UNDER PEER REVIEW

Original Research Article

Pointwise Clique-Safe Domination in the Complement and Complementary Prism of Special Families of Graphs

Abstract

Let $G = (V(G), E(G))$ be any finite, undirected, simple graph. The maximun size of a clique containing a vertex $x \in V(G)$ is called the clique centrality of x, denoted by $\omega_G(x)$. A set $D \subseteq V(G)$ is said to be a pointwise clique-safe dominating set of G for every vertex $y \in D^c$ there exists a vertex $x \in D$ such that $xy \in E(G)$ where $\omega_{\langle D\rangle_G}(x) \geq \omega_{\langle D^c\rangle_G}(y)$. The smallest obtainable cardinality of a pointwise clique-safe dominating set of G is called the pointwise clique-safe domination number of G, denoted by $\gamma_{pcs}(G)$. This study aims to generate some properties of the parameter and to characterize the minimum pointwise clique-safe dominating sets of the complement of some special families of graphs as well as their complementary prism.

Keywords: clique-safe domination, pointwise clique-safe domination number, clique centrality

2020 Mathematics Subject Classification: 05C69, 05C75

1 Introduction

The investigation of domination in graphs was first done through the study of games and recreational mathematics. An attempt by De Jaenisch [1] to determine the number of queens required to cover an $n \times n$ chess board was one of the domination-related problems that were introduced from more or less a century before the formal study of domination in graphs. In 1962, the coefficient of external stability was introduced by Claude Berge [2] which is known later as domination number. Also, during this year, Oystein Ore [3] formally introduced the terms dominating set and domination number. To date, numerous studies have been done in relation to domination in graphs.

Let $G = (V(G), E(G))$ be any finite, undirected, simple graph. A dominating set of G is a nonempty set $D \subseteq V(G)$ such that for every vertex $y \in D^c$, there exists $x \in D$ adjacent to y in G. The smallest cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. Any dominating set of G of cardinality equal to $\gamma(G)$ is called a minimum dominating set of G or a γ-set of G.

Example 1.1. Consider graph G in Figure 1. Let $D = \{v_2\}$. Observe that every vertex in $V(G) \setminus D$ is adjacent to v_2. Hence, D is a dominating set of G and subsequently $\gamma(G) = 1$.

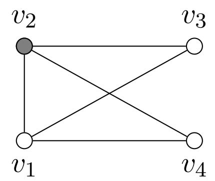

Figure 1: Domination in the graph G

A clique in G is a set $W \subseteq V(G)$ such that the subgraph $\langle W \rangle_G$ induced by W in G is complete. In 1988, Cozzens and Kelleher [4] introduced the dominating cliques in graphs, where they defined a clique dominating set as a set of vertices that dominates G and induces a complete subgraph of G.

In [5], Madriaga and Eballe introduced the clique centrality of a vertex $v \in V(G)$, denoted by $\omega_G(v)$, as the maximum size of a clique containing vertex v.

Another related study was done by Liwat and Eballe [6] in which they introduced the clique-safe domination in graphs. They defined a clique-safe dominating set of G as a nonempty $D \subseteq V(G)$ such that D is dominating and the size of the largest clique in $\langle D \rangle_G$ is greater than or equal to the size of the largest clique in $\langle D^c \rangle_G$. They also introduced the pointwise clique-safe domination in graphs in [7] where they defined the *pointwise clique-safe dominating set* of G as a nonempty $D \subseteq V(G)$ such that for every vertex $y \in V(G) \setminus D = D^c$ there exists a vertex $x \in D$ such that $xy \in E(G)$ where $\omega_{\langle D \rangle_G}(x) \geq \omega_{\langle D^c \rangle_G}(y)$. The minimum cardinality obtainable from among all pointwise clique-safe dominating sets of G is referred to as the pointwise clique-safe domination number of G, denoted by $\gamma_{pcs}(G)$. Any pointwise clique-safe dominating set D of G such that $|D| = \gamma_{pcs}(G)$ is called a minimum pointwise clique-safe dominating set of G or a γ_{pcs}-set of G.

