Original Research Article

Particle in a Box with Generalized Uncertainty Principle

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ABSTRACT

The minimal length is a fundamental length for position measurement, although there is no lower bound in ordinary quantum mechanics for the position uncertainty since the minimum value for the uncertainty can be tend to zero. Different theories of quantum gravity, such as string theory, loop quantum gravity, and black hole physical, confirm a minimal observable length. Kempf et al have formulated a generalized uncertainty principle (GUP) by modifying the Heisenberg uncertainty principle to include an appearance of the length in quantum mechanics. In this study, we have found out the influence of the GUP on the particle for its confinement in a 1D potential box and calculated eigenvalues for the particle in a handy manner. We have also graphically explored the modification in the energy spectrum due to the GUP characterized by the presence of the minimal length.

Keywords: Minimal length; Generalized uncertainty principle; Particle in a box; Energy spectrum;

1. INTRODUCTION

The concept of a fundamental length, a lower bound for space resolution, has arisen from a common aspect of different theories of quantum gravity for example string theory [1-5], loop quantum gravity [6-10] and black hole physics [11-14]. In quantum mechanics, the minimal observable length is considered as an additional uncertainty in position measurement and becomes a fundamental length for position measurement recognized as a minimal length. The form of the Heisenberg uncertainty principle is not compatible with this inferring. Therefore, the uncertainty product in Heisenberg uncertainty principle is modified to the Generalized uncertainty principle (GUP). The generalization leads to a modification of the commutator of the position and momentum operators and consequently, to the Hamiltonian for a quantum mechanical system.

The simplest form of generalized uncertainty principle for the dimension one which infers the existence of a nonvanishing minimal position uncertainty is [15]

$$\Delta X\,\Delta P \;\geq\; \frac{\hbar}{2}\left(1+\beta(\Delta P)^2+\gamma\right), \qquad (1.1)$$

where ΔX and $\Delta P$are the uncertainty in position and momentum respectively, β is the parameter for GUP, and β and γ both are positive and ΔX and ΔP independent although they may depend generally on the expectation values of both X and P. The smallest uncertainty in position or the minimal length has the value

$$\Delta X_{min} = \hbar\sqrt{\beta} \qquad (1.2)$$

in one dimension. As a result, the related commutation relationship is also generalized between position and momenta as

$$[X, P] = i\hbar(1 + \beta P^2), \quad 0 \leq \beta \leq 1. \tag{1.3}$$

The case $\beta \to 0$ corresponds to the ordinary quantum mechanics regime whereas $\beta \to 1$ is a limit for extreme quantum gravity.

The notion of GUP has various leading aspects, for example, providing frameworks for unifying general relativity with quantum mechanics [6], regularizing various divergences in quantum field theory [16], to count the surprising UV/IR mixing in quantum mechanics [17, 18], to depict a fundamental gage for measuring the finite size of a system [16, 19] and to construct an alternative operational description for the complex systems made by quasiparticles or composite particles or various collective excitons in a solid [20, 21].

With the context of minimal length, many quantum mechanical problems have been solved, such as quantum wells [22], quantum tunneling [23, 24], the coulomb-like problem [25], arbitrary dimensional harmonic oscillator [26], d-dimensional free particle [27], Ramsauer-Townsend Effect [28], three-dimensional isotropic harmonic interaction [29], the quadrupole moment of deuteron [30] and the binding energy of deuteron [31] have been studied. The thermodynamic quantities such as specific heat, entropy, men energy, and mean free energy [32] and the linear momentum developed on the non-locality due to GUP [33] have been discussed. In relativistic quantum mechanics, e.g. the ground state of hydrogen atom [34], Dirac oscillator in one [35] and three [36] dimensions, the Dirac equation with a linear scaler potential [37] and with a mixed scaler and vector linear potential [38], the Dirac equation with a static magnetic field [39] and combined static electric and magnetic field [40], (2+1) dimensional Dirac oscillator in presence of a magnetic field [41, 42] and a fermion in a potential box [43] were studied. In quantum field theory, several formulations under the GUP framework for instance quantization of a free scaler field theory [44], the Lagrangian for quantum electrodynamics of a complex scalar field [45], Feynman rules for quantum electrodynamics [46], a toy model [47] and a connection between the GUP parameter and Feynman propagator of gravity [48] have been formulated.

In this work, we consider a familiar and elementary system by which one can realize the energy quantization in ordinary quantum mechanics that a particle in the potential box in one dimension. The eigenfunctions and eigenvalues of the particle confined in the box under the framework of GUP have been calculated. In literature, the problem has been considered by Nozari et al. [49]. In ref. [49] the author solved a fourth-order differential equation and obtained the particle's energy levels to the first order in β. Here, we solve the Schrödinger equation for the particle under the framework of minimal length uncertainty relation in a completely different procedure from the procedure found in the literature. We reduce the Schrödinger equation into a differential equation of second order instead of a fourth order by choosing a suitable auxiliary wave function. Therefore, our solution procedure of the Schrödinger equation becomes very simple and handy. Moreover, our obtained energy spectrum is a general solution and the solution in ref. [49] is a subset of our solution. Because when we expand our obtained energy spectrum in different orders in β, up to the first order of β the energy spectrum is the same found in [49].

