Original Research Article

Anti-excedance on the Γ_1-non Deranged Permutations

Abstract**—**In this paper, we investigated anti-excedance statistics in Γ_1-non deranged permutations, the permutation which fixes the first element in the permutations. This was done first through some computation on this scheme using prime numbers $p \geq 5$. In this work, we redefine the anti-excedance on Γ_1-non deranged permutations. We generated recursion formula for the anti-excedance number, we also show that anti-excedance tops sum for any $\omega_i^{-1} \in G_p^{\Gamma_1}$ is equal to the excedance tops sum of $\omega_i \in G_p^{\Gamma_1}$. Similarly, we observed that anti-excedance bottoms sum for any $\omega_i^{-1} \in G_p^{\Gamma_1}$ is equal to the excedance bottoms sum of $\omega_i \in G_p^{\Gamma_1}$ other properties were also observed.

Keywords: *Anti-excedance, Γ_1-non deranged permutations, anti-excedance tops sum, anti-excedance bottoms sum and anti-excedance difference.*

I. INTRODUCTION

Anti-excedance set of the permutation π of i such that $\pi(i) \leq i$ which is denoted as $Ax(\pi)$ and the number of anti-excedance π denoted by $ax(\pi) = |Ax(\pi)|$. The term "permutation statistics" was originally used by [1], who described it in terms of a permutation , S_n, as well as the number of descents (des), excedances (exc), inversions (inv), and major indices (maj). He demonstrated that over the symmetric group S_n, the *exc* is equally distributed with *des* and the *inv* is equally distributed with *maj*. Due to its use in engineering and science, the study of permutation statistics has become a popular field of study among authors. It deals with permutations whose constituent numbers are analyzed in terms of their arrangement. A range of conclusions, spanning from its avoidance class to its qualities, have been examined since the Aunu permutation pattern was first developed in an effort to provide combinatorial interpretations of particular succession schemes. [2] created a method using both the prime numbers $p \geq 5$ and $\Omega \subseteq N$ employing the catalan numbers as well. The configurations are chosen using a cycle of permutation patterns generated by this approach. Researchers have studied permutation groups with specific characteristics over time; one that comes to mind is the permutation patterns with fixed elements or without fixed elements, in which case the concepts of deranged and non-deranged permutation surfaces are used. This interpretation is supported by [3] modification of the [2] approach to two line notation, which produced a set of permutations with a fix at 1 (which produced the natural identity). This obtained set of permutations is known as the "Γ_1-non deranged permutation group" and is denoted as $G_p^{\Gamma_1}$. [4] added fuzziness to the Γ_1- non-deranged permutation group $G_p^{\Gamma_1}$ and found that it is a one-sided fuzzy ideal (only right fuzzy but not left). They also found that the α-level cut of f matches with if $\alpha = \frac{1}{p}$. [5] Showed the linkages and patterns on the structures and fixed point of the permutations produced by these operations using the direct and skew sum operation on the constituents of the Γ_1-non deranged permutation group. Additionally, the collection of permutations of the form of π is an abelian group, denoted by $G_p^{\Gamma_{m\oplus}}$, if π is the direct sum of these Γ_1-non deranged permutations. [6] have demonstrated the Excedance set of all ω_i in $G_p^{\Gamma_1}$ such that $\omega_i \neq e$ is $\frac{1}{2}(p-1)$ and provided highly useful theoretical features of the Γ_1-non deranged permutations. According to [7], the intersection of the Descent set and all Γ_1-non derangements is empty. They also noted that the Descent number is strictly lower than the Ascent number by $p-1$. [8] demonstrated that the statistics *maz* (sum of the descent of *ZDer*) and *maj* (sum of the descent set) are equally distributed, demonstrating that the statistics *dez* (cardinality of the set *Dez*) is an Eulerian statistics on Γ_1-non deranged permutations. [9] Proved that the admissible inversion set $Ai(\omega_i)$ and admissible inversion set $Ai(\omega_{p-i})$ are disjoint and that the admissible inversion descent $aid(\omega_{p-1})$ is equi-distributed with descent number $des(\omega_{p-1})$. Ibrahim and Aremu [10] noted that any $\chi(G(\omega_{P-I}))$ in $\mathrm{G}_p^{\Gamma_1}$ has a chromatic number of $p-1$ and any $\chi(G(\omega_I))$ has a chromatic number of one. Similar to this, any $\chi'(G(\omega_{P-I}))$ in $\mathrm{G}_p^{\Gamma_1}$ has a chromatic index of $p-2$, and any $\chi'(G(\omega_I))$ in $\mathrm{G}_p^{\Gamma_1}$ has a chromatic value of zero. According to [11], Lmap and Lmal are comparable in all Γ_1- non deranged permutations. The Rmal of ω_1 is empty for any $\mathrm{G}_p^{\Gamma_1}$, it was also discovered.

