*Original Research Article*

# THE *T*-EXPONENTIATED EXPONENTIAL{FRECHET} FAMILY OF DISTRIBUTIONS: THEORY AND APPLICATIONS


**ABSTRACT***:*

This article introduces a new family of Generalized Exponentiated Exponential distribution. Using the T-R{Y} framework, a new family of T-Exponentiated Exponential{Y} distributions named T-Exponentiated Exponential{Frechet} family of distributions is proposed. Some general properties of the family such as hazard rate function, quantile function, non-central moment, mode, mean absolute deviations and Shannon's entropy are discussed. A new continuous univariate probability distribution which is a member of the T-Exponentiated Exponential{Frechet} family of distributions is introduced. From the general properties of the family, expressions are derived for some specific properties of the new distribution. To show the usefulness of the T-Exponentiated Exponential{Frechet} family of distributions, the new distribution is fitted to two real life data sets and the results are compared with the results of some other existing distributions.



**Keywords**: Exponentiated-Exponential distribution, *T-R*{*Y*} framework, Moments, Quantile Function, generalized distribution.


## INTRODUCTION

The quest for more flexible probability distributions has provoked among researchers an active interest to develop new probability distributions that promise more satisfactory fit to the growing number of complex data [1]. There are well recognized standard theoretical distributions in the literature such as the Normal distribution, Chi-square distribution, Student-t distribution, Exponential distribution, Uniform distribution, Gamma distribution, Beta distribution, Rayleigh distribution, Pareto distribution, Weibull distribution, Gumbel distribution, Lomax distribution, Frechet distribution, Burr system of distributions, Gompertz distribution, Dagum distribution, Cauchy distribution, Lindley distribution, Kumaraswamy distribution, Binomial distribution, Geometric distribution and Poisson distribution. These distributions have played and continue to play important roles in the development of statistical theory and applications. Very notable among them is the Gaussian or Normal distribution whose usage is dominant in most practical statistical works. Also popular is the exponential distribution, widely used to model waiting times and lifetime data. However, these distributions have been strongly challenged in application by newer distributions generated using recently developed methodologies for generating families of continuous univariate probability distributions [2].

Lately, the development of new methods for generating more flexible families of continuous univariate probability distributions has witnessed an active interest among researchers. These methods are significant in the advancement of distribution theory and involve the modification and generalization of existing standard theoretical distributions. They induce more skewness either by adding new parameter(s) to an existing distribution or by combining two or more existing distributions thereby making the resulting distribution more flexible and robust.

## LITERATURE REVIEW

There are many well-established generalized families (G families) of continuous univariate probability distributions widely accepted in literature. Among these G families are Azzalini's skewed family [3], Marshall-Olkin Extended (MOE) family [4], Exponentiated family [5], Beta-generated family [6,7], Transmuted family [8], Kumaraswamy generalized (Kw G) family [9], Transformed-Transformer (T-X) family [10], Weibull G family [11] and the T-R{Y} family [12].

These methods of generalization have been used to generalize many of the well-recognized standard theoretical distributions. Some of the generalized distributions include: Exponentiated Generalized Normal distribution and Exponentiated Exponential distribution by [13], Exponentiated Generalised Inverse Weibull distribution [14], Transmuted Weibull distribution [15], Transmuted Lomax distribution [16], Transmuted Pranav distribution [17], Beta Normal distribution [6], Beta Gumbel distribution [18], Beta Weibull distribution [19], Weibull-Uniform distribution and Weibull-Weibull distribution by [11], Weibull Rayleigh distribution [20], Kumaraswamy Normal distribution [9], Kumaraswamy Generalised Pareto distribution [21], ExponentiatedKumaraswamy-Dagum distribution [22], Marshall Olkin Geometric distribution [23], Marshall-Olkin Exponential Pareto distribution [24] and Marshall-Olkin Extended Weibull-Exponential distribution [25]. The comparative performance of these generalized distributions is promising and encourages the use of these generalization methods to enhance the capabilities of existing distributions and to obtain more flexible families of distributions.