Example 1.2. Consider the path P_4 in Figure 2. Let $D = \{v_2, v_4\}$. Observe that D dominates P_4 and that the $\langle D \rangle_{P_4} = \overline{K}_2$, $\langle D^c \rangle_{P_4} = \overline{K}_2$. It can be seen in the diagram that $\omega_{\langle D \rangle_G}(v_2) = \omega_{\langle D \rangle_G}(v_4) = 1$, $\omega_{\langle D^c \rangle_G}(v_1) = \omega_{\langle D^c \rangle_G}(v_3)$. Clearly, D is a pointwise clique-safe dominating set of P_4 and that $\gamma_{pcs}(P_4) = 2$.

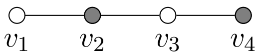

Figure 2: The pointwise clique-safe domination in path P_4

This study investigates the concept of pointwise clique-safe domination in the complement and complementary prism of some special families of graphs. It aims to generate some observable properties of pointwise clique-safe domination in those special families of graphs under the aforementioned unary operations as well as obtain their corresponding pointwise clique-safe domination number.

Throughout this paper, every graph is considered in the context of being simple, finite and undirected. Other terminologies not specifically defined in this paper may be found in [8].

2 Pointwise Clique-safe Domination in the Complement of Some Special Families of Graphs

This section contains the definitions of the special families of graphs involved in this study as well as the definition of the complement $\overline{G}$ of the graph G. Also, some results involving the pointwise clique-safe domination in the aforementioned graphs are in this section.

Definition 2.1. [8] A graph G is said to be *complete* if every pair of distinct vertices are adjacent. A complete graph of order n is denoted by K_n.

Definition 2.2. [8] A graph G is called a *bipartite graph* if the vertex-set $V(G)$ of G can be partitioned into two nonempty subsets V_1 and V_2, called *partite sets* of G, such that every edge in G joins a vertex in V_1 with a vertex in V_2. If each vertex in V_1 is adjacent to every vertex in V_2, then G is called a *complete bipartite graph*; in this case, $G = K_{m,n}$ if $|V_1| = m$ and $|V_2| = n$. A *star* of order $n+1$ is the complete bipartite graph $K_{1,n}$.

Definition 2.3. [9] The star graph S_{n-1} is a tree on n nodes with one node having vertex degree $n-1$ which is called the apex vertex and the other $n-1$ having vertex degree 1.

Example 2.1. *Figure 3 below shows the complete graph K_4, the complete bipartite graph $K_{5,4}$ and the star graph S_4.*

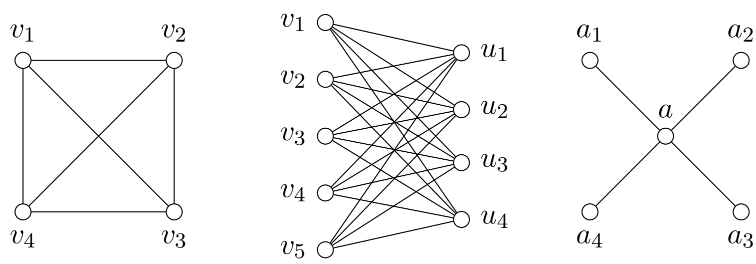

Figure 3: The complete graph K_4, complete bipartie graph $K_{5,4}$ and star graph S_4

Definition 2.4. [8] The *complement* $\overline{G}$ of a graph G is that graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G.

Example 2.2. Figure 4 below shows a graph G and its complement.

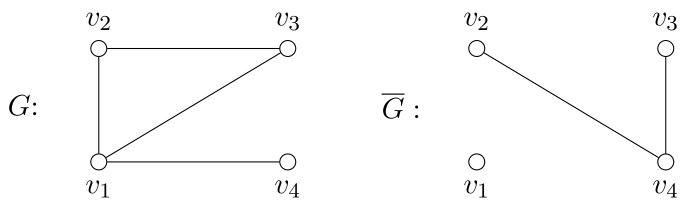

Figure 4: The graph G and $\overline{G}$

The following results are of the complement $\overline{K}_n$ of the complete graph K_n, $\overline{K}_{m,n}$ and $\overline{S}_{n-1}$.