In this paper, we show that anti-excedance tops sum for any $\omega_i^{-1} \in \mathrm{G}_p^{\Gamma_1}$ is equal to the excedance tops sum of $\omega_i \in \mathrm{G}_p^{\Gamma_1}$ .We also show anti-excedance bottoms sum for any $\omega_i^{-1} \in \mathrm{G}_p^{\Gamma_1}$ is equal to the excedance bottoms sum of $\omega_i \in \mathrm{G}_p^{\Gamma_1}$.

II. PRELIMINARIES

In this section, we give some definitions and preliminaries which are useful in our work.

Definition 2.1 [9]

Let Γ be a non empty set of prime cardinality $p \geq 5$ such that $\Gamma \subset N$ A bijection ω on Γ of the form

$$\omega_i = \begin{pmatrix} 1 & 2 & 3 & . & . & . & p \\ 1 & (1+i)_{mop} & (1+2i)_{mop} & . & . & . & (1+(p-1)i)_{mop} \end{pmatrix}$$

is called a Γ_1-non deranged permutation. We denoted G_p to be the set of all Γ_1-non deranged permutations.

$\mathrm{G}_p = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$ is the set of all Γ_1-non deranged permutations where $p = 7$

By definition 2.1, G_7 is generated as follows

$$\omega_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

$$\omega_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 7 & 2 & 4 & 6 \end{pmatrix}$$

$$\omega_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 7 & 3 & 6 & 2 & 5 \end{pmatrix}$$

$$\omega_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 5 & 2 & 6 & 3 & 7 & 4 \end{pmatrix}$$

$$\omega_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 6 & 4 & 2 & 7 & 5 & 3 \end{pmatrix}$$

$$\omega_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 7 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}$$

Definition 2.2 [12]

The pair G_p and the natural permutation composition forms a group which is denoted as $\mathrm{G}_p^{\Gamma_1}$ .This is a special permutation group which fixes the first element of Γ.

Definition 2.3

The excedance set denoted by $Exc(\pi) = \{i : \pi(i) > i\}$.The number of excedance of a permutation π is denoted by $exc(\pi) = |Exc(\pi)|$ that is the cardinality of $Exc(\pi)$.

Definition 2.4

The anti-excedance set is the opposite of excedance set is denoted by $Ax(\pi) = \{i : \pi(i) \le i\}$ .The number of ant-excedance of a permutation π is denoted by $ax(\pi) = |Ax(\pi)|$ that is the cardinality of $Ax(\pi)$.

Definition 2.5

Given a permutation $\pi \in \mathrm{G}_p^{\Gamma_1}$, anti-excedance in π is an integer i with $1 \le i \le n$ such that $a_i \le i$. Here a_i .is called the anti-excedance top and i is called the anti-excedance bottom.

Definition 2.6

The anti-excedance top sum of a permutation π is the summation of its anti-excedance tops. It is denoted by $Axtop(\pi)$.

Definition 2.7

The anti-excedance bottom sum of a permutation π is the summation of its anti-excedance bottom. It is denoted by $Axbot(\pi)$.

Definition 2.8

The anti-excedance difference of a permutation π is the difference between anti-excedance top sum of a permutation π and anti-excedance bottom sum of a permutation π that is $Axdif(\pi) = Axtop(\pi) - Axbot(\pi)$.

III. RESULTS AND DISCUSSION

In this section, we discuss the details of the investigations and results obtain.

Proposition 3.1

Let $\mathrm{G}_p^{\Gamma_1}$ be a Γ_1-non deranged permutation group, then the cardinality of the anti-excedance set of all ω_i in $\mathrm{G}_p^{\Gamma_1}$ such that $\omega_i \ne 1$ is $\frac{p+1}{2}$

Proof:

By the non derangement property of $\mathrm{G}_p^{\Gamma_1}$ that the cardinality of the anti-excedance set of ω_i which is the identity is always p .since p is the identity in permutation it implies that the cardinality of anti-excedance set of identity is always p .since the cardinality of $\mathrm{G}_p^{\Gamma_1}$ is $p-1$ and identity permutation of $\mathrm{G}_p^{\Gamma_1}$ is p then we left with $p-2$ which is $\frac{p+1}{2}$, this complete the proof. □

Proposition 3. 2

Let $\mathrm{G}_p^{\Gamma_1}$ be a Γ_1-non deranged permutation group then the anti-excedance number of ω_i is p where $i-1.$

Proof:

By the non derangement property of $\mathrm{G}_p^{\Gamma_1}$. Known of the element exist in the excedance set of the arbitrary $\omega_1 \in G_p^{\Gamma_1}$ where $i-e$ and so their exist a set in anti-excedance of Γ_1-non deranged permutation of $\mathrm{G}_p^{\Gamma_1}$ being the anti-excedance opposite of excedance, the set contain in anti-excedance is p. This completes the proof. □