## METHODOLOGY

This article studies the generalization of the Exponentiated Exponential distribution using the T-R{Y} framework of [12]. Whereas other methods [26] have been used to generalize the Exponentiated exponential distribution, to the best of our knowledge, the distribution has not been generalized using the *T-R*{*Y*} framework as considered in this article. Suffice it to say that each method of generalization adds a uniqueness to the resulting new distribution.
According to [27], given the random variables *T*, *R* and *Y* with cumulative distribution functions $F_T(x) = P(T \le x)$, $F_R(x) = P(R \le x)$ and $F_Y(x) = P(Y \le x)$ respectively and the corresponding quantile functions $Q_T(u)$, $Q_R(u)$ and $Q_Y(u)$ where $Q_k(u) = \inf\{k : F_k(k) \ge u\}, u \in (0,1)$ with their respective probability density functions $f_T(x)$, $f_R(x)$ and $f_Y(x)$ (where they exist), then the *T-R*{*Y*} framework is defined as

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)dt = F_T\{Q_Y(F_R(x))\} \quad (1)$$

where $F_X(x)$ is the C.D.F. of the new distribution resulting from the *T-R*{*Y*} framework and $T, Y \in (a,b)$ for $-\infty \le a < b \le \infty$. Consequently, the P.D.F. associated with (1) is given as

$$f_X(x) = f_R(x) \times \frac{f_T\{Q_Y(F_R(x))\}}{f_Y\{Q_Y(F_R(x))\}}. \quad (2)$$

According to the authors, different families of generalized *R*-distributions result from different choices of the *T* and *Y* random variables.

## RESULTS

### The *T*-Exponentiated Exponential{Frechet} Family of Distributions (Proposed)

We first derive the parent family: the*T*-Exponentiated Exponential{$Y$} ($T$-EE{$Y$}) Family of Distributions (Proposed). Supposing *R* is a random variable following the Exponentiated Exponential (EE) distribution with shape parameter, $\alpha$ and scale parameter 1. That is, R~EE($\alpha$,1). The probability density function (P.D.F.), the cumulative distribution function (C.D.F.) and the quantile function of the random variable *R*, are respectively given below:

$$f_R(x) = \alpha e^{-x}(1-e^{-x})^{\alpha-1} \tag{3}$$

$$F_R(x) = (1-e^{-x})^{\alpha} \tag{4}$$

$$Q_R(p) = -\ln\left(1-p^{\frac{1}{\alpha}}\right) \tag{5}$$

for $x > 0, \alpha > 0$.

Let *T* and *Y* be two other random variables with P.D.F., C.D.F. and quantile functions respectively (as the case may be) as follows:

$f_T(x)$, $F_T(x)$ and $Q_T(x)$ for the *T* random variable and

$f_Y(x)$, $F_Y(x)$ and $Q_Y(x)$ for the *Y* random variable.

Then the C.D.F. and P.D.F. respectively of the $T$-EE{$Y$} family of distributions are

$$F_X(x) = F_T\{Q_Y[(1-e^{-x})^{\alpha}]\} \tag{6}$$

$$f_X(x) = \frac{\alpha e^{-x}(1-e^{-x})^{\alpha-1} f_T\{Q_Y[(1-e^{-x})^{\alpha}]\}}{f_Y\{Q_Y[(1-e^{-x})^{\alpha}]\}} \tag{7}$$

where$X$ is the $T$-EE{$Y$} random variable, $x > 0, \alpha > 0$ and $\alpha$ is a shape parameter.

Proof: Substituting (4) for $F_R(x)$ in (1) yields (6). Similarly, substituting (3) and (4) for $f_R(x)$ and $F_R(x)$ respectively in (2) yields (7).

The support of the $T$-EE{$Y$} random variable *X* is the same with the support of the random variable *R*. However, *T* and *Y* random variables must have the same support.

**Remark 4.1:** (The transformation of the *T* random variable to the $T$-$R${$Y$} random variable *X*). For the random variable *T*, (1) suggests that $F_X(x) = F_T(Q_Y(F_R(x)))$ $\Rightarrow Q_Y(F_R(x)) = t$.

Therefore, $F_Y(Q_Y(F_R(x)))=F_Y(t)\Rightarrow F_R(x)=F_Y(t)$ and $Q_R(F_R(x))=Q_R(F_Y(t))$
$\Rightarrow x=Q_R(F_Y(t))$. (8)

**Remark 4.2:** The quantile function of the *T*-*R*{*Y*} family of distributions is
$Q_X(u)=Q_R(F_Y(Q_T(u)))$. (9)

Considering (1) and setting $F_T(Q_Y(F_R(x)))=u$, where $u\in(0,1)$, (9) can be obtained by applying successive inverse functions of $F_T(\cdot)$, $Q_Y(\cdot)$ and $F_R(\cdot)$ to both sides of the equation.

