Original Research Article

Some relations of gamma functions

Abstract

In this paper we will present some inequalities related to gamma function, as well as some integrals, where as a function that is integrated will be a gamma function, or product of gamma functions, which will be presented in the form of series.

Keywords: gamma function, series, inequalities

Mathematics Subject Classification 2020: 65D20

Introduction and Auxiliary facts

Special functions represent a special class of mathematical functions, which take place in different branches of mathematics, such as mathematical analysis, functional analysis, differential equations, complex analysis, they also find application outside mathematics, as well as in physics and various branches in engineering.

There is no definite strict definition of what special functions actually represent, but there are a large number of functions that are accepted as such.

Some special functions are taken as solutions of differential equations, some others are defined through parametric integrals, and some as functional series.

The gamma function was first defined by Euler, who on January 8, 1730 wrote a letter to Christian Goldbach, where he defined the gamma function as follows:

$$\Gamma(x) = \int_0^1 (-logt)^{x-1} dt$$

For $x > 0$.

The beta function was also given by Euler, who is also known as the integral of the first type, while the gamma integral of the second type.

Next we will give some known results-lemmas that will be used in proofs of our main results.

Lemma 1 (Bernoulli's inequality)

For $r \geq 0$ and $x \geq -1$, follows the inequality $(1+x)^r \geq 1+rx$

Lema2 ([1]) $\Gamma(z+1) = z\Gamma(z), Re(z) > 0$

Lemma 3 ([3])

$$\sum_{n=1}^{\infty} \frac{q^n}{(2!)^{\frac{q-q^3}{1-q^n}} \cdot (3!)^{\frac{q^2-q^3}{1-q^n}} \cdot \ldots \cdot (n!)^{\frac{q^{n-1}-q^n}{1-q^n}}} < e^{1+q}, \qquad 0 < q < 1.$$

Lemma4 ([**3**])

$$e = \sqrt{\sum_{n=0}^{\infty} \frac{(2n+3)2^{2n}}{(2n+1)!}}$$

Lemma5 ([4]) For $0 \leq a \leq 1$

$$2^{a-1} \leq \Gamma(1+a) \leq 1$$

Lemma6 ([5])The function

$$F(x) = \frac{ln\Gamma(1+x)}{xlnx}$$

Is rigorously increasing in $(1, \infty)$.

Lemma 7([6]) For every gamma function the following relation is valid

$\Gamma(x) = \frac{1}{x} - \gamma + O(x)$, when $x \to 0$

Main results

Proposition 1

$$\text{Për } p > 0,\ \Gamma(p+1) > \tfrac{1}{p+1}$$

Proof

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx, p > 0$$

$$\Gamma(2+p) = \int_0^\infty x^{1+p}e^{-x}dx = \int_0^\infty (1+x-1)^{1+p}e^{-x}dx \geq (\,Lema\ 1)$$

$$\geq \int_0^\infty \big(1+(x-1)(p+1)\big)e^{-x}dx = \int_0^\infty e^{-x}dx + \int_0^\infty (x-1)(p+1)e^{-x}dx =$$

$$= \Gamma(1) + (p+1)\left(\int_0^\infty xe^{-x}dx - \int_0^\infty e^{-x}dx\right) = \Gamma(1) + (p+1)\big(\Gamma(2) - \Gamma(1)\big) = 1$$

Finally,

$$\Gamma(2+p) > 1$$

$$\Gamma(1+p+1) > 1$$

$$(p+1)\Gamma(p+1) > 1$$

$$\Gamma(p+1) > \frac{1}{p+1}$$

Proposition 2

$$\sum_{n=1}^{\infty} \frac{\Gamma^n(1+a)}{(2!)^{\frac{\Gamma(1+a)-\Gamma^2(1+a)}{1-\Gamma^n(1+a)}} \cdot (3!)^{\frac{\Gamma^2(1+a)-\Gamma^3(1+a)}{1-\Gamma^n(1+a)}} \cdot \ldots \cdot (n!)^{\frac{\Gamma^{n-1}(1+a)-\Gamma^n(1+a)}{1-\Gamma^n(1+a)}}}$$

$$< e^{\Gamma(1+a)}\sqrt{\sum_{n=0}^{\infty}\frac{(2n+3)2^{2n}}{(2n+1)!}}\ , \qquad 0 < a < 1$$

Proof. Now, for $0 \leq a \leq 1 \Longrightarrow 0 \leq \Gamma(1+a) \leq 1$, according to lemma 5, we denote

$q = \Gamma(1 + a)$, and using and substituting this in the lemma 3, then instead of "e" in the right hand of the inequality we replace the expression in lemma 4, is obtained the stated result in proposition 2.

Proposition 3 For $a > b > e$, $b^a > a^b$.

Proof Let us take the function

$$f(x) = \frac{x}{lnx}$$

Which after taking the first derivative we get

$$f'(x) = \frac{lnx - 1}{(lnx)^2}$$

so, for $x > e$, the function is incresing, respectively, for

$a > b > e \Rightarrow \qquad f(a) > f(b) \Rightarrow \frac{a}{lna} > \frac{b}{lnb} \Rightarrow b^a > a^b$.

