Original Research Article

Fuzzy Fixed Point Theorems in Normal Cone Metric Space

.

ABSTRACT

In this paper, we proved a few fuzzy fixed point theorems in whole regular cone metric spaces, which can be the generalization of a few current consequences within side the literature.

Keywords: Normal cone, cone metric space, fixed point, fuzzy

2020 AMS subject classifications: 54H25, 46S40, 47H10

1. INTRODUCTION

Many researchers make the research under the fixed point theorems [1], [4], [6]. There exist some of generalizations of metric spaces, and one in all them is the cone metric spaces [2]. The notation of cone metric space is initiated via way of means of Huang and Zhang [5] and additionally they mentioned a few homes of the convergence of sequences and proved the fuzzy fixed point theorems of a contraction mappings cone metric spaces [3]. Many authors have studied the life and forte of strict fuzzy constant factors for single valued mappings and multi valued mappings in metric spaces [7], [8]. In this paper speak life and precise fixed point factor in entire ordinary cone metric spaces, which might be the generalization of a few current contraction principle.

Definition 1.1:

A subset S of E is called a cone if and only if :

1. S is closed, nonempty and $S \neq 0$
2. $ax + by \in S$ for all $x, y \in S$ and nonnegative real numbers a, b
3. $S \cap S^{-} = \{0\}$.

Given a cone $S \subset E$, we define a partial ordering $\leq$ with respect to S by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ that $x \leq y$ but $x \neq y$, while x, y will stand for $y - x \in int\, S$, where $int\, S$ denotes the interior of S. The cone P is called normal if there is a number $L > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq L\|y\|$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant. The cone L is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \ldots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0\ as\ n \to 0$.

Equivalently the cone S is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. Suppose E is a Banach space, S is a cone in E with $int\, S \neq 0\ and \leq$ is partial ordering with respect to S.

Example 1.1:

Let $L > 1$ be given. Consider the real vector space with

$$E = \left\{ax + b : a, b \in \mathbb{R}; x \in \left[1 - \frac{1}{k}, 1\right]\right\}$$

With supremum norm and the cone $S = \{ax + b : a \geq 0, b \geq 0\}$ in E. the cone S is ordinary and so normal.

Definition 1.2:

Suppose that E is real Banach space, then S is a cone in E with $int\, S \neq \emptyset$, and $\leq$ is partial ordering with respect to S. Let $\mathbb{X}$ be a nonempty set, a function $d: \mathbb{X} \times \mathbb{X} \to E$ is called a fuzzy cone metric on $\mathbb{X}$ if it satisfies the following conditions with

1. $d(x, y) \geq 0,$ and $d(x, y) = 0$ if and only if $x = y\ \forall x, y \in X,$
2. $d(x, y) = d(y, x), \forall\, x, y \in X,$
3. $d(x, y) \leq d(x, z) + d(z, y), \forall\, x, y \in X,$

Then $(\mathbb{X}, d)$ is called a cone metric space $(\mathbb{C}_F\mathbb{M})$.

Definition 1.3:

A fuzzy cone metric space is a 3-tuple $(\mathbb{X}, \mathbb{C}_F\mathbb{M}, *)$ such that S is a cone of E, $\mathbb{X}$ is nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $\mathbb{X} \times \mathbb{X} \times int(S)$ satisfying the following conditions, for all $x, y, z \in X$ and $t, s \in int(P)$ (that is $t \gg \Theta, s \gg \Theta$) .

1. $\mathbb{C}_F\mathbb{M}(x, y, t) > 0,$
2. $\mathbb{C}_F\mathbb{M}(x, y, t) = 1$ if and only if $x = y,$
3. $\mathbb{C}_F\mathbb{M}(x, y, t) = \mathbb{C}_F\mathbb{M}(y, x, t),$
4. $\mathbb{C}_F\mathbb{M}(x, y, t) * \mathbb{C}_F\mathbb{M}(y, z, s) \leq \mathbb{C}_F\mathbb{M}(x, z, t + s),$
5. $\mathbb{C}_F\mathbb{M}(x, y, .): int(P) \to [0,1]$ is continuous.

If $(\mathbb{X}, \mathbb{C}_F\mathbb{M}, *)$ is a fuzzy cone metric space, we will say that M is a fuzzy cone metric on $\mathbb{X}$.

Definition 1.4:

Let$(\mathbb{X}, \mathbb{C}_F\mathbb{M}, *)$ be a fuzzy cone metric space, $x \in \mathbb{X}$ and $\{x_n\}$ be a sequence in $\mathbb{X}$. Then

$\{x_{\mathfrak{n}}\}$ is said to converge to x if for any $t \gg \Theta$ and any $\mathfrak{r} \in (0, 1)$ there exists a natural number $\mathfrak{n}_0$ such that $\mathcal{M}\ (x_{\mathfrak{n}};\ x;\ t)\ >\ 1\ -\ r$ for all $\mathfrak{n}\ \geq\ \mathfrak{n}_0$. We denote this by

$\lim_{\mathfrak{n} \longrightarrow \infty} x_{\mathfrak{n}}\ =\ x$ or $x_{\mathfrak{n}} \longrightarrow\ x$ as $\mathfrak{n}\ \to\ \infty$.

Let $(\mathbb{X}, \mathbb{C}_F\mathbb{M}, *)$ be a fuzzy cone metric space, $x \in \mathbb{X}$ and $\{x_{\mathfrak{n}}\}$ be a sequence in $\mathbb{X}$. $\{x_{\mathfrak{n}}\}$

converges to x if and only if $\mathcal{M}\ (x_{\mathfrak{n}};\ x;\ t) \longrightarrow\ 1\ as\ \mathfrak{n} \longrightarrow\ \infty$, for each $t\ \gg \Theta$.