Theorem 2.3. *Let K_n be a complete graph of order n. A set $D \subseteq V(\overline{K}_n)$ is a pointwise clique-safe dominating set of the complement $\overline{K}_n$ of K_n if and only if $D = V(\overline{K}_n)$.*

Proof. Notice that the complement of the complete graph K_n is a null graph $\overline{K}_n$ of order n. This means that if D is the pointwise clique-safe dominating set of a null graph $\overline{K}_n$, then $D = V(\overline{K}_n)$. The converse is straightforward. □

Corollary 2.4. *The pointwise clique-safe domination number of the complement $\overline{K}_n$ of K_n is given by $\gamma_{pcs}(\overline{K}_n) = n$.*

Proof. This is a direct consequence of Theorem 2.3. □

Theorem 2.5. *Let $K_{m,n}$ be a complete bipartite graph of order $m+n$ with partite sets A and B such that $|A| = m$ and $|B| = n$. A set $D \subseteq V(K_{m,n})$ is a pointwise clique-safe dominating set of the complement $\overline{K}_{m,n}$ of $K_{m,n}$ if and only if $D = \gamma_{pcs}-$set of $K_n \cup \gamma_{pcs}-$set of K_m.*

Proof. Notice that the complement of the complete bipartite graph $K_{m,n}$ will result to two complete graphs .These complete graphs are $\langle A \rangle_{\overline{K}_{m,n}} = K_m$ and $\langle B \rangle_{\overline{K}_{m,n}} = K_n$ in which for every vertex $v \in A$ and $u \in B$, $u, v \notin V(\overline{K}_{m,n})$. This implies that we must obtain the pointwise clique-safe dominating sets of K_m and K_n, respectively. Hence, D is the union of the pointwise clique-safe dominating sets of K_m and K_n.
The converse is straightforward. □

Corollary 2.6. *The pointwise clique-safe domination number of the complement $\overline{K}_{m,n}$ of $K_{m,n}$ is given by $\gamma_{pcs}(\overline{K}_{m,n}) = \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil$.*

Proof. Notice that by Theorem 2.5, $\gamma_{pcs}(K_m) = \lceil \frac{m}{2} \rceil$ and $\gamma_{pcs}(K_n) = \lceil \frac{n}{2} \rceil$. Hence, $\gamma_{pcs}(\overline{K}_{m,n}) = \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil$. □

Theorem 2.7. *Let S_{n-1} be a star graph of order n with a as an apex vertex. A set $D \subseteq V(\overline{S}_{n-1})$ is a pointwise clique-safe dominating set of the complement $\overline{S}_{n-1}$ of S_{n-1} if and only if D is the union of the pointwise clique-safe dominating set of K_{n-1} and $\{a\}$.*

Proof. Observe that all pendant vertices of S_{n-1} will form a complete graph K_{n-1} in $\overline{S}_{n-1}$. This means that for every vertex $v \in V(K_{n-1})$, $a, v \notin \overline{S}_{n-1}$. This implies that the pointwise clique-safe dominating set D of $\overline{S}_{n-1}$ will contain the vertex a and the pointwise clique-safe dominating set of K_{n-1}.
The converse is straightforward. □

Corollary 2.8. *The pointwise clique-safe domination number of the complement $\overline{S}_{n-1}$ of S_{n-1} is given by $\gamma_{pcs}(\overline{S}_{n-1}) = \lceil \frac{n-1}{2} \rceil + 1$.*

Proof. Notice that by Theorem 2.7, the pointwise clique-safe dominating set with minimum cardinality of $\overline{S}_{n-1}$ contains the $\gamma_{pcs}-$set of K_{n-1} and the vertex a. Hence, $\gamma_{pcs}(\overline{S}_{n-1}) = \lceil \frac{n-1}{2} \rceil + 1$. □

Remark 2.9. *The pointwise clique-safe domination of a self-complentary graph G is equal pointwise clique-safe domination number of its complement.*

3 Pointwise Clique-safe Domination in the Complementary Prism of Some Special Families of Graphs

Definition 3.1. [10] For a graph $G = (V, E)$, the *complementary prism*, denoted $G\overline{G}$, is formed from the disjoint union of G and its complement $\overline{G}$ by adding an edge between corresponding vertices u and $u^{'}$ of G and $\overline{G}$, respectively.

