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Properties and Characterizations of Norm-Attainable Operators in Compact and Self-Adjoint Settings

Abstract

In this research paper, we investigate the properties and characterizations of norm-attainable operators in the context of compactness and self-adjointness. We present a series of propositions, a lemma, a theorem, and a corollary that shed light on the nature of these operators and provide insights into their behavior in various settings. Our results contribute to the understanding of norm-attainable operators and their implications in functional analysis.

[1]

Keywords:compact operators, numerical range, function spaces, normal operators, self-adjoint operators, functional calculus, reflexive spaces, finite-dimensional spaces..

1 Introduction

In this paper, the concept of norm-attainable operators is investigated within the framework of functional analysis, with a particular emphasis on their relevance in the study of compact and self-adjoint operators. Norm attainment refers to the property of an operator being able to achieve its norm on certain vectors in its domain. The properties and characterizations of norm-attainable operators have been extensively studied in operator theory, and this paper explores their behavior and implications, especially in the context of compactness and self-adjointness. Several researchers have contributed to the literature on norm-attainable operators, including Kinyanjui [1] who focused on characterizations, Lindenstrauss [2] who studied operators attaining their norms, Okelo [3] who investigated absolutely norm-attaining compact hyponormal operators, Okelo [4] who examined norm-attainability in normed spaces, Okelo, Agure, and Ambogo [5] who explored elementary operators' norms and norm-attainable operator characterizations, Okelo and Aminer [6] who studied norm inequalities and orthogonal extensions of norm-attainable operators, Satish and Vern [7] who provided a spectral characterization, and Shkran [8] who focused on norm-attaining operators and pseudo-spectrum. Overall, this paper provides a comprehensive overview of the theories and concepts related to compactness, self-adjointness, and norm attainment in the context of operator theory.

2 Preliminaries

Before delving into the main results, we provide the necessary definitions and background information.

Definition 2.1. *A linear operator T between two normed vector spaces X and Y is said to be compact if it maps bounded sets in X to sets that are relatively compact (i.e., have compact closures) in Y. In other words, for any bounded set B in X, the image $T(B)$ is relatively compact in Y.*

Definition 2.2. *A linear operator T on a complex or real inner product space H is said to be self-adjoint if it satisfies the condition $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all vectors $x, y \in H$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in H. For a complex inner product space, a self-adjoint operator is often referred to as Hermitian.*

Definition 2.3. *Let T be a linear operator on a normed vector space X. T is said to be norm-attainable if there exists a vector x in the domain of T such that $\|T\| = \|Tx\|$. In other words, the operator achieves its norm on a particular vector in its domain.*

The notion of norm attainment is significant in operator theory as it allows for a deeper understanding of the behavior and properties of operators.

3 Methodology

In this research, we adopt a theoretical approach to analyze the properties and characterizations of norm-attainable operators in compact and self-adjoint settings. Using mathematical frameworks and notation from functional analysis and operator theory, we leverage techniques from spectral theory, functional calculus, and numerical methods to investigate norm-attainable operators. By carefully examining conditions and assumptions, we establish our main results through rigorous proofs and logical arguments. Although studying norm-attainable operators may pose challenges in dealing with complex mathematical structures, our methodology focuses on mathematical analysis and reasoning to provide a comprehensive understanding of these operators. Our aim is to contribute to the understanding of norm-attainable operators in compact and self-adjoint settings by employing mathematical techniques, rigorous proofs, and logical reasoning.

4 Results

In this section, we present and discuss the main results of our research, which include propositions, a lemma, a theorem, and a corollary. We begin by stating and discussing Proposition 1, which provides insights into norm-attainable operators when both compactness and self-adjointness are present. We analyze the implications and consequences of this result in the broader context of functional analysis.

Proposition 1. *If a mapping T from a vector space H to itself is both compact and self-adjoint, then it belongs to the set of norm-attainable operators $NA(H)$.*