Let $(\mathbb{X}, \mathbb{C}_F\mathbb{M}, *)$ be a fuzzy cone metric space and $\{x_{\mathfrak{n}}\}$ be a sequence in $\mathbb{X}$.

Then $\{x_{\mathfrak{n}}\}$ is said to be a Cauchy sequence if for any $0\ <\ \varepsilon <\ 1$ and any $t\ \gg \Theta$.

There exists a natural number $\mathfrak{n}_0$ such that $\mathcal{M}\ (x_{\mathfrak{n}};\ x_{\mathfrak{m}};\ t) >\ 1 - \varepsilon$ for all $\mathfrak{n}, \mathfrak{m} \geq \mathfrak{n}_0$.

2. MAIN RESULT

Theorem 2.1:

Let $(\mathbb{X}, \mathbb{C}_F\mathbb{M}, *)$ be a complete fuzzy cone metric space and S be a normal cone with normal constant L. suppose the mapping $T: \mathbb{X} \times \mathbb{X} \times (0, \infty) \to [0, \infty)$ satisfies the following conditions:

$$\mathbb{C}_F\mathbb{M}(T_x, T_y, t) \le \left(\frac{\mathbb{C}_F\mathbb{M}(x,T_x,t)+\mathbb{C}_F\mathbb{M}(y,T_y,t)}{\mathbb{C}_F\mathbb{M}(x,T_x,t)+\mathbb{C}_F\mathbb{M}(y,T_y,t)+l}\right)\mathbb{C}_F\mathbb{M}(x, y, t) \qquad (1)$$

For all $x, y \in \mathbb{X}, where\ l \ge 1\ \&\ t \in \mathbb{X}. then$

i. T has fuzzy unique fixed point in $\mathbb{X}$.

ii. $T^n x^{'}$ converges to a fuzzy fixed point, for all $x^{'} \in \mathbb{X}$.

Proof :

i. Let $x_0 \in \mathbb{X}$ be arbitrary and choose a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$.

$$\mathbb{C}_F\mathbb{M}(x_{n+1,}x_n, t) = \mathbb{C}_F\mathbb{M}(Tx_n, Tx_{n-1}, t)$$
$$\le \left(\frac{\mathbb{C}_F\mathbb{M}(x_n, Tx_n, t) + \mathbb{C}_F\mathbb{M}(x_{n-1}, Tx_{n-1}, t)}{\mathbb{C}_F\mathbb{M}(x_n, Tx_n, t) + \mathbb{C}_F\mathbb{M}(x_{n-1}, Tx_{n-1}, t) + l}\right)\mathbb{C}_F\mathbb{M}(x_n, x_{n-1,}t)$$
$$\le \left(\frac{\mathbb{C}_F\mathbb{M}(x_{n,}x_{n+1}, t) + \mathbb{C}_F\mathbb{M}(x_{n-1,}x_n, t)}{\mathbb{C}_F\mathbb{M}(x_{n,}x_{n+1}, t) + \mathbb{C}_F\mathbb{M}(x_{n-1}, x_n, t) + l}\right)\mathbb{C}_F\mathbb{M}(x_{n,}x_{n-1}, t)$$

Take

$$\lambda_n = \frac{\mathbb{C}_F\mathbb{M}(x_{n,}x_{n+1,}t)+\mathbb{C}_F\mathbb{M}(x_{n-1,}x_n,t)}{\mathbb{C}_F\mathbb{M}(x_{n,}x_{n+1,}t)+\mathbb{C}_F\mathbb{M}d(x_{n-1,}x_n,t)+l}\ ,$$

We have

$$\begin{aligned}\mathbb{C}_F\mathbb{M}(x_{n+1}, x_n, t) &\le \lambda_n \mathbb{C}_F\mathbb{M}(x_n, x_{n-1}, t)\\ &\le (\lambda_n\lambda_{n-1})\ \mathbb{C}_F\mathbb{M}(x_{n-1}, x_{n-2}, t)\\ &.\\ &.\\ &.\\ &\le (\lambda_n\lambda_{n-1} \dots \lambda_1)\ \mathbb{C}_F\mathbb{M}(x_1, x_0, t).\end{aligned}$$

Observe that (λ_n) is non increasing, with positive terms. So,$\lambda_1 \dots \lambda_n \le {\lambda_1}^n$ and ${\lambda_1}^n \to 0$.
It follows that

$$\lim_{n\to\infty}(\lambda_1\lambda_2 \dots \lambda_n) = 0.$$

Thus, it is verified that

$$\lim_{n\to\infty} \mathbb{C}_F\mathbb{M}(x_{n+1,}x_n, t) = 0$$

Now for all $m, n \in \mathbb{N}\ and\ m > n\ we\ have$

$$\mathbb{C}_F\mathbb{M}(x_{m,}x_n, t) \le \mathbb{C}_F\mathbb{M}(x_n, x_{n+1}, t) + \mathbb{C}_F\mathbb{M}(x_{n+1}, x_{n+2}, t) + \cdots \mathbb{C}_F\mathbb{M}(x_{m-1}, x_m, t)$$

