then the bipolar fuzzy graph has $\gamma_{whe}(W_n)$.*
  2. *If $n \geq 4$ then the bipolar fuzzy graph has no $\gamma_{whe}(W_n)$.*

- *If bipolar fuzzy graph be a star S_n with $n \geq 3$, then the star has $\gamma_{whe}(S_n)$.*

- *If bipolar fuzzy graph be a double star $S_{m,n}$, then the double star has $\gamma_{whe}(S_{m,n})$.*

Theorem 3.12. *A BFG $G = (V, E)$ be a tree (T) and it has one whole edge domination number if the number of vertices not more than 3.*

Proof. Let T be a tree and let D be a whole edge dominating set with minimum cardinality.
Suppose that D contains two edges say $\{e_1, e_2\}$, then there are two cases arises.

Case:(i)

If the number of the remained edges in $G - D$ is one say $\{e_3\}$, then since the graph is a tree so it must be a path of some order with an edge $\{e_3\}$ which the incident vertices on it of some degree.
Thus, $\{e_3\}$ is a whole edge dominating set in BFG and this is a contradiction with the set D is the minimum cardinality.

Case:(ii)
If the number of the remained edges in $G - D$ more than one, then there is a cycle contains the edges in D and the other edges.
Again, this is a contradiction with our assumption.
Therefore, D has one edge, then the middle edge in this tree is whole edge dominating set, which is strong to all edges.
So, that edge has minimum number of whole edge dominating graph.
Thus, the required is satisfied. □

Theorem 3.13. *Let G be BFG and $\bar{G}$ be the complement of BFG G with the nodes and arcs as in G or not. If D is the whole edge dominating set of G then $\bar{G}$ also has atleast one whole edge dominating set.*

Proof. Let G and $\bar{G}$ be BFG . Let us assume that G contains less number of nodes and arcs than G or equal number of nodes and arcs of G.
Suppose e_i and e_j are any two edges adjacent in G then they may be adjacent (or) non adjacent in $\bar{G}$.
This implies there exists distinct edge dominating sets in $\bar{G}$ but which does not equals D. □

Theorem 3.14. *If D be whole edge dominating set of a complete BFG G, then the edges of whole edge dominating set D incident with the nodes containing maximum degree.*

Proof. Let D be a whole edge dominating set in G.
Assume that the edges of whole edge dominating set D is not incident with the nodes having maximum degree. Then arcs of whole edge dominating set D are strong, which are incident with the node containing minimum degree.
By definition of edge dominating set, for each $e_j \in E - D$ there exists $e_i \in D$ such that e_i is strong to e_j. Hence whole edge dominating set D must contain atleast one strong arc.
This implies D is not minimum, then it leads to contradiction.
Hence edges of whole edge dominating set D should incident with the nodes containing maximum degree. □

Theorem 3.15. *Consider a bipolar fuzzy graph $G = (V, E)$, then G has $\gamma_{whe}(G)$ if $\gamma_{whe}(G/e)$.*

Proof. **Case:(i)**
a) If G contains a spanning subgraph isomorphic to star. This graph becomes a star too when we contract the edge e. So, the remains BFG G also has $\gamma_{whe}(G)$.
b) If BFG G be a double star, then we contract the edge e of the BFG G which is belong to the $\gamma_{whe}(G)$-set and that edge e separate the BFG G into the two stars. Then the BFG G is not connected.So it has no $\gamma_{whe}(G)$.
c) If the BFG G be a path, then we contract the edge e of the BFG G which is belong to the $\gamma_{whe}(G)$-set then the BFG G becomes not connected. So,the BFG G has no $\gamma_{whe}(G)$.
d) If the BFG G be a cycle, For example C_4, Then when we contract the edge e graph G turn to C_3, which means it has the whole edge domination number.
Case :(ii)
If e does not belong to any $\gamma_{whe}(G)$-set, then contracting an edge e do not influence to whole domination number of G. □

Remark:
1. If BFG $G - v$ has whole edge domination number, then BFG G is not necessary has whole edge domination number.
2. If $G - e$ has a whole edge domination number, then G is not necessary has whole edge domination number (as an example see Figure 4).