Proof. To prove the proposition, we need to show that if a mapping $T : H \to H$ is both compact and self-adjoint, then T is a norm-attainable operator, i.e., there exists a vector $x \in H$ such that $\|Tx\| = \|T\|$.
1. Compactness: Since T is compact, it maps bounded sets in H to relatively compact sets. This implies that the image of the unit ball in H, denoted as $B(0, 1)$, under T is relatively compact.
2. Self-adjointness: T is self-adjoint, which means for all vectors $x, y \in H$, the inner product of Tx and y is equal to the inner product of x and Ty, i.e., $\langle Tx, y\rangle = \langle x, Ty\rangle$. Now, we aim to show that there exists a vector $x_0 \in H$ such that $\|Tx_0\| = \|T\|$. To do this, we will use the following properties of compact operators:
Property 1: For any compact operator T, there exists a sequence of unit vectors $\{x_n\}$ in H such that $\|Tx_n\| \to \|T\|$ as $n \to \infty$.
Property 2: Since T is self-adjoint, the set $\{x \in H : \|x\| = 1\}$ is compact. Now, let's use these properties to construct the desired vector x_0: Consider the set $K = \{x \in H : \|x\| = 1\}$. By Property 2, K is compact. Define $f : K \to \mathbb{R}$ as $f(x) = \|Tx\|$. Since T is a continuous operator (as it is compact), f is continuous as well. Since K is compact and f is continuous, by the extreme value theorem, f attains its maximum on K. In other words, there exists a vector $x_0 \in K$ such that $\|Tx_0\| = \|T\|$. Now, we have found a vector x_0 with $\|x_0\| = 1$ such that $\|Tx_0\| = \|T\|$. Therefore, T is norm-attainable, and the proposition is proved. □

Moving forward, the next Proposition addresses the case where a mapping T from a vector space H to itself is normal, with the real part being Hermitian and the imaginary part exhibiting certain properties. We explore the norm-attainability of such operators and investigate the conditions under which they belong to the class of norm-attainable operators.

Proposition 2. *If T is a normal operator expressed as $T_1 = T_2 + iT_3$, where T_2 is Hermitian and T_3 is a diagonal operator with diagonal entries α_n for a* *positive integer n, then T belongs to the class of normal-attainable operators, denoted as $NA(H)$.*

Proof. Consider the operator T_3 with diagonal entries α_n for all positive integers n. Let T_{3n} be the diagonal operator with diagonal entries $\alpha_n, \ldots, \alpha_{n-1}, 0, 0, \ldots$. The difference between T_3 and T_{3n} has a diagonal with elements $0, \ldots, 0, \alpha_{n+1}, \ldots$. It is shown that $|T_3 - T_{3n}| = \sup_n |\alpha_{n+1}|$, and since $\alpha_n \to 0$, $|T_3 - T_{3n}| \to 0$. As the limit of compact operators must also be compact, if $\alpha_n \to 0$, then T_3 is compact. Next, we demonstrate that $T = T_2 + iT_3$ is a normal operator. Suppose f is an eigenvector of T_3 with corresponding eigenvalue f. Then the corresponding real and imaginary components of T are given by T_2 and T_3, respectively. Since $(T_2 + iT_3)^*$ is normal, we have $|Tf| = |(T_2 + iT_3)f| = |(T_2 + iT_3)^* f| = |T^* f|$. Let V be the set of vectors for which $|Tf| = |T^* f|$, and it contains all eigenvectors of T_3. As $|Tf|$ being equal to $|T^f|$ implies $\langle Tf, Tf \rangle = \langle T^T f, f \rangle = \langle TT^f, f \rangle$, we find $\langle (T^*T - TT^*)f, f \rangle = 0$. By taking $T^*T - TT^*$ to be a positive operator, we have $\langle (T^*T - TT^*)f, f \rangle = 0$, implying $(T^*T - TT^*)f = 0$. Thus, $V = \mathrm{Ker}(T^*T - TT^*)$ is a subspace, and since the eigenvectors of T_3 span the whole space H, we conclude that T is a normal operator and therefore norm-attainable. □

Next Proposition focuses on the scenario where T is a normal operator, and its fractional power $T^{p>0}$ is compact. We delve into the norm attainment properties of these operators and analyze the relationship between compactness and norm-attainability.

Proposition 3. *In the context of Hilbert spaces H_1 and H_2, consider a normal operator T that maps from H_1 to H_2. If the composition $T^{p>0}$ (meaning T raised to the power of p for some p greater than zero) is a compact operator, then the operator T is norm-attainable (NA).*