$$\le [(\lambda_n\lambda_{n-1} \dots \lambda_1) + (\lambda_{n+1}\lambda_n \dots \lambda_1) + \cdots + (\lambda_{m-1}\lambda_{m-2} \dots \lambda_1)]\ \mathbb{C}_F\mathbb{M}(x_1, x_0, t)$$
$$= \textstyle\sum_{k=n}^{m-1}(\lambda_k\lambda_{k-1} \dots \lambda_1)\ \mathbb{C}_F\mathbb{M}(x_1, x_0, t)$$

$$\|\mathbb{C}_F\mathbb{M}(x_m, x_n, t)\| \le L\left\|\sum_{k=n}^{m-1}(\lambda_k\lambda_{k-1} \dots \lambda_1)\ \mathbb{C}_F\mathbb{M}(x_1, x_0, t)\right\|$$

$$\|\mathbb{C}_F\mathbb{M}(x_m, x_n, t)\| \le L\sum_{k=n}^{m-1}(\lambda_k\lambda_{k-1} \dots \lambda_1)\ \|\ \mathbb{C}_F\mathbb{M}(x_1, x_0, t)\|$$

$\|\mathbb{C}_F\mathbb{M}(x_m, x_n, t)\| \quad \le L\sum_{k=n}^{m-1} a_k\ \|\mathbb{C}_F\mathbb{M}(x_1, x_0, t)\|$,

Where $a_{k=(\lambda_k\lambda_{k-1}\dots\lambda_1)}$ and L is normal constant of S.

Now $\lim_{k\to\infty}\frac{a_{k+1}}{a_k} < 1\ and\ \sum_{k=1}^{\infty} a_k$is finite,

and $\sum_{k=n}^{m-1}(\lambda_k\lambda_{k-1} \dots \lambda_1) \to 0, as\ m, n \to \infty$.

Hence $\{a_k\}$ is convergent by D' Alembert's ratio test, therefore $\{x_n\}$ is a cauchy sequence.
There is $x^{'} \in \mathbb{X}$ such that $x_n \to x^{'}\ as\ n \to \infty$.

$$\mathbb{C}_F\mathbb{M}(Tx^{'}, x^{'}, t) \le \mathbb{C}_F\mathbb{M}(Tx^{'}, Tx_n, t) + \mathbb{C}_F\mathbb{M}(Tx_n, x^{'}, t)$$
$$\le \left(\frac{\mathbb{C}_F\mathbb{M}(x^{'},Tx^{'},t)+\mathbb{C}_F\mathbb{M}(x_n,Tx_n,t)}{\mathbb{C}_F\mathbb{M}(x^{'},Tx^{'},t)+\mathbb{C}_F\mathbb{M}(x_n,Tx_n,t)+l}\right)\mathbb{C}_F\mathbb{M}(x_n, x^{'}, t) + \mathbb{C}_F\mathbb{M}(Tx_n, x^{'}, t)$$
$$\le \left(\frac{\mathbb{C}_F\mathbb{M}(x^{'},Tx^{'},t)+\mathbb{C}_F\mathbb{M}(x_n,Tx_{n+1},t)}{\mathbb{C}_F\mathbb{M}(x^{'},Tx^{'},t)+\mathbb{C}_F\mathbb{M}(x_n,Tx_{n+1},t)+l}\right)\mathbb{C}_F\mathbb{M}(x_n, x^{'}, t) + \mathbb{C}_F\mathbb{M}(Tx_{n+1}, x^{'}, t)$$
$$\mathbb{C}_F\mathbb{M}(Tx^{'}, x^{'}, t) \le 0\ \ as\ n \to \infty$$

Therefore $\left\|\mathbb{C}_F\mathbb{M}\left(Tx', x', t\right)\right\| = 0$.
Thus, $Tx' = x'$.

Uniqueness

Suppose x' and y' are two fixed points of T.

$$\begin{aligned} \mathbb{C}_F\mathbb{M}\left(x', y', t\right) &= \mathbb{C}_F\mathbb{M}\left(Tx', Ty', t\right) \\ &\leq \left(\frac{\mathbb{C}_F\mathbb{M}\left(x', Tx', t\right) + \mathbb{C}_F\mathbb{M}\left(y', Ty', t\right)}{\mathbb{C}_F\mathbb{M}\left(x', Tx', t\right) + \mathbb{C}_F\mathbb{M}\left(y^1, Ty', t\right) + l}\right)\mathbb{C}_F\mathbb{M}\left(x', y', t\right) \\ &\leq 0 \end{aligned}$$

Therefore $\left\|\mathbb{C}_F\mathbb{M}\left(x', y', t\right)\right\| = 0$. Thus $x' = y'$.
Hence x' is an unique fuzzy fixed point of T.

ii. Now

$$\begin{aligned} \mathbb{C}_F\mathbb{M}\left(\boldsymbol{T^n x', x', t}\right) &= \mathbb{C}_F\mathbb{M}\left(\boldsymbol{T^{n-1} T x', x', t}\right) = \mathbb{C}_F\mathbb{M}\left(\boldsymbol{T^{n-1} x', x', t}\right) = \mathbb{C}_F\mathbb{M}\left(\boldsymbol{T^{n-2}\left(Tx'\right), x', t}\right) \ldots \\ &= \mathbb{C}_F\mathbb{M}\left(\boldsymbol{Tx', Tx', t}\right) = \boldsymbol{0} \end{aligned}$$

Hence $\boldsymbol{T^n x'}$ converges to a fuzzy fixed point, for all $\boldsymbol{x'} \in \mathbb{X}$.