To establish a subfamily of T-Exponentiated Exponential{Y} family of distributions, the *T*-Exponentiated Exponential{Frechet} (*T*-EE{F}) family of distributions, we choose the Frechet distribution and let *Y* be a Frechet random variable.

Supposing *Y* is a random variable from the Frechet distribution with P.D.F., C.D.F. and quantile function respectively as follows:

$$f_Y(x)=\beta x^{-\beta-1}e^{-x^{-\beta}}, \qquad x>0,\beta>0 \tag{10}$$

$$F_Y(x)=e^{-x^{-\beta}} \tag{11}$$

and

$$Q_Y(x)=[-\ln(p)]^{-\frac{1}{\beta}}, \quad p\in(0,1). \tag{12}$$

Then the CDF and PDF respectively of the *T*-EE{F} family of distributions are

$$F_X(x)=F_T\{[-\ln(1-e^{-x})^{\alpha}]^{-\frac{1}{\beta}}\} \tag{13}$$

and

$$f_X(x)=\frac{\alpha e^{-x}(1-e^{-x})^{-1}f_T\{[-\ln(1-e^{-x})^{\alpha}]^{-\frac{1}{\beta}}\}}{\beta[-\ln(1-e^{-x})^{\alpha}]^{\frac{\beta+1}{\beta}}} \tag{14}$$

for $x>0,\alpha,\beta>0$.

Proof. Substituting (12) for $Q_Y(\cdot)$ in (6) yields (13).Likewise, substituting (10) for $f_Y(\cdot)$ and (12) for $Q_Y(\cdot)$ in (7) and simplifying leads to (14).

## Some Properties of the *T*-Exponentiated Exponential{Frechet} family of Distributions

We present some general properties of the *T*-EE{F} family of distributions in this section. For brevity, hints are given for the proof of some results.

**Lemma**. (Transformation of Random Variables). Let $T$ be a random variable with C.D.F. $F_T(x)$. Then the random variable $X = -\ln\left(1-e^{-\frac{1}{\alpha}\left(T^{-\beta}\right)}\right)$ follows the $T$-EE{F} distribution. This result is straightforward from substituting (11) for $F_Y(\cdot)$ and (5) for $Q_R(\cdot)$ in (8).

**Lemma.** (Quantile function of the $T$-EE{F} family of distributions). The quantile function of the T-Exponentiated Exponential{Frechet} family of distributions is

$$Q_X(u) = -\ln\left\{1-\left(e^{-\frac{1}{\alpha}[Q_T(u)]^{-\beta}}\right)\right\} \tag{15}$$

By substituting (11) for $F_Y(\cdot)$ and (5) for $Q_R(\cdot)$ in (9), the result in (15) can be obtained.

**Proposition.** The $r^{th}$ non-central moment of the $T$-EE{F} family of distributions is

$$E\left(X^r\right) = r\sum_{i=0}^{r}\binom{i-r}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{r-j}\binom{i}{j}q_{j,i}E\left\{\left(e^{-\left(\frac{r+i}{\alpha}\right)\left(T^{-\beta}\right)}\right)\right\}. \tag{16}$$

Proof. Generally, the $r^{th}$ non-central moment is represented mathematically as

$$E\left(X^r\right) = \int_x x^r f(x)dx \tag{17}$$

where $f(x)$ is the probability density function of the random variable, $X$. Since (8) in Remark 2.1 indicates that $x \xrightarrow{d} Q_R\left(F_Y(T)\right)$, the $r^{th}$ non-central moment of the $T$-EE{F}family of distributions (where it exists) can be obtained using the relation

$$E\left(X^r\right) = E\left\{\left(Q_R\left(F_Y(T)\right)\right)^r\right\} \tag{18}$$

where $E(\cdot)$ is the Expectation of the $T$-EE{F} random variable $X$. The result in (16) can therefore be obtained by substituting (5) for $Q_R(\cdot)$ and (11) for $F_Y(\cdot)$ in (18) and using the series expansion

$$\left(-\ln(1-z)\right)^a = a\sum_{i=0}^{\infty}\binom{i-a}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{a-j}\binom{i}{j}q_{j,i}Z^{a+i} \tag{19}$$

where $a > 0$ is real, $|Z| < 1$ and $q_{j,i}$ is a constant which can be calculated recursively as

$$q_{j,i}=\frac{1}{i}\sum_{m=1}^{i}\frac{(-1)^m(jm-i+m)}{m+1}q_{j,i-m}\text{ ; for } i=1,2,3,\ldots \text{and } q_{j,0}=1.$$