Example 3.16. Consider a BFG, (Figure 4:),

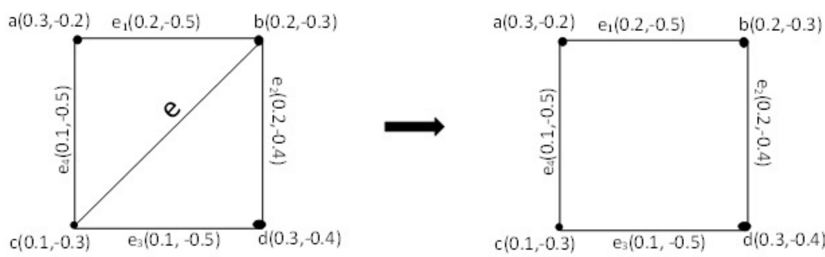

FIGURE 4 Bipolar fuzzy graph (BFG) no 4.

In this example, If we remove the edge e, then $\gamma_{whe}(G - e) < \gamma(G)$.

3. If $G + e$ has whole edge domination number, then G not necessary has whole edge domination number (as an example see Figure 5).

Example 3.17. Consider a BFG, (Figure 5:),

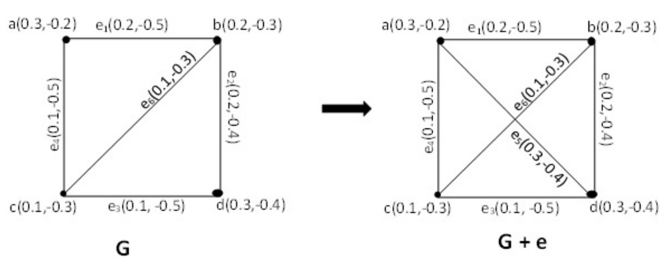

FIGURE 5 Bipolar fuzzy graph (BFG) no 5

In this example, If we add an edge e_5, then $\gamma_{whe}(G + e) < \gamma(G)$.

4. If Ge has a whole edge domination number, then G is not necessary has a whole edge domination number.

Theorem 3.18. *If a BFG G has γ_{whe}, then $\gamma_{whe}(G - v) = \gamma_{whe}(G)$,where $v \in V$ (or) $(G - v)$ has no whole edge domination set.*

Proof. There are two cases as follows.

Case :(i)
If $(G - v)$ is disconnected, then $(G - v)$ has no whole dominating set.
Case :(ii)
If $(G - v)$ is connected,there are two cases as follows
i) If $\gamma_{whe}(G) = 1$, then BFG G includes a spanning subgraph either it is a star or double star.
Now, If a BFG G includes a spanning subgraph isometric to star, then there are two cases as follows.

a) If a bipolar fuzzy graph G has maximum number of edges, which means there is an edge say $e = vu$, such that u and v are adjacent to all other vertices.

Thus, if we delete any other vertex from this graph the whole edge dominating is not influenced by this deletion, that means a bipolar fuzzy graph G has $\gamma_{whe}(G)$ (as an example, see figure 6).

Example 3.19. Consider a BFG, (Figure 6:),

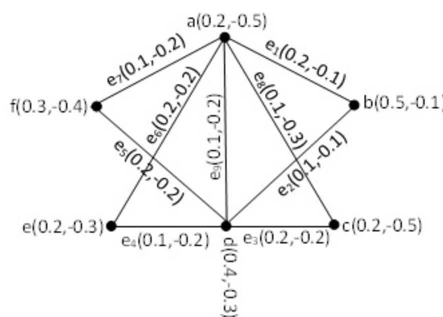

FIGURE 6 Bipolar fuzzy graph (BFG) no 6

In this example, If we remove the edge e_3, then $\gamma_{whe}(G - e) < \gamma(G)$.

b) If a bipolar fuzzy graph G has no maximum number of strong edges, which means the vertex u that is incident with the edge e is not adjacent to some other vertices in G, so if we delete the vertex v, then we get an isolated vertex,

so $(G - v)$ has no whole edge dominating set.