Corollary 2.1:

Let $(\mathbb{X}, \mathbb{C}_F\mathbb{M}, *)$ be a complete cone fuzzy metric space and $\boldsymbol{S}$ be a normal cone with normal constant L. suppose the mapping $\boldsymbol{T}: \mathbb{X} \to \mathbb{X}$ satisfies the following conditions:

$$\mathbb{C}_F\mathbb{M}\left(\boldsymbol{T_x, T_y, t}\right) \leq \left(\frac{\mathbb{C}_F\mathbb{M}(\boldsymbol{x, T_x, t}) + \mathbb{C}_F\mathbb{M}\left(\boldsymbol{y, T_y, t}\right)}{\mathbb{C}_F\mathbb{M}(\boldsymbol{x, T_x, t}) + \mathbb{C}_F\mathbb{M}\left(\boldsymbol{y, T_y, t}\right) + \boldsymbol{1}}\right)\mathbb{C}_F\mathbb{M}(\boldsymbol{x, y, t}) \qquad (2)$$

For all $\boldsymbol{x, y} \in \mathbb{X}$. Then

i. T has fuzzy unique fixed point in $\mathbb{X}$.
ii. $\boldsymbol{T^n x'}$ Converges to a fuzzy fixed point, for all $\boldsymbol{x'} \in \mathbb{X}$.

Proof :

The proof of the corollary immediate by
Taking $\boldsymbol{l = 1}$ in the above theorem.

Theorem 2.2:

Let $(\mathbb{X}, \mathbb{C}_F\mathbb{M}, *)$ be a complete fuzzy metric space and let T be a mapping from $\mathbb{X}$ into itself. Suppose that T satisfies the following condition:

$$\mathbb{C}_F\mathbb{M}\left(\boldsymbol{T_x, T_y, t}\right) \leq \left(\frac{\mathbb{C}_F\mathbb{M}\left(\boldsymbol{y, T_y, t}\right)}{\mathbb{C}_F\mathbb{M}(\boldsymbol{x, T_x, t}) + \mathbb{C}_F\mathbb{M}\left(\boldsymbol{y, T_y, t}\right) + \boldsymbol{l}}\right)\mathbb{C}_F\mathbb{M}(\boldsymbol{x, y, t}) \qquad (3)$$

For all $\boldsymbol{x, y} \in \mathbb{X}, \boldsymbol{where\ l \geq 1}\ \&\ \boldsymbol{t} \in \mathbb{X}$. Then

i. T has unique fuzzy fixed point in $\mathbb{X}$.
ii. $\boldsymbol{T^n x'}$ Converges to a fuzzy fixed point, for all $\boldsymbol{x'} \in \mathbb{X}$.

Proof :

i. Let $\boldsymbol{x_0} \in \mathbb{X}$ be arbitrary and choose a sequence $\{\boldsymbol{x_n}\}$ such that $\boldsymbol{x_{n+1} = T\, x_n}$

We have

$$\begin{aligned} \mathbb{C}_F\mathbb{M}(\boldsymbol{x_{n+1}, x_n, t}) &= \mathbb{C}_F\mathbb{M}(\boldsymbol{T\, x_n, T\, x_{n-1}, t}) \\ &\leq \left(\frac{\mathbb{C}_F\mathbb{M}\left(\boldsymbol{x_{n-1}, Tx_{n-1}, t}\right)}{\mathbb{C}_F\mathbb{M}\left(\boldsymbol{x_n, x_{n+1}, t}\right) + \mathbb{C}_F\mathbb{M}(\boldsymbol{x_{n-1}, x_n, t}) + \boldsymbol{l}}\right)\mathbb{C}_F\mathbb{M}\left(\boldsymbol{x_n, x_{n-1}, t}\right) \end{aligned}$$

$$\leq \left(\frac{\mathbb{C}_F\mathbb{M}\left(\boldsymbol{x_{n-1}, x_n, t}\right)}{\mathbb{C}_F\mathbb{M}\left(\boldsymbol{x_n, x_{n+1}, t}\right) + \mathbb{C}_F\mathbb{M}(\boldsymbol{x_{n-1}, x_n, t}) + \boldsymbol{l}}\right)\mathbb{C}_F\mathbb{M}\left(\boldsymbol{x_n, x_{n-1}, t}\right)$$

$$\leq \left(\frac{\mathbb{C}_F\mathbb{M}\left(\boldsymbol{x_{n-1}, x_n, t}\right)}{\mathbb{C}_F\mathbb{M}\left(\boldsymbol{x_n, x_{n+1}, t}\right) + \mathbb{C}_F\mathbb{M}(\boldsymbol{x_{n-1}, x_n, t}) + \boldsymbol{l}}\right)\mathbb{C}_F\mathbb{M}\left(\boldsymbol{x_n, x_{n-1}, t}\right)$$

Take

$$\boldsymbol{\lambda_n} = \frac{\mathbb{C}_F\mathbb{M}(\boldsymbol{x_n, x_{n+1}, t}) + \mathbb{C}_F\mathbb{M}(\boldsymbol{x_{n-1}, x_n, t})}{\mathbb{C}_F\mathbb{M}(\boldsymbol{x_n, x_{n+1}, t}) + \mathbb{C}_F\mathbb{M}(\boldsymbol{x_{n-1}, x_n, t}) + \boldsymbol{l}}$$

We have

$$\mathbb{C}_F\mathbb{M}(x_{n+1}, x_n, t) \leq \lambda_n \mathbb{C}_F\mathbb{M}(x_n, x_{n-1}, t)$$

$$\leq (\lambda_n\lambda_{n-1})\mathbb{C}_F\mathbb{M}(x_{n-1}, x_{n-2}, t)$$

.

.