This paper is arranged as follows: In chapter 2, we present a solution of a particle in the one-dimensional box under the generalized uncertainty principle. We obtain the energy spectrum and auxiliary wave functions of the particle. In chapter 3, we briefly discuss and conclude this work.

2. PARTICLE IN A BOX UNDER THE INFLUENCE OF GENERALIZED UNCERTAINTY PRINCIPLE

We consider that a particle having mass *m* is confined to the following 1D box shown in figure-1

$$V(x) = \begin{cases} 0 & for\ -a < x < +a \\ +\infty & at\ x = -a\ and\ x = +a \end{cases}. \quad (2.1)$$

The time-independent Schrödinger equation under the generalized uncertainty principle (GUP) framework is [15]

$$\left[\frac{P^2}{2m} + V(X)\right]\psi(x) = E\psi(x). \quad (2.2)$$

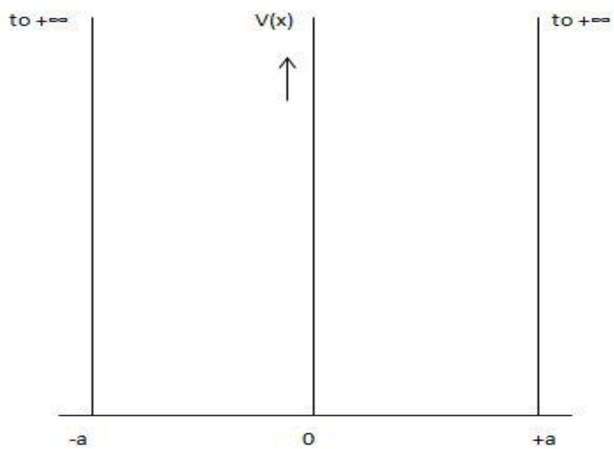

Fig.1: Geometry of particle in a box of width 2a.

The modified momentum and the position operator are respectively

$$P = p\left(1 + \frac{1}{3}\beta p^2\right) \quad (2.3a)$$

and

$$X = x, \quad (2.3b)$$

where

$$p = -i\hbar\frac{\partial}{\partial x}. \quad (2.3c)$$

Then we get from equation (2.2),

$$\left[\frac{p^2}{2m} + \frac{\beta}{3m}p^4 + V(x)\right]\psi(x) = E\psi(x). \quad (2.4)$$

Now we consider an auxiliary wave function $\varphi(x)$ as [50]

$$\psi(x) = \left(1 - \frac{2}{3}\beta p^2\right)\varphi(x). \quad (2.5)$$

Putting the expression of $\psi(x)$ in equation (2.4), we get,

$$\left[\left\{1 + \frac{4m}{3}\beta\big(E - V(x)\big)\right\}\frac{P^2}{2m} + (V(x) - E)\right]\varphi(x) = 0 \quad (2.6)$$

Equation (2.6) is a representation of a Schrödinger equation with general uncertainty principle (GUP) for the auxiliary wave function $\varphi(x)$.

As the particle resides in the region $-a < x < +a$, then for *V(x) = 0*, we get the following Schrödinger equation from equation (2.6) in presence of minimal length

$$\left[(1 + \frac{4m}{3}\beta E)\frac{P^2}{2m} - E\right]\varphi(x) = 0. \quad (2.7)$$

The equation (2.7) becomes

$$\frac{d^2\varphi}{dx^2} + \alpha^2\varphi = 0, \tag{2.8}$$

where

$$\alpha^2 = \frac{2mE}{\hbar^2\left(1+\frac{4m}{3}\beta E\right)}. \tag{2.9}$$

The solution of the equation (2.8) is

$$\varphi(x) = A\cos(\alpha x) + B\sin(\alpha x). \tag{2.10}$$

Using the boundary condition for the auxiliary wave function $\varphi(x)$ that the functions must be vanishes at walls of the infinite potential at $x = a$ and $x = -a$, and doing some algebra, we get,

$$2A\cos(\alpha x)=0 \tag{2.11a}$$

and

$$2B\sin(\alpha x)=0. \tag{2.11b}$$

We cannot have

$$A = B = 0 \tag{2.12}$$

because then for all values of *x* in the region -*a* < *x* < +*a*, φ becomes zero.
We cannot also have

$$\cos(\alpha x)= \sin(\alpha x)=0 \tag{2.13}$$

because

$$\sin^2\theta + \cos^2=1. \tag{2.14}$$

Thus, we can have

(1) A=0 and $\sin(\alpha a)$=0

and

(2) B= 0 and $\cos(\alpha a)$= 0.

Then we obtain

$$\alpha = n\frac{\pi}{2a}, \tag{2.15}$$

where *n* is a positive integer and $n \neq 0$, otherwise $\varphi(x)= 0$.
Then normalized auxiliary wave function becomes

$$\varphi_n(x)= \sqrt{\frac{2}{a}}\sin\left(n\frac{\pi}{2a}x\right) \quad ,\text{(2.16a)}$$

when *n* is an even positive integer and

$$\varphi_n(x)= \sqrt{\frac{2}{a}}\cos\left(n\frac{\pi}{2a}x\right), \tag{2.16b}$$

when *n* is an odd positive integer.