Otherwise, deleting any vertex from graph G do not influence the whole edge domination.

Now, if BFG G contains a spanning subgraph isometric to double star, then there are two cases as follows.
c) If $e = vu$ is the edge that is strong from all the edges in G, and if we delete u

or v, then the graph $(G-v)$ or $(G-u)$ has an isolated vertices. Thus $(G-v)$ or $(G-u)$ has no whole edge dominating set.
d) If the deleted vertex is not adjacent to the edge e which is dominating the graph edges, then this deletion do not influence to whole edge domination edge.
If a BFG G, it has strong edges $G = C_4$ or K_4.

Thus, $(G-v)$ is a path of order three, so it has $\gamma_{whe}(G-v)$.

Therefore, $\gamma_{whe}(G-v) = \gamma_{whe}(G)$.
For all cases above, one can see that $\gamma_{whe}(G-v) = \gamma_{whe}(G)$ or $(G-v)$ has no whole edge dominating set. □

4. Perfect Whole Edge Domination in Bipolar Fuzzy Graphs

Definition 4.1. A vertex v in a bipolar fuzzy graph $G = (V, E)$ is called a perfect bipolar fuzzy vertex if $\mu^P(v) = 1$ and $\mu^N(v) = -1$ (i.e.,) $\mu(v) = (1, -1)$ for all $v \in V$.

Definition 4.2. An edge $e = v, w$ (simply vw) in a bipolar fuzzy graph $G = (V, E)$ is called a perfect bipolar fuzzy edge if $\rho^P(vw) = 1$ and $\rho^N(vw) = -1$ (i.e.,) $\rho(vw) = (1, -1)$ for all $vw \in E$

Proposition 4.3. Every complete bipolar fuzzy graph is strong bipolar fuzzy graph.

Proposition 4.4. Every semi-complete bipolar fuzzy graph is semi-regular bipolar fuzzy graph.

Some dimension of perfect whole edge dominating set in bipolar fuzzy graph are mentioned.

In constructing a bipolar fuzzy graph in which the bipolar or double-sided (positive and negative, or effect and side effect) nature of human perception occurs, we need to consider both extreme values (positive extreme 1 and negative extreme -1) and develop a concept of perfectness in bipolar fuzzy graphs.

In this situation, we have to analyze the perfectness of each vertex (point/junction/node) and each edge (line/route/path) independently.
This section introduces the concepts of perfect Whole Edge Domination in bipolar fuzzy graph, complete perfect Whole Edge Domination in bipolar fuzzy graph, semi-perfect Whole Edge Domination in bipolar fuzzy graph, and investigates some results.

In this article, we consider $G = (V, E)$ is always a connected undirected simple bipolar fuzzy graph.

Definition 4.5. A vertex v in a whole edge domination bipolar fuzzy graph $G = (V, E)$ is called a perfect whole edge domination bipolar fuzzy vertex if $\mu^P(v) = 1$ and $\mu^N(v) = -1$ (i.e.,) $\mu(v) = (1, -1)$ for all $v \in V$.

Example 4.6. Consider a graph $G = (V, E)$ is a BFG, (Figure 1:),
In this example, $G = (V, E)$ be a whole edge domination bipolar fuzzy graph, where $V = \{d, e, f, g\}$ and $E\{e_1, e_2, e_3, e_4\}$. Here g is the only perfect whole edge domination bipolar fuzzy vertex; here there is no perfect whole edge domination bipolar fuzzy vertex.