.

$$\leq (\lambda_n\lambda_{n-1} \dots \lambda_1)\, \mathbb{C}_F\mathbb{M}(x_1, x_0, t).$$

Observe that $\{\lambda_n\}$ is non- increasing, with positive terms.

So, $(\lambda_1 \dots \lambda_n) \leq {\lambda_1}^n \rightarrow 0$. It follows that

$$\lim_{n\to\infty}(\lambda_1\lambda_2 \dots \lambda_n) = 0.$$

Thus, it is verified that

$$\lim_{n\to\infty} \mathbb{C}_F\mathbb{M}(x_{n+1,} x_n, t) = 0.$$

Now for all $m, n \epsilon \mathbb{N}$ we have

$$\mathbb{C}_F\mathbb{M}(x_{m,} x_n, t) \leq \mathbb{C}_F\mathbb{M}(x_n, x_{n+1}, t) + \mathbb{C}_F\mathbb{M}(x_{n+1}, x_{n+2}, t) + \cdots \mathbb{C}_F\mathbb{M}(x_{m-1}, x_m, t)$$

$$\leq [(\lambda_n\lambda_{n-1} \dots \lambda_1) + (\lambda_{n+1}\lambda_n \dots \lambda_1) + \cdots + (\lambda_{m-1}\lambda_{m-2} \dots \lambda_1)]\, \mathbb{C}_F\mathbb{M}(x_1, x_0, t)$$

$$= \sum_{k=n}^{m-1}(\lambda_k\lambda_{k-1} \dots \lambda_1)\, \mathbb{C}_F\mathbb{M}(x_1, x_0, t)$$

$$\|\mathbb{C}_F\mathbb{M}(x_m, x_n, t)\| \leq L \left\| \sum_{k=n}^{m-1} (\lambda_k\lambda_{k-1} \dots \lambda_1)\, \mathbb{C}_F\mathbb{M}(x_1, x_0, t) \right\|$$

$$\|\mathbb{C}_F\mathbb{M}(x_m, x_n, t)\| \leq L \sum_{k=n}^{m-1} (\lambda_k\lambda_{k-1} \dots \lambda_1)\, \|\mathbb{C}_F\mathbb{M}(x_1, x_0, t)\|$$

$\|\mathbb{C}_F\mathbb{M}(x_m, x_n, t)\| \quad \leq L \sum_{k=n}^{m-1} a_k \, \|\mathbb{C}_F\mathbb{M}(x_1, x_0, t)\|,$

Where $a_{k=(\lambda_k\lambda_{k-1}\dots\lambda_1)}$ and L is normal constant of S.

Now $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1 \text{ and } \sum_{k=1}^{\infty} a_k$is finite, and

$\sum_{k=n}^{m-1} (\lambda_k\lambda_{k-1} \dots \lambda_1) \rightarrow 0, as\ m, n \rightarrow \infty.$

Hence $\{a_k\}$ is convergent by D' Alembert's ratio test, therefore $\{x_n\}$ is a cauchy sequence.

There is $x' \in \mathbb{X}$ such that $x_n \rightarrow x'$

$\mathbb{C}_F\mathbb{M}(Tx', x', t) \leq \mathbb{C}_F\mathbb{M}(Tx', Tx_n, t) + \mathbb{C}_F\mathbb{M}(Tx_n, x', t)$

$\leq \left(\frac{\mathbb{C}_F\mathbb{M}(x', Tx', t) + \mathbb{C}_F\mathbb{M}(x_n, Tx_n, t)}{\mathbb{C}_F\mathbb{M}(x', Tx', t) + \mathbb{C}_F\mathbb{M}(x_n, Tx_n, t) + l}\right) \mathbb{C}_F\mathbb{M}(x_n, x', t) + \ \mathbb{C}_F\mathbb{M}(Tx_n, x', t)$

$$\leq \left(\frac{\mathbb{C}_F\mathbb{M}(x', Tx', t) + \mathbb{C}_F\mathbb{M}(x_n, Tx_{n+1}, t)}{\mathbb{C}_F\mathbb{M}(x', Tx', t) + \mathbb{C}_F\mathbb{M}(x_n, Tx_{n+1}, t) + l}\right) \mathbb{C}_F\mathbb{M}(x_n, x', t) + \ \mathbb{C}_F\mathbb{M}(Tx_{n+1}, x', t)$$

$\mathbb{C}_F\mathbb{M}(Tx', x', t) \leq 0$ as $n \rightarrow \infty$

Therefore $\|\mathbb{C}_F\mathbb{M}(Tx', x', t)\| = 0$. Thus, $Tx' = x'$.

Uniqueness

Suppose x' and y' are two fuzzy fixed points of T.

$$\mathbb{C}_F\mathbb{M}(x', y', t) = \mathbb{C}_F\mathbb{M}(Tx', Ty', t)$$

$$\leq \left(\frac{\mathbb{C}_F\mathbb{M}(x', Tx', t) + \mathbb{C}_F\mathbb{M}(y', Ty', t)}{\mathbb{C}_F\mathbb{M}(x', Tx', t) + \mathbb{C}_F\mathbb{M}(y', Ty', t) + l}\right) \mathbb{C}_F\mathbb{M}(x', y', t)$$

$$\leq 0$$

Therefore $\|\mathbb{C}_F\mathbb{M}(x', y', t)\| = 0.$ Thus $x' = y'$.