Example 3.1. Consider the graphs G and $\overline{G}$ in Figure 5, the complementary prism of G is given below:

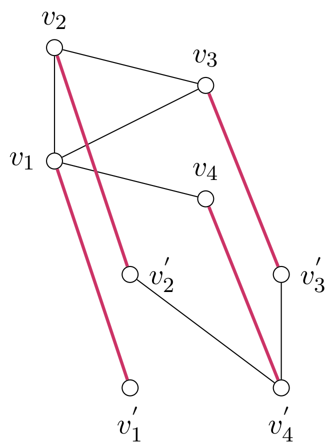

Figure 5: The complementary prism $G\overline{G}$ of the graph G

Theorem 3.2. *Let $G\overline{G}$ be the complementary prism of the graph G. A set $D = V(G) \subseteq V(G\overline{G})$ is the pointwise clique-safe dominating set of $G\overline{G}$ if and only if for corresponding vertices $x \in V(G)$ and $y \in V(\overline{G})$, we have $\omega_G(x) \geq \omega_{\overline{G}}(y)$.*

Proof. Recall that $D = V(G)$ is a pointwise clique-safe dominating set of the graph G. Now, if $D = V(G)$ is also a pointwise clique-safe dominating set of $G\overline{G}$ then every vertex $y \in V(\overline{G})$ is pointwise clique-safe dominated by its corresponding vertex $x \in G$. This implies that for corresponding vertices $x \in V(G)$ and $y \in V(\overline{G})$, we have $\omega_G(x) \geq \omega_{\overline{G}}(y)$.
The converse is straightforward. □

The next results are of the complementary prisms of some specific special families of graphs.

Theorem 3.3. *Let $K_n\overline{K}_n$ of order $2n$ be the complementary prism of the complete graph K_n. A set $D \subseteq V(K_n\overline{K}_n)$ is the pointwise clique-safe dominating set of $K_n\overline{K}_n$ if and only if D takes one of the following forms:*

a.) $D = V(K_n)$;

b.) $D = W \cup \{v_i\}$ *such that W is a pointwise clique-safe dominating set of K_n and $v_i \in \overline{K}_n$ does not corresponds to a vertex in W.*

Proof. Part a is an application of Theorem 3.2. For part b, let W be a pointwise clique-safe dominating set of K_n. Notice that if D is a pointwise clique-safe dominating set of $K_n\overline{K}_n$, then D

must also pointwise clique-safe dominate K_n. This means that D must contain W. This implies that D must also contain those vertices in $\overline{K}_n$ that were not pointwise clique-safe dominated by W. Hence, $D = W \cup \{v_i\}$ such that $v_i \in \overline{K}_n$ does not corresponds to a vertex in W.
The converse is straightforward. □

Observe that the pointwise clique-safe dominating set D in parts a and b in Theorem 3.3 have the same cardinality. Hence, the next result is obtained.

Corollary 3.4. *The poitwise clique-safe domination number of the complementary prism $K_n\overline{K}_n$ is given by $\gamma_{pcs}(K_n\overline{K}_n) = n$.*

Proof. Part a of Theorem 3.3 shows that $|D| = n$ while part b of the same theorem asserts that $|D| = |W| + |\{v_i\}|$. This implies that if $|W| = \lceil \frac{n}{2} \rceil$ then $|\{v_i\}| = \lfloor \frac{n}{2} \rfloor$. Hence,

$$|D| = |W| + |\{v_i\}| = \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor = n \tag{3.1}$$