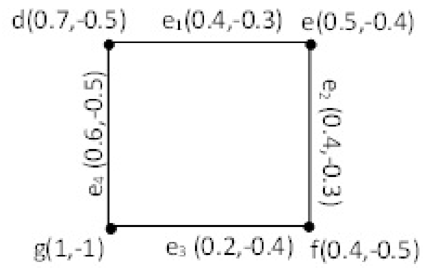

FIGURE 7 Bipolar fuzzy graph (BFG) no 7

Definition 4.7. An edge $e = \{v, w\}$ (simply vw) in a whole edge domination bipolar fuzzy graph $G = (V, E)$ is called a perfect whole edge domination bipolar fuzzy edge if $\rho^P(vw) = 1$ and $\rho^N(vw) = -1$ (i.e.,) $\rho(vw) = (1, -1)$ for all $vw \in E$

Example 4.8. Consider a graph $G = (V, E)$ is a BFG, (Figure 2:),

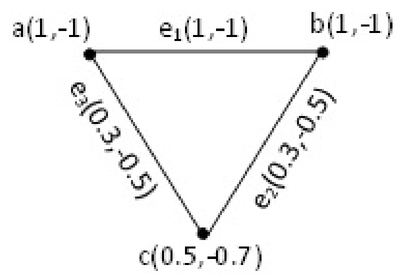

FIGURE 8Bipolar fuzzy graph (BFG) no 8

In this example, $G = (V, E)$ be a whole edge domination bipolar fuzzy graph, where $V = \{a, b, c\}$ and $E = \{e_1, e_2, e_3\}$. Here e_1 is the only perfect whole edge domination bipolar fuzzy edge; but here there is no perfect whole edge domination bipolar fuzzy edge.

Definition 4.9. A whole edge domination bipolar fuzzy graph $G = (V, E)$ is called an μ-perfect whole edge domination bipolar fuzzy graph if all vertices in G are perfect whole edge domination bipolar fuzzy vertices.

Example 4.10. Consider a graph $G = (V, E)$ is a BFG, (Figure 3:),

Definition 4.11. A whole edge domination bipolar fuzzy graph $G = (V, E)$ is called an ρ-perfect whole edge domination bipolar fuzzy graph if all the edges in G are perfect whole edge domination bipolar fuzzy edges.

Example 4.12. Consider a graph $G = (V, E)$ is a BFG, (Figure 4:),
Consider a graph $G = (V, E)$ is a whole edge domination BFG, (Figure 5:),
Here all the vertices are perfect whole edge domination bipolar fuzzy vertices and

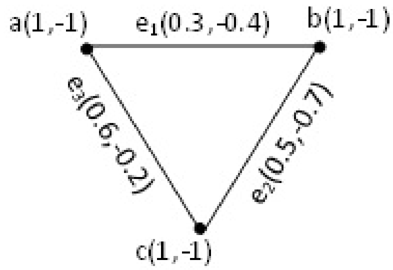

FIGURE 9 Bipolar fuzzy graph (BFG) no 9

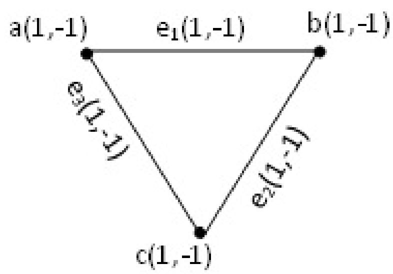

FIGURE 10 Bipolar fuzzy graph (BFG) no 10

all the edges are perfect whole edge domination bipolar fuzzy edges. Therefore, this is an μ-perfect and ρ-perfect whole edge domination bipolar fuzzy graph.

Proposition 4.13. Every complete whole edge domination in bipolar fuzzy graph is strong whole edge domination bipolar fuzzy graph.

Proposition 4.14. Every ρ-perfect whole edge domination bipolar fuzzy graph is an μ-perfect whole edge domination bipolar fuzzy graph.

Proof. Let $G = (V, E)$ be a ρ-perfect bipolar fuzzy graph. Then all edges in G have the bipolar fuzzy values (1,-1),

(i.e.,) $\rho(vw) = (1, -1)$ for all $vw \in E$.

By the definition of bipolar fuzzy graph, we have $\rho(vw) = (\rho^P(vw), \rho^N(vw))$ where $\rho^P(vw) \le min(\mu^P(v), \mu^P(w))$ and $\rho^N(vw) \le min(\mu^N(v), \mu^N(w))$.

This implies that $1 \le min(\mu^P(v), \mu^P(w))$ and $-1 \ge max(\mu^N(v), \mu^N(w))$.