Hence x' is an unique fuzzy fixed point of T.

ii. Now

$$\mathbb{C}_F\mathbb{M}(T^n x', x', t) = \mathbb{C}_F\mathbb{M}(T^{n-1}T', x', t) = \mathbb{C}_F\mathbb{M}(T^{n-1}x', x', t) = \mathbb{C}_F\mathbb{M}(T^{n-2}(Tx'), x', t) \dots$$

$$= \mathbb{C}_F\mathbb{M}(Tx', Tx', t) = 0$$

Hence $T^n x'$ converges to a fuzzy fixed point, for all $x' \in \mathbb{X}$.

Corollary 2.2:

Let $(\mathbb{X}, \mathbb{C}_F\mathbb{M}, *)$ be a complete fuzzy metric space and let T be a mapping from $\mathbb{X}$ into itself. Suppose that T satisfies the following condition:

$$\mathbb{C}_F\mathbb{M}(T_x, T_y, t) \le \left(\frac{\mathbb{C}_F\mathbb{M}(y, T_y, t)}{\mathbb{C}_F\mathbb{M}(x, T_x, t) + \mathbb{C}_F\mathbb{M}(y, T_y, t) + l}\right) \mathbb{C}_F\mathbb{M}(x, y, t) \tag{4}$$

For all $x, y \in \mathbb{X}$,where $l \ge 1$ & $t \in \mathbb{X}$. Then

i. T has Specific fuzzy fixed point in $\mathbb{X}$.
ii. $T^n x'$ converges to a fuzzy fixed point, for all $x' \in \mathbb{X}$.

Proof :

The proof of the corollary immediate by
Taking $l = 1$ in the above theorem.

Theorem 2.3:

Let $(\mathbb{X}, \mathbb{C}_F\mathbb{M}, *)$ be a complete cone metric space and P be a normal cone with ordinary constant L. suppose the mapping $T: \mathbb{X} \to \mathbb{X}$ satisfies the following conditions:

$$\mathbb{C}_F\mathbb{M}(T_x, T_y, t) \le \left(\frac{\mathbb{C}_F\mathbb{M}(x, T_y, t) + \mathbb{C}_F\mathbb{M}(y, T_x, t)}{\mathbb{C}_F\mathbb{M}(x, T_x, t) + \mathbb{C}_F\mathbb{M}(y, T_y, t) + l}\right) \left(\mathbb{C}_F\mathbb{M}(x, T_y, t) + \mathbb{C}_F\mathbb{M}(y, T_x, t)\right) \tag{5}$$

For all $x, y \in \mathbb{X}, \text{where } l \ge 1$ & $t \in \mathbb{X}$. Then

i. T has unique fuzzy fixed point in $\mathbb{X}$.
ii. $T^n x'$ converges to a fuzzy fixed point, for all $x' \in \mathbb{X}$.

Proof :

Let $x_0 \in \mathbb{X}$ be arbitrary and choose a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$.

$$\mathbb{C}_F\mathbb{M}(x_n, x_{n+1}, t) = \mathbb{C}_F\mathbb{M}(Tx_n, T\,x_{n-1}, t)$$

$$\le \left(\frac{\mathbb{C}_F\mathbb{M}(x_n, Tx_{n-1}, t) + \mathbb{C}_F\mathbb{M}(x_{n-1}, Tx_n, t)}{\mathbb{C}_F\mathbb{M}(x_n, Tx_n, t) + \mathbb{C}_F\mathbb{M}(x_{n-1}, Tx_{n-1}, t) + l}\right) \left(\mathbb{C}_F\mathbb{M}(x_n, Tx_n, t) + \mathbb{C}_F\mathbb{M}(x_{n-1}, Tx_{n-1}, t)\right)$$

$$\le \left(\frac{\mathbb{C}_F\mathbb{M}(x_n, x_n, t) + \mathbb{C}_F\mathbb{M}(x_{n-1}, x_{n+1}, t)}{\mathbb{C}_F\mathbb{M}(x_n, x_{n+1}, t) + \mathbb{C}_F\mathbb{M}(x_{n-1}, x_n, t) + l}\right) \left(\mathbb{C}_F\mathbb{M}(x_n, x_{n+1}, t) + \mathbb{C}_F\mathbb{M}(x_n, x_{n-1}, t)\right)$$

$$\le \left(\frac{\mathbb{C}_F\mathbb{M}(x_{n-1}, x_{n+1}, t)}{\mathbb{C}_F\mathbb{M}(x_n, x_{n+1}, t) + \mathbb{C}_F\mathbb{M}(x_{n-1}, x_n, t) + l}\right) \left(\mathbb{C}_F\mathbb{M}(x_n, x_{n+1}, t) + \mathbb{C}_F\mathbb{M}(x_n, x_{n-1}, t)\right)$$

$$\le \left(\frac{\mathbb{C}_F\mathbb{M}(x_{n-1}, x_n, t) + \mathbb{C}_F\mathbb{M}(x_n, x_{n+1}, t)}{\mathbb{C}_F\mathbb{M}(x_n, x_{n+1}, t) + \mathbb{C}_F\mathbb{M}(x_{n-1}, x_n, t) + l}\right) \left(\mathbb{C}_F\mathbb{M}(x_n, x_{n+1}, t) + \mathbb{C}_F\mathbb{M}(x_n, x_{n-1}, t)\right)$$

Take

$$\lambda_n = \frac{\mathbb{C}_F\mathbb{M}(x_{n-1}, x_n, t) + \mathbb{C}_F\mathbb{M}(x_n, x_{n+1}, t)}{\mathbb{C}_F\mathbb{M}(x_n, x_{n+1}, t) + \mathbb{C}_F\mathbb{M}(x_{n-1}, x_n, t) + l}$$