**Proposition.** The mode(s) of the T-Exponentiated Exponential{Frechet} family of distributions are the solutions to the equation

$$x=\ln\left\{\frac{\alpha(\beta^{-1})f_T'\left((-\ln(F_R(x)))^{-1/\beta}\right)}{f_T\left((-\ln(F_R(x)))^{-1/\beta}\right)(-\ln(F_R(x)))^{(\beta+1)/\beta}}-\frac{\alpha\left(\frac{\beta+1}{\beta}\right)}{\ln(F_R(x))}\right\} \tag{20}$$

Proof. With respect to (3), if we consider the fact that $f_R'(x)=\frac{[f_R(x)]^2}{F_R(x)}\left\{\frac{\alpha e^{-x}-1)}{\alpha e^{-x}}\right\}$, the derivative $f_X'(x)$ of (7) w.r.t. $x$ can be reduced to $f_X'(x)=\frac{\beta^{-1}[f_R(x)]^2}{[F_R(x)]^2(-\ln(F_R(x)))^{(\beta+1)/\beta}}*A(x)$, where

$$A(x)=\left\{\frac{(\alpha e^{-x}-1)f_T((-\ln(F_R(x)))^{-1/\beta})}{\alpha e^{-x}}\right.$$
$$+\frac{\beta^{-1}f_T'((-\ln(F_R(x)))^{-1/\beta})}{(-\ln(F_R(x)))^{(\beta+1)/\beta}}-f_T((-\ln(F_R(x)))^{-1/\beta})$$
$$\left.-\frac{\left(\frac{\beta+1}{\beta}\right)f_T((-\ln(F_R(x)))^{-1/\beta})}{\ln(F_R(x))}\right\}.$$

On setting $f_X'(x)=0$ and solving the equation $A(x)=0$, the result in (20) is obtained.

**Proposition.** The mean absolute deviation from the mean and median (denoted by $MAD_{(\mu)}$ and $MAD_{(M)}$) of the $T$-EE{F} random variable are respectively

$$MAD_{(\mu)}=2\mu F_X(\mu)-2\sum_{i=0}^{1}\binom{i-1}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{1-j}\binom{i}{j}P_{j,i}S_{\exp(-(1/\alpha)(v^{-\beta}))}(\mu,0,i+1) \tag{21}$$

$$MAD_{(M)}=\mu-2\sum_{i=0}^{1}\binom{i-1}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{1-j}\binom{i}{j}P_{j,i}S_{\exp(-(1/\alpha)(v^{-\beta}))}(M,0,i+1) \tag{22}$$

where $S_{\Phi(v)}(q,s,z)=\int_s^{(-\log(F_R(q)))^{-1/\beta}}(\Phi(v))^z f_T(v)dv$.

Proof. If we let $MAD_{(\mu)}$ and $MAD_{(M)}$ denote the mean absolute deviation of a continuous random variable $X$ from its mean and median respectively such that $x \in [a,b]$, then by definition

$MAD_{(\mu)} = E(|x-\mu|) = \int_a^b |x-\mu| f(x) dx$ and upon further operations

$$MAD_{(\mu)} = 2\mu F(\mu) - 2\int_a^{\mu} x f(x) dx . \tag{23}$$

Likewise, $MAD_{(M)} = E(|x-M|) = \int_a^b |x-M| f(x) dx$ and upon further operations

$$MAD_{(M)} = \mu - 2\int_a^{M} x f(x) dx . \tag{24}$$

Let $I_q = \int_0^q x f_X(x) dx$. Therefore, $MAD_{(\mu)} = 2\mu F(\mu) - 2I_{\mu}$ and $MAD_{(M)} = \mu - 2I_M$ .