Further, this implies that $\mu^P(v) = 1, \mu^P(w) = 1$, because greater than 1 is not possible; similarly $\mu^N(v) = -1, \mu^N(w) = -1$, because less than -1 is not possible.

Therefore $(\mu^P(v), \mu^N(v)) = (1, -1)$ and $(\mu^P(w), \mu^N(w)) = (1, -1)$, that is, $\mu(v) = (1, -1)$ and $\mu(w) = (1, -1)$.

In general, $\mu(v_i) = (1, -1)$ for all $v_i \in V$.

Hence, every ρ-perfect whole edge domination bipolar fuzzy graph is an μ-perfect whole edge domination bipolar fuzzy graph.

Note that the converse of this need not be true. □

Definition 4.15. A whole edge domination bipolar fuzzy graph $G = (V, E)$ is called a perfect bipolar fuzzy graph if it is an ρ-perfect whole edge domination bipolar fuzzy graph.

Definition 4.16. A whole edge domination bipolar fuzzy graph $G = (V, E)$ is called a complete perfect whole edge domination bipolar fuzzy graph if $\rho^P(vw) = 1$ and $\rho^N(vw) = -1$, (i.e.,) $\rho(vw) = (1, -1)$ for all $v, w \in V$

Definition 4.17. A whole edge domination bipolar fuzzy graph $G = (V, E)$ is called semi-perfect whole edge domination bipolar fuzzy graph if all vertices in G are μ-perfect whole edge domination bipolar fuzzy vertices.

Theorem 4.18. *Every complete perfect whole edge domination bipolar fuzzy graph is a perfect whole edge domination bipolar fuzzy graph.*

Proof. Let $G = (V, E)$ be a complete perfect whole edge domination bipolar fuzzy graph. Since G is complete, all vertices are connected together. Since G is perfect, all edges are ρ-perfect. That is, $\rho(rw) = (1, -1)$, for all $vw \in E$.

Therefore, obviously G is a perfect whole edge domination bipolar fuzzy graph.

Note that every perfect whole edge domination bipolar fuzzy graph is not necessarily a complete perfect whole edge domination bipolar fuzzy graph □

Theorem 4.19. *Every complete perfect whole edge domination bipolar fuzzy graph is a semi-perfect whole edge domination bipolar fuzzy graph.*

Proof. Since $G = (V, E)$ is complete perfect whole edge domination bipolar fuzzy graph, all edges are ρ-perfect and all vertices are joined by an edge. Clearly, G is μ-perfect. Therefore, G is a semi-perfect. Hence, every complete perfect whole edge domination bipolar fuzzy graph is a semi-perfect whole edge domination bipolar fuzzy graph. □

Proposition 4.20. If $G = (V, E)$ is a perfect whole edge domination bipolar fuzzy graph, then (i) $d(v_i) = d_E(vi) = d_N(vi)$, for all $v_i \in V$; (ii) $\sum_{v_i} d(v_i) = \sum_{v_i} d_E(vi) = \sum_{v_i} d_N(vi)$ for all $v_i \in V$;

Proof. Since $G = (V, E)$ is perfect whole edge domination bipolar fuzzy graph, it has μ-perfect and ρ-perfect, by definitions of degree, effective degree and neighbourhood degree, the results are immediate. □

Let us verify the Proposition by the following example.

Example 4.21. Let $G = (V, E)$ be a perfect whole edge domination bipolar fuzzy graph, where, $V = \{v_1, v_2, v_3\}$ and $E = \{e_1, e_2, e_3\}$ with $\mu(a) = \mu(b) = \mu(c) = (1, -1)$; $\rho(e_1) = \rho(e_2) = \rho(e_3) = (1, -1)$.

This is an μ-perfect and ρ-perfect whole edge domination bipolar fuzzy graph.