We have

$$\mathbb{C}_F\mathbb{M}(x_{n+1}, x_n, t) \le \lambda_n \left(\mathbb{C}_F\mathbb{M}(x_n, x_{n+1}, t) + \mathbb{C}_F\mathbb{M}(x_n, x_{n-1}, t)\right)$$

$$(1 - \lambda_n)\, \mathbb{C}_F\mathbb{M}(x_{n+1}, x_n, t) \le \lambda_n\, \mathbb{C}_F\mathbb{M}(x_n, x_{n-1}, t)$$

$$\mathbb{C}_F\mathbb{M}(x_{n+1}, x_n, t) \le \frac{\lambda_n}{(1 - \lambda_n)} \mathbb{C}_F\mathbb{M}(x_n, x_{n-1}, t)$$

$$\le \frac{\lambda_n \lambda_{n-1}}{(1 - \lambda_n)(1 - \lambda_{n-1})}\, \mathbb{C}_F\mathbb{M}(x_{n-1}, x_{n-2}, t)$$

$$\vdots$$

$$\le \frac{\lambda_n \lambda_{n-1} \dots \lambda_1}{(1 - \lambda_n)(1 - \lambda_{n-1}) \dots (1 - \lambda_1)} \mathbb{C}_F\mathbb{M}(x_1, x_0, t).$$

$$\le \gamma_n\, \mathbb{C}_F\mathbb{M}(x_1, x_0, t)$$

Where

$$\gamma_n = \frac{\lambda_n\lambda_{n-1}\ldots\lambda_1}{(1-\lambda_n)(1-\lambda_{n-1})\ldots(1-\lambda_1)}$$

Observe that $\{\lambda_n\}$ is non increasing, with positive terms. So, $(\lambda_1 \ldots \lambda_n) \le {\lambda_1}^n \to 0.$
It follows that

$$\lim_{n\to\infty} (\lambda_1\lambda_2 \ldots \lambda_n) = 0.$$

Therefore

$$\lim_{n\to\infty} \gamma_n = 0$$

Thus, it is verified that

$$\lim_{n\to\infty} \mathbb{C}_F\mathbb{M}(x_{n+1,} x_n, t) = 0.$$

Now for all $m, n \,\epsilon \mathbb{N}$ we have

$$\mathbb{C}_F\mathbb{M}(x_{m,} x_n, t) \le \mathbb{C}_F\mathbb{M}(x_n, x_{n+1}, t) + \mathbb{C}_F\mathbb{M}(x_{n+1}, x_{n+2}, t) + \cdots \mathbb{C}_F\mathbb{M}(x_{m-1}, x_m, t)$$

$$\le [(\gamma_n + \gamma_{n+1} + \cdots + \gamma_{m-1})]\, \mathbb{C}_F\mathbb{M}(x_1, x_0, t)$$

$$\le \sum_{k=n}^{m-1} \gamma_k\, \mathbb{C}_F\mathbb{M}(x_1, x_0, t)$$

$$\|\mathbb{C}_F\mathbb{M}(x_m, x_n, t)\| \le L \left\| \sum_{k=n}^{m-1} \gamma_k\, \mathbb{C}_F\mathbb{M}(x_1, x_0, t) \right\|$$

$$\|\mathbb{C}_F\mathbb{M}(x_m, x_n, t)\| \le L \sum_{k=n}^{m-1} \gamma_k\, \|\mathbb{C}_F\mathbb{M}(x_1, x_0, t)\|$$

$\|\mathbb{C}_F\mathbb{M}(x_m, x_n, t)\| \quad \le L \sum_{k=n}^{m-1} a_k\, \|\mathbb{C}_F\mathbb{M}(x_1, x_0, t)\|,$
where $a_{k=}\gamma_k$ and L is normal constant of S.
Now $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 0 \text{ and } \sum_{k=1}^{\infty} a_k$ is finite.
Since $\sum_{k=n}^{m-1} \gamma_k$ is convergent by D' Alembert's ratio test as $m \to \infty$.
Therefore $\{x_n\}$ is a cauchy sequence.
There is $x' \in \mathbb{X}$ such that $x_n \to x'$ as $n \to \infty$

$\mathbb{C}_F\mathbb{M}(Tx', x', t) \le \mathbb{C}_F\mathbb{M}(Tx', Tx_n, t) + \mathbb{C}_F\mathbb{M}(Tx_n, x', t)$

$\le \left(\frac{\mathbb{C}_F\mathbb{M}(x', Tx_n, t) + \mathbb{C}_F\mathbb{M}(x_n, Tx', t)}{\mathbb{C}_F\mathbb{M}(x', Tx_n) + \mathbb{C}_F\mathbb{M}(x_{n,}, Tx', t) + l}\right) \mathbb{C}_F\mathbb{M}(x_n, x', t) + \mathbb{C}_F\mathbb{M}(Tx_n, x', t)$

$$\le \left(\frac{\mathbb{C}_F\mathbb{M}(x', x_{n+1}, t) + \mathbb{C}_F\mathbb{M}(x_{n,} Tx', t)}{\mathbb{C}_F\mathbb{M}(x', x_{n+1}, t) + \mathbb{C}_F\mathbb{M}(x_n, Tx', t) + l}\right) \mathbb{C}_F\mathbb{M}(x_n, x', t) + \mathbb{C}_F\mathbb{M}(Tx_{n+1}, x', t)$$

$\mathbb{C}_F\mathbb{M}(Tx', x', t) \le 0$ as $n \to \infty$
Therefore $\|\mathbb{C}_F\mathbb{M}(x', Tx', t)\| = 0.$
Thus, $Tx' = x'$.