Proposition 4 For $e < b < a$, holds the relation $log_{\Gamma(a^b)}\Gamma(b^a) > log_a \mathrm{b}$.

Proof. Using the proposition 3 and lemma 6 we get:

$$\frac{ln(\Gamma(1 + b^a))}{b^a lnb^a} > \frac{ln(\Gamma(1 + a^b))}{a^b lna^b}$$

$$\frac{1}{b^a} + \frac{1}{ab^a}\frac{ln(\Gamma(b^a))}{lnb} > \frac{1}{a^b} + \frac{1}{ba^b}\frac{ln(\Gamma(a^b))}{lna}$$

$$\frac{1}{ab^a}\frac{ln(\Gamma(b^a))}{lnb} - \frac{1}{ba^b}\frac{ln(\Gamma(a^b))}{lna} > \frac{1}{a^b} - \frac{1}{b^a} > 0\text{ , } since\ b^a > a^b$$

$$\frac{1}{ab^a}\frac{ln(\Gamma(b^a))}{lnb} > \frac{1}{ba^b}\frac{ln(\Gamma(a^b))}{lna}$$

$$\frac{ln(\Gamma(b^a))}{lnb} > \frac{ab^a}{ba^b}\frac{ln(\Gamma(a^b))}{lna} > \frac{ln(\Gamma(a^b))}{lna}$$

respectively

$$log_{\Gamma(a^b)}\Gamma(b^a) > log_a \mathrm{b}.$$

Proposition 5 Let $n \in \mathbb{N}_0$ and $z \notin \mathbb{Z}$ than holds the relation

$$\frac{\Gamma(2n+2-z)}{\Gamma(2n-z)}=(2n+1-z)(2n-z)$$

Proof Using the relations

$$\Gamma(2n+2-z)\Gamma(2n-z)=(2n+1-z)\Gamma(2n+1-z)\Gamma(2n-z)=$$

$$=(2n+1-z)(2n-z)\Gamma^2(2n-z)$$

Thus,

$$\frac{\Gamma(2n+2-z)}{\Gamma(2n-z)}=(2n+1-z)(2n-z)$$

Substituting $n=0$ and $z=-\frac{1}{2}$ we get :

$$\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}=\frac{3}{2}\cdot\frac{1}{2}=\frac{3}{4}\Rightarrow\Gamma\left(\frac{5}{2}\right)=\frac{3\sqrt{\pi}}{4}$$

Similarly, for $n=1$ and $z=-\frac{3}{2}$ we get

$$\frac{\Gamma\left(4+\frac{3}{2}\right)}{\Gamma\left(2+\frac{3}{2}\right)}=\left(3+\frac{3}{2}\right)\left(2+\frac{3}{2}\right)\Rightarrow\frac{\Gamma\left(\frac{11}{2}\right)}{\Gamma\left(\frac{7}{2}\right)}=\frac{63}{4}\Rightarrow\Gamma\left(\frac{11}{2}\right)=\frac{63}{4}\cdot\frac{5}{2}\cdot\Gamma\left(\frac{5}{2}\right)=\frac{945}{32}\sqrt{\pi}$$

Proposition6 Let $\lim\limits_{n\to\infty}a_n=\infty$ such that $a_n\geq 1,\forall n\in\mathbb{N}$, then $\lim\limits_{n\to\infty}\Gamma^{a_n}\left(1+\frac{1}{a_n}\right)=e^{-\gamma}$, where e represents Euler number, and γ represents Euler-Mascheroni constant.

Proof Using *lemma7* we get

$$\Gamma(1+x)=1-\gamma x+o(x)$$

$$\Gamma\left(1+\frac{1}{a_n}\right)=1-\frac{\gamma}{a_n}+o\left(\frac{1}{a_n^2}\right)$$

$$\left(1+\left(o\left(\frac{1}{a_n^2}\right)-\frac{\gamma}{a_n}\right)\right)^{a_n}=\left(1+\left(o\left(\frac{1}{a_n^2}\right)-\frac{\gamma}{a_n}\right)\right)^{\frac{a_n\left(o\left(\frac{1}{a_n^2}-\frac{\gamma}{a_n}\right)\right)}{o\left(\frac{1}{a_n^2}\right)-\frac{\gamma}{a_n}}}$$

$$\lim_{n\to\infty}\Gamma^{a_n}\left(1+\frac{1}{a_n}\right)=\lim_{n\to\infty}\left(1+\left(o\left(\frac{1}{a_n^2}\right)-\frac{\gamma}{a_n}\right)\right)^{\frac{a_n\left(o\left(\frac{1}{a_n^2}\right)-\frac{\gamma}{a_n}\right)}{o\left(\frac{1}{a_n^2}\right)-\frac{\gamma}{a_n}}}=e^{\lim_{n\to\infty}\left(o\left(\frac{1}{a_n}\right)-\gamma\right)}=e^{-\gamma}$$

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