Proof. Let $T : H_1 \to H_2$ be a bounded linear operator induced by a linear and bounded measurable $m \times n$ function ψ on a suitable measurable space. If T^p is also a multiplication operator induced by ψ^p, resulting in a diagonal operator whose diagonal entries converge to zero, then T can be expressed as the direct sum of itself with a diagonal operator whose diagonal elements tend to zero. This decomposition implies that T is a compact operator. Let M be the closed unit ball of a Hilbert space H, i.e., $M = x \in H : |x| \leq 1$. Since $T : H_1 \to H_2$ is compact, the image of the unit ball, $T(M) \subset H_2$, is also compact under the norm topology. Additionally, the norm function $|\cdot|H_2 : T(H_2) \to (0, \infty)$ is a continuous $m \times n$ function on $T(H_2)$. Therefore, the norm of T is attained as $\sup |x|H_1 \leq 1 |Tx|H_2 = \max |Tx|H_2 : |x|H_1 \leq 1$, and there exists $x_0 \in T(H_1)$ such that $|Tx_0|_{H_2} = |T|$. □

The following Lemma provides an essential result that establishes the relationship between self-adjoint contractions and norm-attainability. We explore the conditions under which a self-adjoint contraction can be classified as a norm-attainable operator and discuss the significance of this relationship.

Lemma 1. *Let $T : H \to H$ be a self-adjoint contraction operator on a Hilbert space H. Then, the operator T is said to be norm-attainable (NA) if and only if either $-|T|$ or $|T|$ belongs to the spectrum of T ($\sigma(T)$).*

Proof. Assume that either $-|T|$ or $|T|$ is an element of the spectrum of T ($\sigma(T)$). In this case, we can find a corresponding eigenvector $x \in H$. By normalizing x, we obtain a new eigenvector $x_0 = \frac{x}{|x|}$ through orthonormalization. It follows that $|T(\frac{x}{|x|})| = |Tx_0| \leq 1$, since T is a contraction. Consequently, we have $|T| = |Tx_0| \leq 1$. This establishes the first part of the proposition. Conversely, suppose there exists a normalized vector x_0 in the domain D such that $|T(x_0)| = |T|$. We can show that $(1 - T^2)x_0$ is strictly positive by observing that $\langle (1 - T^2)x_0, x_0\rangle = |x_0|^2 > |Tx_0|^2$. Since $(1 - T^2)x_0$ is strictly positive, we have $(1+T)(x_0 - Tx_0) > 0$, as $x_0 - Tx_0 > 0$. Let $x' = x_0 - Tx_0$ and $y' = \frac{x'}{|x'|}$. Then, we have $Ty' = -y'$, which implies that y' is a unit eigenvector corresponding to the eigenvalue $-|T| = -1$. □

Furthermore, the next proposition extends the previous results by considering a self-adjoint contraction $T \in NA(H)$, which is also p'-normal. We investigate the norm-attainability of $\alpha T^{p'}$, where α is a scalar and $p' \leq 1$. We analyze the conditions under which norm-attainability is preserved and establish connections with the farthest point of the numerical range of $T^{p'}$.

Proposition 4. *Consider a self-adjoint contraction operator T on a Hilbert space H, where H is not isomorphic to $L^1(0,1)$. Suppose T is also p'-normal for some p'. Then, $\alpha T^{p'}$ is also a norm-attaining operator for some $p' \leq 1$ and $0 < \alpha < 1$ if and only if the norm $\|T^{p'}\|$ or its negative $-\|T^{p'}\|$ corresponds to the farthest point in the numerical range $W(T^{p'})$ of the operator $T^{p'}$.*

Proof. Let M be a positive operator defined as either $M = (I + T)|T^{p'}|$ or $M = (I - T)|T^{p'}|$. Consider the inequality:

$$\langle T^{p'} x_o, x_o\rangle \leq \langle T^{p'} x_o, x_o\rangle + \langle Mx_o, x_o\rangle = |T^{p'}| \tag{1}$$

or

$$\langle T^{p'} x_o, x_o\rangle \geq \langle T^{p'} x_o, x_o\rangle - \langle Mx_o, x_o\rangle = -|T^{p'}| \tag{2}$$

Given the assumption that $T^{p'}$ is a norm-attainable operator on H, it implies that $|T^{p'}|$ or $-|T^{p'}|$ is an extreme point of the numerical range $W(T^{p'})$, since $\langle T^{p'} x_o, x_o\rangle \leq |T^{p'}|$ and $\langle T^{p'} x_o, x_o\rangle \geq -|T^{p'}|$ for all $x_o \in D$. Conversely, if $-|T^{p'}|$ or $|T^{p'}|$ is an extreme point of $W(T^{p'})$, then there exists $x_0' \in D$ such that $|T^{p'}| = \langle T^{p'} x_0', x_0'\rangle$ or $-|T^{p'}| = -\langle T^{p'} x_0', x_0'\rangle$. From inequalities 1 and 2, it is evident that $\langle Mx_0', x_0' \rangle = 0$. Since M is positive, it follows that $Mx_0' = 0$. Hence, $T^{p'}x_0' = |T^{p'}|x_0'$ or $T^{p'}x_0' = -|T^{p'}|x_0'$. Let $0 < \alpha < 1$ be given, then considering the definition of M, we have:

$$\langle \alpha T^{p'} x_o, x_o \rangle \leq \langle \alpha T^{p'} x_o, x_o \rangle + \langle M x_o, x_o \rangle \quad = \quad |T^{p'}|$$

or

$$\langle \alpha T^{p'} x_o, x_o \rangle \leq \langle \alpha T^{p'} x_o, x_o \rangle - \langle M x_o, x_o \rangle \quad = \quad -|T^{p'}|$$

□

The theorem in the sequel builds upon the previous results and explores the relationship between norm-attainable operators, compactness, self-adjointness, and functional calculus. We consider a compact and self-adjoint operator T with a positive measure $d\mu_x$ and investigate the norm attainment properties of the operator $w(T)$ within the algebra of rational functions $R(D)$. We establish bounds on the functional calculus of T and discuss the implications of these findings.

Theorem 1. *If T is a non-attainable, p-normal, self-adjoint, and compact operator on a Hilbert space H, with its domain D containing the spectrum of T, and there exists a positive measure $d\mu_x$, then the numerical range of T is non-attainable, $|f(T)| \leq |f|$ for all functions f in the resolvent set of D.*

Proof. Consider a non-attainable, p-normal, self-adjoint, and compact operator T on a Hilbert space H, with its domain D containing the spectrum of T, and let there exist a positive measure $d\mu_x$. We want to show that the condition $Re(1 - zT)^{-1} \geq 0$ for every $z \in \mathbb{C}$ with $|z| < 1$ is equivalent to $w(T) \leq 1$. First, we note that the resolvent of T can be expressed as a series expansion $(1 - zT)^{-1} = 1 + \sum_{n=1}^{\infty} z^n T^n$. By considering the inner product $\langle Re(1 - zT)^{-1}x, x_0 \rangle$, where x_0 is an arbitrary vector in H, we obtain:

$$\begin{aligned} \langle Re(1 - zT)^{-1}x, x_0 \rangle &= \langle 1 + \sum_{n=1}^{\infty} z^p T^p x_0, x_0 \rangle \\ &= |x_0| + \sum_{n=1}^{\infty} z^p \langle T^p x_0, x_0 \rangle. \end{aligned}$$

Since T is p-normal, $\langle T^n x_0, x_0 \rangle$ is norm-attainable for $n \equiv p$. This allows us to find a positive measure μ_{x_0} on $[0, 2\pi]$ such that $|x_0| + \sum_{n=1}^{\infty} z^p \langle T^p x_0, x_0 \rangle$ takes the form of the integral $\int \frac{1}{1 - ze^{i\theta}} d\mu_{x_0}\theta$ for $\theta \in [0, 2\pi]$ and $|z| < 1$. By expanding this integral, we obtain the equation:

$$\langle T^p x_0, x_0 \rangle = 2 \int e^{in\theta} d\mu_{x_0}(\theta) n = 1, 2, ... \tag{3}$$

Now, we apply equation 3 to a polynomial $f(z) = \sum_{k=1}^{n} \alpha_k z^k$, which generates $\langle f(T)^p x_0, x_0\rangle = 2\int f^n(e^{in\theta})d\mu_{x_0}(\theta), n = 1, 2, \ldots$ As $|f| \leq 1$, $|f(T)^p|$ is bounded, and the inner product can be rewritten as:

$$\begin{aligned} \langle (1 + \sum_{m=1}^{\infty} z^p f(T)^p) x_0, x_0\rangle &= |x_0|^2 + 2\sum_{m=1}^{\infty} z^p \int f(e^{i\theta})^n d\mu_{x_0}(\theta) \\ &= \int \frac{1}{1 - zf(e^{i\theta})} d\mu_{x_0}(\theta). \end{aligned}$$

Since the integrand is positive, we can conclude that $Re(1 - zT)^{-1} \geq 0$ is equivalent to $w(T) \leq 1$, for every $z \in \mathbb{C}$ with $|z| < 1$. □

In the sequel, the lemma addresses the continuity of mappings between L^2 spaces with weak topologies. We examine the continuity of operators between two L^2 spaces, considering the weak topologies of both spaces. This result highlights the importance of weak convergence in the context of norm-attainable operators.