We get from equation (2.15)

$$E_n = \frac{n^2\pi^2\hbar^2}{8ma^2\left(1-\frac{n^2\pi^2\hbar^2}{6a^2}\beta\right)}, \tag{2.17}$$

where

$$n = 1, 2, 3\ldots \tag{2.18}$$

The equation (2.17) gives the desired quantized energies of a particle in a box under the influence of the GUP.

To compare the obtained result with the existing result found in the literature, we use Binomial expansion to expand $\left(1-\frac{n^2\pi^2\hbar^2}{6a^2}\beta\right)^{-1}$ and neglect the terms of higher order of β. Then we get

$$E_n = \frac{n^2\pi^2\hbar^2}{8ma^2} + \frac{n^4\pi^4\hbar^4\beta}{48ma^4} \tag{2.19}$$

which coincides the result found in the literature [49], where the width of the box was considered a and appearing of the GUP parameter β in the expression of the modified momentum operator was only β instead of $\frac{1}{3}\beta$ (see equation (2.3a)). If we set the deformation parameter $\beta = 0$, then the energy spectrum of the particle in the 1D box under the context of minimal length uncertainty relation (Eq. (2.17)) becomes

$$E = \frac{n^2\pi^2\hbar^2}{8ma^2} \tag{2.20}$$

which is the same as found in usual quantum mechanics.

Under the influence of GUP, the energy spectrum of the particle is modified and there is a shift in the energy levels of the particle in the box from the energy levels found in ordinary quantum mechanics. Up to first order β, the shift of the energy levels is $\frac{n^4\pi^4\hbar^4\beta}{48ma^4}$ (see equation (2.19)). Graphical presentation of the modification of energy levels as a result of the existence of a minimal length expressed by equation (2.17) is shown in the figure-2 for various values of β. The figure indicates that increasing of β leads to much modification in energy levels due to GUP.

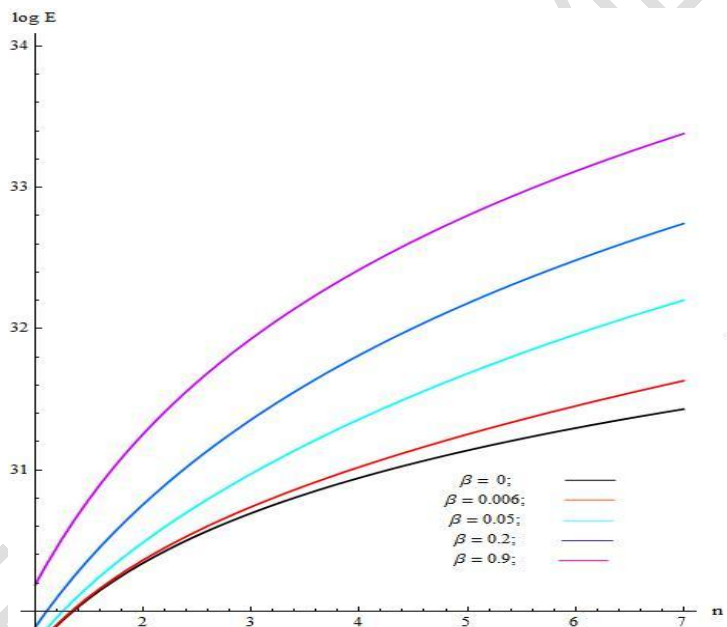

Fig.2: The energy spectrum for a Particle in 1D box in the GUP framework. The variation of energy levels due to variation of the deformed parameter, β is highlighted considering *n* is a continuous parameter. The case *β = 0* corresponds to the regime of ordinary quantum mechanics.

3. DISCUSSION AND CONCLUSION

In this paper, we consider a well-known quantum mechanical system which is a particle in a one- dimensional infinite potential box and find the impact of GUP on the particle. Since the potential of the system is time independent, the time independent Schrödinger equation under the framework of GUP has been converted into a second order differential equation through a transformation for an auxiliary wave function. Using the boundary condition on the auxiliary wave function that at the wall of an infinite potential wave function must be vanished, we obtain our desired energy spectrum of the particle under the context of the minimal length uncertainty principle. Due to the presence of the length, the energy levels of the particle are modified. The style of the modification of the energy spectrum with increasing the value of GUP parameter β has been explored graphically also. The modified energy spectrum recovers the energy spectrum found in ordinary quantum mechanics when the parameter β reduces to zero.

When we expand our obtained modified energy spectrum in higher orders of β, the spectrum up to the first order coincides with the spectrum found in the literature [49]. Moreover, our solution procedure turns into an easier and handy one, since we convert the equation of motion of the particle in the box into second order instead of the fourth order differential equation. Finally, we conclude that the paper provides an effective method for the solution of the considered problem and counts the more energy correction accomplished with the GUP parameter.

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