Proposition 3.3

Let $\mathrm{G}_p^{\Gamma_1}$ be a Γ_1-non derangement permutations, then the sum of excedance numbers is

$$\sum_{i=1}^{p-1} exc(\omega_i) = \frac{P(P-1)}{2} - (p-1).$$

Proof:

The order of $\mathrm{G}_p^{\Gamma_1}$ and the number of positions with excedance and anti-excedance are both $p-1$. We know the cardinality of excedance is less than the cardinality of anti-excedance by 1 in $\mathrm{G}_p^{\Gamma_1}$ where $\omega_i \neq e$ .Also the sum of excedance number of is less than anti-excedance by $2(p-1)$. Since the sum of anti-excedance number is $\frac{p(p-1)}{2} + p - 1$, then

$$\frac{p(p-1)}{2} + (p-1) - [2(p-1)]$$

$$\frac{p(p-1)}{2} + p - 1 - 2p + 2$$

$$\frac{p(p-1)}{2} - (p-1) \ .$$

□

Proposition 3.4

Let $\mathrm{G}_p^{\Gamma_1}$ be a Γ_1-non derangement permutations, then the sum of Anti-excedance numbers is

$$\sum_{i=1}^{p-1} ax(\omega_i) = \frac{p(p-1)}{2} + (p-1).$$

Proof:

The order of $\mathrm{G}_p^{\Gamma_1}$ and the number of positions with excedance and anti-excedance are both $p-1$. We know the cardinality of anti-excedance is greater than the cardinality of excedance by 1 in $\mathrm{G}_p^{\Gamma_1}$ where $\omega_i \neq e$. Also, the sum of anti-excedance number is greater than sum of excedance number by $2(p-1)$. From proposition 3 the sum of excedance numbers is $\frac{p(p-1)}{2} - (p-1)$, Then

$$\frac{p(p-1)}{2}-(p-1)+[2(p-1)]$$
$$\frac{p(p-1)}{2}-(p-1)+2p-2$$
$$\frac{p(p-1)}{2}+p-1\ .$$
□

Proposition 3.5
Let $\omega_i \in G_p^{\Gamma_1}$ Then for any $i, j \neq 1$.

$$ax(\omega_i)=ax(\omega_j)\,.$$

Proof:
For any $\omega_i \in G_p^{\Gamma_1}$, $i \neq 1$. Only 1 is fixed in ω_i . Therefore, since p is prime, we have that $P-1$ are non-fixed letters. Since p is prime, then half of the letters $\left(\frac{p+1}{2}\right)$ will be greater than their images, while half will be less than their images. Therefore, we have

$$ax(\omega_i)=\frac{p+1}{2},\, i\neq 1.$$
□

Proposition 3.6
Let $\omega_i \in G_p^{\Gamma_1}$. Then

$$ax\left(\omega_{\frac{p+1}{2}}\right)=\frac{p+1}{2}\,.$$

Proof:
For $\omega_{\frac{p+1}{2}} \in G_p^{\Gamma_1}$. It can be represented as

$$\omega_{\frac{p+1}{2}}=\begin{pmatrix}1 & 2 & 3 & 4 & . & . & . & p-1 & p \\ 1 & (p-\frac{p-3}{2}) & 2 & (p-\frac{p-3}{2}-1). & . & . & & p & \frac{p+1}{2}\end{pmatrix}.$$

From this we can observe that the permutation has $\frac{p+1}{2}$ where the first block has one element while all other blocks have two element each. Obviously there will be $\frac{p+1}{2}$ proper block with two elements. Hence $ax\left(\omega_{\frac{p+1}{2}}\right)=\frac{p+1}{2}$. □

Proposition 3.7

Let $\omega_i \in G_p^{\Gamma_1}$, for $i = \dfrac{p+1}{2}$ .Then the

$$Ax(\omega_i) = \bigcup_{k=0}^{\frac{p-1}{2}} \{2k+1\} .$$

Proof:

Let the representation of $\omega_{\frac{p+1}{2}}$ be as in Proof of proposition 3.6. Since the first block has one element while all other blocks have two elements each. By definition of $Ax,$ we can observed that the position of images is increasing by two Hence $Ax(\omega_i) = \bigcup_{k=0}^{\frac{p-1}{2}} \{2k+1\}$. □

Proposition 3.8

Let $\omega_i \in G_p^{\Gamma_1}$. Then the

$$Axtop(\omega_i^{-1}) = Etop(\omega_i).$$

Proof:

Given $\omega_i \in G_p^{\Gamma_1}$. The $Axtop(\omega_i^{-1})$ .is the sum of anti-excedance tops of ω_i^{-1} . By the definition of inverse of a permutation and definition of excedance top, we have that the anti-excedance tops of ω_i^{-1} are the same as excedance tops of ω_i Hence, $Axtop(\omega_i^{-1}) = Etop(\omega_i)$. □