Lemma 2. *Let H_1 and H_2 be two L^2 spaces equipped with their respective weak topologies H_1^w and H_2^w. The proposition states that if an operator T is continuous from H_1^w to H_2^w, then it is also continuous in the opposite direction, i.e., from H_2^w to H_1^w.*

Proof. Consider the dual space H_2^* of H_2, and let $l \in H_2^*$. The map $x \mapsto l[T(x)]$ is both continuous and linear, mapping elements from H_1 to the field $\mathbb{K}$ and, therefore, it is also continuous when applied to elements in H_1^w. Now, let T be a continuous and linear operator from H_1^w to H_2^w. For every $l \in H_2^*$, the composition $l \circ T$ is continuous from H_1^w to $\mathbb{K}$. Since T is continuous from H_1^w to H_2^w and $l \circ T$ is continuous for all $l \in H_2^*$, we can apply the closed graph theorem. Thus, the graph $G(T)$ of operator T is closed in $H_1 \times H_2$ with respect to the strong topology $(H_1 \times H_2)^s$. This closure of the graph ensures that T is continuous from H_1 to H_2 according to the closed graph theorem □

Finally in the Proposition below, we consider the conditions under which an operator $T \in B(H_1, H_2)$ belongs to the class of norm-attainable operators. We establish that T is a norm-attainable operator if and only if H_2 is a finite-dimensional L^2 space, and H_1 is a reflexive Banach space. This result provides a criterion for identifying norm-attainable operators in specific settings.

Proposition 5. *An operator T belongs to the class of norm-attainable operators, denoted as $NA(H)$, if and only if the following two conditions hold:*

(i). The target space H_2 is a finite-dimensional L^2 space.

(ii). The input space H_1 is a reflexive space.

Proof. Let H_2 be a finite-dimensional space, and let $T \in B(H_1, H_2)$ be a bounded linear operator mapping from H_1 to H_2. Since H_1 is reflexive, the unit ball U_x in H_1 becomes compact in the weak topology. If T is continuous with respect to the norms on H_1 and H_2 and the transformation from H_1 to H_2 is continuous, then T is also continuous when considering the weak topologies on H_1 and H_2. Since H_2 is finite-dimensional, the norm topology and the weak topology coincide on H_2. Therefore, $T(U_x)$ is compact in the weak topology of H_2. Since $T(U_x)$ is also mapped by T from compact and weakly closed sets in H_1 to compact and weakly closed sets in H_2, we conclude that T is norm-attainable. Conversely, if all $T \in B(H_1, H_2)$ are norm-attainable for finite-dimensional H_2, then according to James' theorem, H_1 must be reflexive. □

5 Conclusions

In this research paper, we have investigated the properties and characterizations of norm-attainable operators in the context of compactness and self-adjointness. Through a series of propositions, a lemma, a theorem, and a corollary, we have shed light on the nature of norm-attainable operators and provided insights into their behavior in various settings. Our results contribute to the understanding of norm-attainable operators and their implications in functional analysis.

References

[1] **Kinyanjui J. N.** , Characterization of Norm-Attainable Operators, International Journal of Mathematics Archives, Vol.9, No.7, (2018), 124-129.

[2] **Lindenstrauss J.**, On operators which attain their norm, *Israel J. Math.*, 1(1963), 139-148.

[3] **Okelo N. B.**, Characterization of Absolutely Norm Attaining Compact Hyponormal Operators, *proceedings of international mathemacal sciences*, 2(2020).

[4] **Okelo N. B.**, Various notions of norm-attainability in normed spaces, *math.FA* 1(2020), 1-13.

[5] **Okelo N. B.**, **Agure J .O. and Ambogo D. O.**, Norms of elementary operators and characterization of norm - attainable operators, *Int. Journal of Math. Analysis*, 24(2010), 687-693.

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[6] **Okelo N. B.**, **Aminer J. O.T**, Norm inequalities of norm-attainable operators and their orthogonal extensions, *school of math.Acturial sci.JOUST Bondo Kenya*.

[7] **Satish K.P and Vern I.P**, A spectral characterization of AN operators, *J.Aust.Math.soc.*, 102(2017), 369-391.

[8] **Shkran S.**, Norm-attaining operators and Pseudo spectrum, *Queens university Belfast, department of pure mathematics, University Road BTT INN BELFAST UK.*