Rewriting (14) for brevity as $f_X(x) = \dfrac{\beta^{-1} f_R(x) f_T\{[-\ln(F_R(x)]^{-\frac{1}{\beta}}\}}{F_R(x)[-\ln(F_R(x))^{\alpha}]^{\frac{\beta+1}{\beta}}}$ gives

$$I_q = \int_0^q \left\{ \frac{x\beta^{-1} f_R(x) f_T\left\{(-\ln(F_R(x)))^{-1/\beta}\right\}}{F_R(x)(-\ln(F_R(x)))^{(\beta+1)/\beta}} \right\} dx . \tag{25}$$

The substitution $v = (-\ln(F_R(x)))^{-(1/\beta)}$ yields $x = -\ln\left(1-\left(e^{-(1/\alpha)(v^{-\beta})}\right)\right)$ and $F_R(x) = e^{-v^{-\beta}}$. Furthermore, $F_R(x) = e^{-v^{-\beta}} \Rightarrow f_R(x)dx = \beta v^{-(\beta+1)} e^{-v^{-\beta}} dv$ where $\left|\frac{dx}{dv}\right|$ is the Jacobian of the transformation. Making these substitutions in (25) and simplifying results to

$$I_q = \int_0^{(-\ln(F_R(q)))^{-(1/\beta)}} \left\{ -\ln\left(1-e^{-(1/\alpha)(v^{-\beta})}\right) f_T(v) \right\} dv \cdot \tag{26}$$

Applying (19) in (26) yields

$$I_q = \sum_{i=0}^{1} \binom{i-1}{i} \sum_{j=0}^{i} \frac{(-1)^{i+j}}{1-j} \binom{i}{j} P_{j,i} S_{\exp\left(-(1/\alpha)(v^{-\beta})\right)}(q,0,i+1) \tag{27}$$

and the results in (21) and (22), where

$$S_{\Phi(v)}(q,s,z) = \int_s^{(-\log(F_R(q)))^{-1/\beta}} (\Phi(v))^z f_T(v) dv .$$

**Proposition.** The Shannon's entropy of a random variable $X$ following the $T$-EE{F} distribution is defined as

$$\eta_X = \eta_T - \log(\alpha) + \mu_X + \log(\beta) - (\beta+1)\,\mathrm{E}\{\log(T)\} - \alpha^{-1}\,\mathrm{E}\{T^{-\beta}\} \tag{28}$$

where $\eta_T$ is the Shannon's entropy of the random variable $T$ with PDF $f_T(x)$.

Proof. Let $X$ be a $T$-EE{F} random variable with C.D.F. as in (1). It is easy to see that $T \overset{d}{=} Q_Y(F_R(x))$ and $x \overset{d}{=} Q_R(F_Y(T))$. Therefore we can rewrite (2) as

$$f_X(x) = \frac{f_R(x) f_T(t)}{f_Y(t)} \tag{29}$$

Generally, the Shannon's entropy of a random variable $X$ with P.D.F. $f_X(x)$ is defined as

$$\eta_X = \mathrm{E}\{-\log(f_X(x))\}. \tag{30}$$

Putting (29) in (30) and simplifying, we obtain

$$\eta_X = \eta_T + \mathrm{E}\{\log(f_Y(t))\} - \mathrm{E}\{\log(f_R(x))\}. \tag{31}$$

By substituting (3) for $f_R(x)$ in (31) and simplifying, we obtain

$$\eta_X = \eta_T + \mathrm{E}\{\log(f_Y(t))\} - \log(\alpha) + \mu_X + (1-\alpha)\,\mathrm{E}\{\log(1-e^{-x})\}. \tag{32}$$

Since $Y$ is a Frechet random variable with P.D.F. as in (10), we have that

$$\mathrm{E}\{\log(f_Y(T))\} = \log(\beta) - (\beta+1)\,\mathrm{E}\{\log(T)\} - \mathrm{E}\{T^{-\beta}\}. \tag{33}$$

Also since $x \overset{d}{=} Q_R(F_Y(T))$, then $x = -\log\left(1-e^{-(1/\alpha)(T^{-\beta})}\right)$. With $x = -\log\left(1-e^{-(1/\alpha)(T^{-\beta})}\right)$ and further simplifying we have,

$$\mathrm{E}\left(\log(1-e^{-x})\right) = -\alpha^{-1}\,\mathrm{E}(T^{-\beta}). \tag{34}$$

Therefore, by putting (33) and (34) in (32) and simplifying, the result in (28) is obtained.