Uniqueness

Suppose $x' \text{ and } y'$ are two fuzzy fixed points of T.

$$\mathbb{C}_F\mathbb{M}(x', y', t) = \mathbb{C}_F\mathbb{M}(Tx', Ty', t)$$

$$\le \left(\frac{\mathbb{C}_F\mathbb{M}(x', Ty', t) + \mathbb{C}_F\mathbb{M}(y', Tx', t)}{\mathbb{C}_F\mathbb{M}(x', Tx', t) + \mathbb{C}_F\mathbb{M}(y', Ty', t) + l}\right) \left(\mathbb{C}_F\mathbb{M}(x', Tx', t) + \mathbb{C}_F\mathbb{M}(y', Ty', t)\right)$$

$$\le 0$$

Therefore $\|\mathbb{C}_F\mathbb{M}(x', y', t)\| = 0.$ Thus $x' = y'$.
Hence x' is an unique fuzzy fixed point of T.
(ii) Now

$$\mathbb{C}_F\mathbb{M}(T^n x', x', t) = \mathbb{C}_F\mathbb{M}(T^{n-1}(Tx'), x', t) = \mathbb{C}_F\mathbb{M}(T^{n-1}x', x', t) = \mathbb{C}_F\mathbb{M}(T^{n-2}(Tx'), x', t) \ldots$$

$$= \mathbb{C}_F\mathbb{M}(Tx', x', t) = 0$$

Hence $T^n x'$ converges to a fuzzy fixed point, for all $x' \in \mathbb{X}$.

Corollary 2.3:

Let $(\mathbb{X}, \mathbb{C}_F\mathbb{M}, *)$ be a complete fuzzy metric space and let T be a mapping from S be a normal cone with normal constant L. Suppose the mapping $T: \mathbb{X} \to \mathbb{X}$ Satisfies the subsequent condition: $\mathbb{C}_{\boldsymbol{F}}\mathbb{M}(\boldsymbol{T}_{\boldsymbol{x}},\boldsymbol{T}_{\boldsymbol{y}},\boldsymbol{t}) \leq \left(\frac{\mathbb{C}_{\boldsymbol{F}}\mathbb{M}(\boldsymbol{x},\boldsymbol{T}_{\boldsymbol{y}},\boldsymbol{t})+\mathbb{C}_{\boldsymbol{F}}\mathbb{M}(\boldsymbol{y},\boldsymbol{T}_{\boldsymbol{x}},\boldsymbol{t})}{\mathbb{C}_{\boldsymbol{F}}\mathbb{M}(\boldsymbol{x},\boldsymbol{T}_{\boldsymbol{x}},\boldsymbol{t})+\mathbb{C}_{\boldsymbol{F}}\mathbb{M}(\boldsymbol{y},\boldsymbol{T}_{\boldsymbol{y}},\boldsymbol{t})+\boldsymbol{l}}\right)\left(\mathbb{C}_{\boldsymbol{F}}\mathbb{M}(\boldsymbol{x},\boldsymbol{T}_{\boldsymbol{x}},\boldsymbol{t}) + \mathbb{C}_{\boldsymbol{F}}\mathbb{M}(\boldsymbol{y},\boldsymbol{T}_{\boldsymbol{y}},\boldsymbol{t})\right)$ (6)

For all $\boldsymbol{x},\boldsymbol{y} \in \mathbb{X}$. Then

i. T has unique fuzzy fixed point in $\mathbb{X}$.
ii. $\boldsymbol{T}^{\boldsymbol{n}}\boldsymbol{x}'$ converges to a fuzzy fixed point, for all $\boldsymbol{x}' \in \mathbb{X}$.

Proof :

The evidence of the corollary on the spot by taking $\boldsymbol{L} = \mathbf{1}$ within side the above theorem.

Reference:

1. Senthil Kumar. P, Thiruveni. P; Estimation of fuzzy metric spaces from metric spaces, Ratio Mathematica, Volume 41, 2021, pp. 64-70.
2. Krishnakumar.R , Dhamodharan. D; Fixed point theorems in normal cone metric space, International J. of Math. Sci. & Engg. Appls.(IJMSEA), ISSN 0973-9424, Vol. 10 No. III, 2016, pp. 213-224.
3. Saif Ur Rehman, Hong-Xu Li; Fixed point theorems in fuzzy cone metric spaces ; J. Non linear Sci. Appl., 10(2017), ISSN: 2008-1898; pp. 5763-5769.
4. Senthil kumar. P, Thiruveni. P , Dhamodharan. D; Some results on common fixed point in fuzzy 2-normed linear space, Advances And Applications In Mathematical Sciences, Volume 21, Issue 11, September 2022, Pages 6425-6435.
5. Huang L.G., Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., 332 (2007), 1468-1476.
6. Senthil kumar. P, Thiruveni. P , Rostam. K Saeed; Fuzzy Fixed PointTheorems for $\boldsymbol{\gamma}$ – WeakContractive Mappings in 3 Metric Spaces, Journal of algebraic statistics, Volume 13, No. 2, 2022, p. 723-729SN: 1309-3452 .
7. George A and Veeramani P (1997), *On Some Results of Analysis For Fuzzy Metric Spaces,* Fuzzy Sets and Systems (90) 365-368.
8. Karmosil O, Michalek J (1975), *Fuzzy metric and Statistical Metric Spaces*, Kybernetica 11, 326-334.