□

Theorem 3.5. *Let $K_{m,n}\overline{K}_{m,n}$ with $n, m \geq 2$ be the complementary prism of the complete bipartite graph $K_{m,n}$ and W be the pointwise clique-safe dominating set of $\overline{K}_{m,n}$. A set $D \subseteq V(K_{m,n}\overline{K}_{m,n})$ is the pointwise clique-safe dominating set of $K_{m,n}\overline{K}_{m,n}$ if and only if $D = \gamma_{pcs}-$set of $K_{m,n} \cup W$.*

Proof. Notice that we need W and the $\gamma_{pcs}-$set of $K_{m,n}$ to pointwise clique-safe dominate $K_{m,n}\overline{K}_{m,n}$ since W only will not pointwise clique-safe dominate those vertices that do not correspond to its elements. Hence, $D = \gamma_{pcs}-$set of $K_{m,n} \cup W$.
The converse is straightforward. □

Corollary 3.6. *The pointwise clique-safe domination number of the complementary prism $K_{m,n}\overline{K}_{m,n}$ of $K_{m,n}$ is given by $\gamma_{pcs}(K_{m,n}\overline{K}_{m,n}) = \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil + 2$.*

Proof. This is a direct consequence of Theorem 3.5. □

For the complementary prism $S_{n-1}\overline{S}_{n-1}$ of the star graph S_{n-1}, notice that $\overline{S}_{n-1}$ will have a complete subgraph K_{n-1}. This information will be used in the next result.

Theorem 3.7. *Let $S_{n-1}\overline{S}_{n-1}$ be the complementary prism of the star graph S_{n-1} with a as an apex vertex and W be a pointwise clique-safe dominating set of $\overline{S}_{n-1}$. A set D is a pointwise clique-safe dominating set if and only D takes one of the following forms:*

- *a.) $D = \gamma_{pcs}$-set of $K_{n-1} \cup \{a\} \cup \{v_i\}$ such that v_i is a pendant vertex of S_{n-1};*
- *b.) $D = W \cup \{a\}$.*

Proof. Notice that $\gamma_{pcs}-$set of K_{n-1} will also pointwise clique-safe dominate those vertices in S_{n-1} that correspond to its vertices. Now, observe that the clique centrality of those vertices that are not pointwise clique-safe dominated by the $\gamma_{pcs}-$set of K_{n-1} is equal to 2. This means that we need at least two adjacent vertices to pointwise clique-safe dominate those vertices in which the apex vertex a of S_{n-1} happens to be one of these adjacent vertices since it is adjacent to every vertex not pointwise clique-safe dominated by the $\gamma_{pcs}-$set of K_{n-1}. If a is paired with a pendant vertex, then this shows part a. Moreover, if a is paired with its corresponding vertex, then this shows part b.
The converse is straightforward. □

Corollary 3.8. *The pointwise clique-safe domination number of the complementary prism $S_{n-1}\overline{S}_{n-1}$ of the star graph S_{n-1} is given by $\gamma_{pcs}(S_{n-1}\overline{S}_{n-1}) = \lceil \frac{n-1}{2} \rceil + 2$.*

Proof. Observe that the minimum pointwise clique-safe dominating set D in parts a and b of Theorem 3.7 has equal cardinality. Part a asserts that

$$|D| = \gamma_{pcs}(K_{n-1}) + 1 + 1 = \lceil \frac{n-1}{n} \rceil + 2. \quad (3.2)$$

Now for part b, we have

$$|D| = \gamma_{pcs}(\overline{S}_{n-1}) + 1 = \lceil \frac{n-1}{n} \rceil + 1 + 1 = \lceil \frac{n-1}{n} \rceil + 2. \quad (3.3)$$

□

4 Conclusion

In this article, the pointwise clique-safe domination is being investigated in the complement and complementary prism of some special families of graphs such as the complete graph K_n, the complete bipartite graph $K_{m,n}$ and the star graph S_{n-1}. Furthermore, the corresponding expressions for the pointwise clique-safe domination number of the complement and complementary prism of those aforementioned graphs are determined. Finally, the parameter introduced in this paper may be explored further to address some relevant problems as done in [11], [12], [13], [14], [15], [16], [17] and [18].

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