**A New Member of the *T*-EE{F} Family of Distributions**

In this section, we present a new member of the T-Exponentiated Exponential{Frechet} family of Distributions named GumbelTypeII-Exponentiated Exponential{Frechet} Distribution - GumTII-EE{F} for short. We discuss some specific properties of the new distribution following the general structure of the properties of the $T$-EE{F} Family of Distributions presented in section 3.

**The GumbelTypeII-ExponentiatedExponential{Frechet} Distribution**

**Proposition.** If $X$~GumTII-EE{F}($\alpha, b, \lambda$), we say that the random variable $X$ follows the GumbelTypeII-Exponentiated Exponential{Frechet} distribution with parameters $\alpha$ , $b$ and $\lambda$ such that the cumulative distribution function (C.D.F.) and the probability density function (P.D.F.) respectively of the random variable are as proposed below

$$F_X(x) = e^{-b\left(-\ln(1-e^{-x})^{\alpha}\right)^{\lambda}} \tag{35}$$

$$f_X(x) = \alpha\lambda b e^{-x}\left(1-e^{-x}\right)^{-1}\left\{-\ln(1-e^{-x})^{\alpha}\right\}^{\lambda-1} e^{-b\left(-\ln(1-e^{-x})^{\alpha}\right)^{\lambda}} \tag{36}$$

for $x > 0$ and $\alpha, b, \lambda > 0$.

Proof. Let $T$ be a GumbelTypeII random variable with C.D.F. and P.D.F. respectively as

$$F_T(x) = e^{-bx^{-c}} \tag{37}$$

$$f_T(x) = bcx^{-c-1}e^{-bx^{-c}} \tag{38}$$

for $x > 0$ and $b, c > 0$. By putting (37) and (38) in (13) and (14) respectively, simplifying and setting $\lambda = \frac{c}{\beta}$, the results in (35) and (36) can be respectively obtained. The Graphs of the P.D.F. of the GumTII-EE{F} distribution are provided in Figure 1(a-d). As evidenced in the graphs, the GumTII-EE{F} distribution can be unimodal, J shaped or reverse-J shaped.

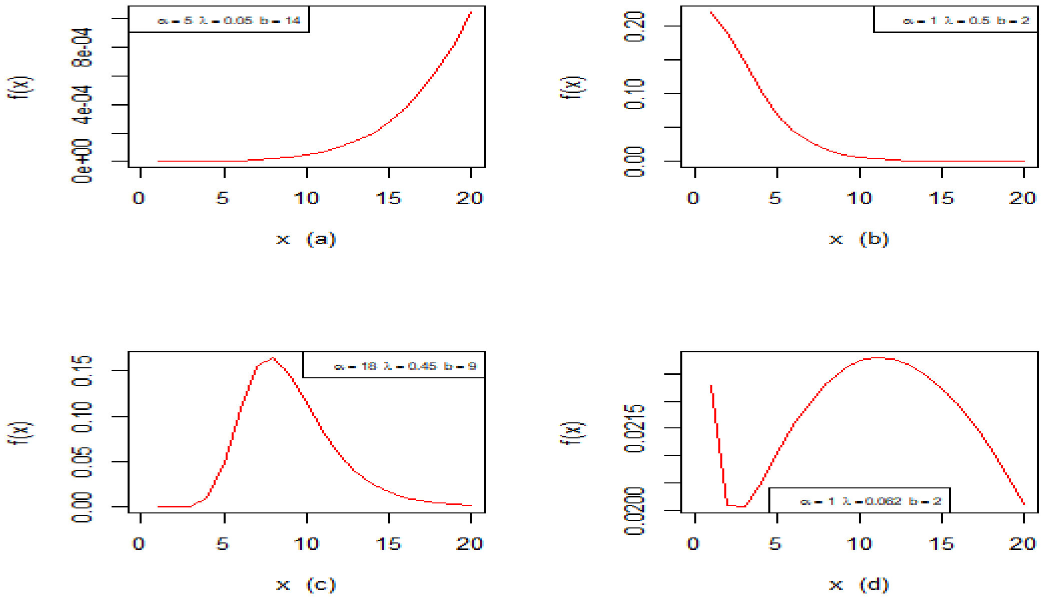


Figure 1: The graphs of the PDF of GumTII-EE{F} Distribution for varying values of alpha, lambda and b.