Proposition 3.9

Let $\omega_i \in G_p^{\Gamma_1}$. Then the

$$Axbot(\omega_i^{-1}) = Ebot(\omega_i).$$

Proof:

Given $\omega_i \in G_p^{\Gamma_1}$ .The $Axbot(\omega_i^{-1})$ is the sum of the anti- excedance bottoms of ω_i^{-1} By the definition of inverse of a permutation and definition of excedance bottoms , we have that the anti-excedance bottoms of ω_i^{-1} are the same as the excedance bottoms of ω_i . Therefore, $Axbot(\omega_i^{-1}) = Ebot(\omega_i).$ □

Proposition 3.10

Let $\omega_i \in G_p^{\Gamma_1}$. Then the

$$Axdif(\omega_i^{-1}) = Edif(\omega_i).$$

Proof:

By Proposition 3.8 and Proposition 3.9, we have that $Axtop(\omega_i^{-1}) = Etop(\omega_i)$ and $Axbot(\omega_i^{-1}) = Ebot(\omega_i)$. Therefore $Axdif(\omega_i^{-1}) = Axtop(\omega_i^{-1}) - Axbot(\omega_i^{-1})$ is equal to the $Edif(\omega_i) = Etop(\omega_i) - Ebot(\omega_i)$. Thus, $Axdif(\omega_i^{-1}) = Edif(\omega_i)$. □

IV. CONCLUSION

In this paper, we computed the statistic anti- excedance on Γ_1-non deranged permutations. We have shown that the anti-excedance tops sum for any $\omega_i^{-1} \in \mathrm{G}_p^{\Gamma_1}$ is equal to the excedance tops sum of $\omega_i \in \mathrm{G}_p^{\Gamma_1}$ and also shown that anti-excedance bottoms sum for any $\omega_i^{-1} \in \mathrm{G}_p^{\Gamma_1}$ is equal to the excedance bottoms sum of $\omega_i \in \mathrm{G}_p^{\Gamma_1}$.

REFERENCES

[1] MacMahon P.A.(1915).*Combinatory Analysiks* Vol. 1 and 2 Cambridge University Press (reprinted by Chesea,New York,1955)

[2] Garba, A.I. and Ibrahim, A.A. (2010). A new method of constructing a variety of finite Group based on some succession scheme. *International Journal of Physical Science* 2(3) , 23-26

[3] Ibrahim, A.A., Ejima, O. and Aremu, K.O., (2016). On the representations of Γ1- Deranged Permutation group $\mathrm{G}_p^{\Gamma_1}$, *Advances i n Pure Mathematics*, 6, 608-614.

[4] Aremu, K.O, Ejima O. and Abdullahi, M.S., (2017) On the fuzzy Γ1-non deranged Permutation group $\mathrm{G}_p^{\Gamma_1}$ *Asian Journal of Mathematics and Computer Research*,18(4),152-157

[5] Aremu, K.O.,Buoro, S., Garba, A.I., and Ibrahim, A.H.(2018). On the Direct and Skew Sums Γ1-non deranged permutations. *Punjab Journal of Mathematics and Computer Research*, 50(3): 43-51.

[6] Ibrahim, M. and Garba, A.I.(2018). Excedance on Γ1-non deranged permutations. *Proceedings of Annual National Conference of Mathematical Association of Nigeria (MAN)*, 197−201

[7] Ibrahim, M. and Garba, A.I.(2019). Descent on Γ1-non deranged Permutation group. *Journal of the Mathematical Association of Nigeria ABACUS*,46(1) ,12 − 18.

[8] Ibrahim, M. and Magami, M.S.(2019). On Fixed Decomposition of Γ1-non deranged Permutations. *International Journal of Scientific and Research Publications*,9(12) ,158 –162

[9] Magami, M.S. and Ibrahim, M. (2021). Admissible inversion on Γ1-non deranged permutations. *Asian Research Journal of Mathematics* ,17(8) : 44 − 53.

[10] Ibrahim, M. and Aremu K.O.(2022). Graph Coloring with Inversion in Γ1-non deranged Permutations. *UMYU Scientifica*, **1**(2), 107-111.

[11] Ibrahim, M. and Musa A.K.(2023). A Study of Some new Statistics on Γ1-non deranged Permutations. *International journal of science for global sustainability*, **9**(1), 136-141.

[12] Ibrahim, M., Ibrahim, A.A, Garba, A.I. and Aremu, K.O.(2017).Ascent on Γ1-non deranged permutation group $\mathcal{G}_p^{\Gamma_1}$. *International journal of science for global sustainability*, **4**(2), 27-32.