By usual calculations, we get,
$d(v_1) = d_E(v_1) = d_N(v_1) = (2, -2)$;
$d(v_2) = d_E(v_2) = d_N(v_2) = (2, -2)$;
$d(v_3) = d_E(v_3) = d_N(v_3) = (2, -2)$;

In general, $d(v_i) = d_E(v_i) = d_N(v_i)$ for all $v_i \in V$

Further, the sum of the degrees, sum of the effective degrees and sum of the

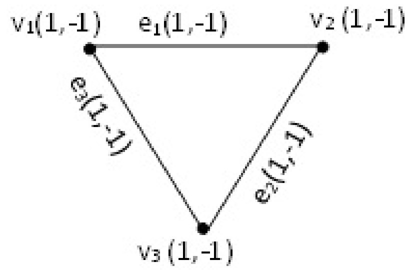

FIGURE 11 Bipolar fuzzy graph (BFG) no 11

neighbourhood degrees are same, that is,

$$\sum_{v_i} d(v_i) = \sum_{v_i} d_E(v_i) = \sum_{v_i} d_N(v_i) = (6, -6) \text{ for all } v_i \in V$$

Theorem 4.22. *Every complete whole edge domination bipolar fuzzy graph is a totally regular whole edge domination bipolar fuzzy graph.*

Theorem 4.23. *Every complete perfect whole edge dominating bipolar fuzzy graph is regular whole edge dominating bipolar fuzzy graph.*

Proof. Since G is complete perfect whole edge dominating bipolar fuzzy graph with n vertices; the (open) neighbourhood degree of any vertex is $(n-1, -(n-1))$, i.e., $d_N(vi) = (n-1, -(n-1)) \times (1, -1)$, for all $v_i \in V$. This is an $(n-1)$-regular bipolar fuzzy graph. Thus, every complete perfect whole edge dominating bipolar fuzzy graph is regular whole edge dominating bipolar fuzzy graph □

Proposition 4.24. Every semi-perfect whole edge dominating bipolar fuzzy graph is not necessarily regular whole edge dominating bipolar fuzzy graph.

Proposition 4.25. In any complete perfect whole edge dominating bipolar fuzzy graph, $d(v_i) = d_E(vi) = d_N(vi)$, for all $v_i \in V$;

5. Conclusion

In this work, we introduced whole edge domination in bipolar fuzzy graphs with some results. Some bounds and theorems were established as well. Further perfect whole edge domination in bipolar fuzzy graphs also introduced, related results and examples were also discussed.

References

1. M. Akram, *Bipolar fuzzy graphs*, Information Sciences, 181 (2011), 5548 - 5564.
2. Ammar Alsinai, Ahmed A. Omran,Thaer Abdul Ameer Ibrahim and Hakeem A. Othman *Some Properties of Whole Edge Domination in Graphs*, An International Journal of Applied Mathematics and Information Sciences, 16, No. 3(2022), 435 - 440.
3. M.G. Karunambigai, M. Akram, K. Palanivel and S. Sivasankar, *Domination in bipolar fuzzy graph* , in Proceedings of the International Conference on Fuzzy Systems FUZZ-IEEE-2013, Hyderabad, India (2013), 1 - 6.

4. S. Mitchell and S. T. Hedetniemi, *Edge domination in trees*, Gongr. Numer., 19 (1977), 489 - 509.
5. A. Rosenfeld,*Fuzzy graphs, Fuzzy Sets and their Applications to Cognitive and Decision Processes* (Proc. U.S.-Japan Sem., Univ. Calif., Berkeley, Calif. (1974)), Academic Press, New York (1975), 77 - 95.
6. A. Somasundaram and S. Somasundaram, *Domination in fuzzy graphs - I, Pattern Recognition Letters*, 19(1998), 787 - 791.
7. L.A. Zadeh, Fuzzy sets, *Information and Control*, 8 (1965), 338 - 353.
8. W.-R. Zhang, *Bipolar fuzzy sets and relations: a computational framework forcognitive modeling and multiagent decision analysis*, Proceedings of IEEE Conf., 1994, pp. 305–309.
9. W.-R. Zhang, *Bipolar fuzzy sets*, Proceedings of FUZZ-IEEE (1998) 835–840.