**Properties of the GumbelTypeII-ExponentiatedExponential{Frechet} Distribution**

Some properties of the GumTII-EE{F} distribution are presented in this subsection.

**Hazard Rate Function.** The hazard rate function denoted by $h(x)$ is generally defined as

$$h(x)=\frac{f(x)}{1-F(x)}. \tag{39}$$

Therefore, by putting (35) and (36) in (39), the hazard rate function of the GumTII-EE{F} distribution denoted by $h_X(x)$ is

$$h_X(x)=\frac{\alpha\lambda b e^{-x}\left(1-e^{-x}\right)^{-1}\left\{-\ln(1-e^{-x})^{\alpha}\right\}^{\lambda-1} e^{-b\left(-\ln(1-e^{-x})^{\alpha}\right)^{\lambda}}}{1-e^{-b\left(-\ln(1-e^{-x})^{\alpha}\right)^{\lambda}}}. \tag{40}$$

Figure2(a-d) presents the graphs of the hazard rate function of the GumTII-EE{F} distribution. The graphs indicate that the distribution can be used to model data that exhibits bathtub, increasing, decreasing or roller-coaster hazard rate behaviour.

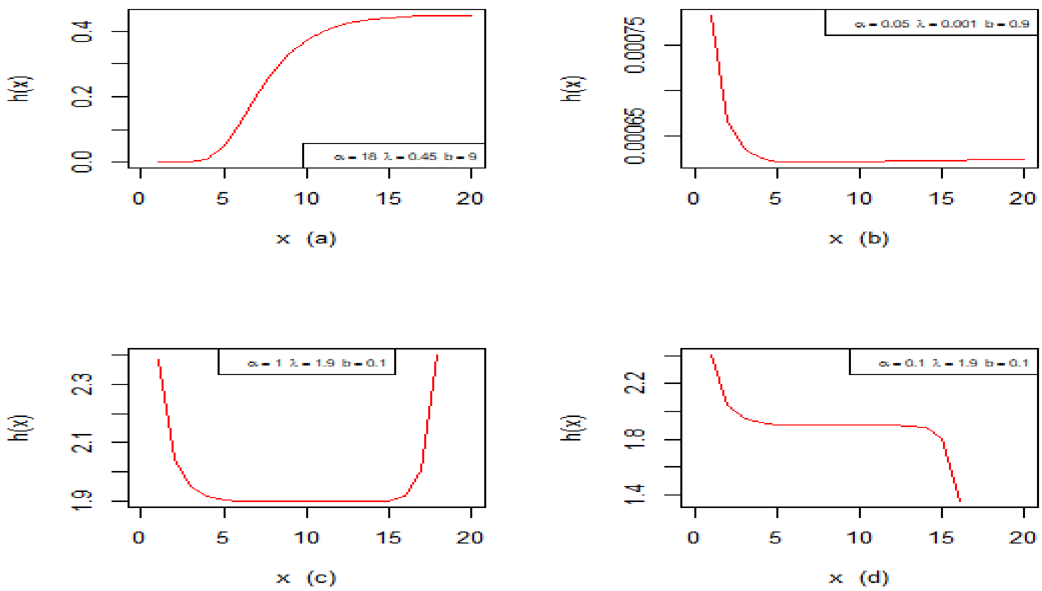


Figure 2: The graphs of the HRF of GumTII-EE{F} Distribution for varying values of alpha, lambda and b.

**Quantile Function.** The quantile function of the GumTII-EE{F} distribution is defined as

$$Q_X(u) = -\ln\left(1 - e^{-(1/\alpha)(b^{-(1/\lambda)})\{-\ln u\}^{1/\lambda}}\right) \tag{41}$$

Proof. The quantile function of the Gumbel Type-II distribution is $Q_T(u) = b^{1/c}(-\ln u)^{-1/c}$. Substituting for $Q_T(u)$ in (15), simplifying and setting $\lambda = \frac{c}{\beta}$, we obtain (41).

**Proposition.** The r[th] Non-central moment of the GumTII-EE{F} distribution is as defined below:

$$E\left(X^r\right) = r\sum_{i=0}^{r}\binom{i-r}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{r-j}\binom{i}{j}q_{j,i}\sum_{v=0}^{\infty}\frac{(-1)^v\left(\frac{r+i}{\alpha}\right)^v}{b^{(v\beta)/c}v!}\Gamma\left(\frac{v\beta}{c}+1\right). \tag{42}$$

Proof. Recall (16). With $T$ following the Gumbel Type-II distribution,

$$E\left\{e^{-\left(\frac{r+i}{\alpha}\right)\left(t^{-\beta}\right)}\right\} = \int_0^\infty e^{-\left(\frac{r+i}{\alpha}\right)\left(t^{-\beta}\right)} bct^{-c-1}e^{-bt^{-c}}\,dt\,. \tag{43}$$

Applying the series expansion $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$, making the substitution $z = bt^{-c}$ and simplifying, we obtain $E\left\{ e^{-\left(\frac{r+i}{\alpha}\right)\left(t^{-\beta}\right)} \right\} = \sum_{v=0}^{\infty} \frac{(-1)^v \left(\frac{r+i}{\alpha}\right)^v}{b^{\frac{v\beta}{c}} v!} \Gamma\left(\frac{vb}{c}+1\right)$ and then the result in (42).

**Proposition.** The mode of the GumTII-EE{F} distribution is the solution(s) of the equation below

$$x = \ln\left\{ \frac{\alpha bc\beta^{-2} f_R(x)\left(-\ln(F_R(x))\right)^{\frac{c-\beta+1}{\beta}} e^{-b\left(-\ln(F_R(x))\right)^{\frac{c}{\beta}}} \left\{ bc\left(-\ln(F_R(x))\right)^{\frac{c}{\beta}} - c - 1 \right\}}{F_R(x) f_T\left(\left(-\ln(F_R(x))\right)^{\frac{-1}{\beta}}\right)\left(-\ln(F_R(x))\right)^{\frac{\beta+1}{\beta}}} - \frac{\alpha\left(\frac{\beta+1}{\beta}\right)}{\ln(F_R(x))} \right\} \tag{44}$$

Proof. Since $T$ follows the Gumbel Type-II distribution, the result in (44) is obtained by evaluating and applying the function $f_T'\left(\left(-\ln(F_R(x))\right)^{-\frac{1}{\beta}}\right)$ in equation (20).

**Proposition.** The mean absolute deviation from the mean and median of the GumTII-EE{F} distribution is as respectively defined below

$$MAD_{(\mu)} = 2\mu F_R(\mu) - 2\sum_{i=0}^{1}\binom{i-1}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{1-j}\binom{i}{j}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{i+1}{\alpha}\right)^k}{b^{\frac{\beta k}{c}}k!}$$

$$*P_{j,i}\Gamma\left(\left(\tfrac{\beta k}{c}+1\right),\left(-\ln\left(F_R(\mu)\right)\right)^{-(1/\beta)}\right) \tag{45}$$

$$MAD_{(M)} = \mu - 2\sum_{i=0}^{1}\binom{i-1}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{1-j}\binom{i}{j}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{i+1}{\alpha}\right)^k}{b^{\frac{\beta k}{c}}k!}$$

$$*P_{j,i}\Gamma\left(\left(\tfrac{\beta k}{c}+1\right),\left(-\ln\left(F_R(M)\right)\right)^{-(1/\beta)}\right) \tag{46}$$

Proof. Recall (27) with

$$S_{\exp\left(-(1/\alpha)(v^{-\beta})\right)}(q,0,i+1) = \int_0^{\left(-\ln\left(F_R(q)\right)\right)^{-(1/\beta)}} \left\{ \left[ e^{-(1/\alpha)(v^{-\beta})} \right]^{i+1} f_T(v) \right\} dv \tag{47}$$

Since $T$ is a Gumbel Type-II random variable, by first substituting for $f_T(v)$ in (47) and then applying the series expansion $e^z=\sum_{k=0}^{\infty}\frac{z^k}{k!}$ to $\left[e^{-(1/\alpha)(v^{-\beta})}\right]^{i+1}$, we obtain

$$S_{\exp(-(1/\alpha)(v^{-\beta}))}(q,0,i+1)=bc\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{i+1}{\alpha}\right)^k}{k!}\int_0^{(-\ln(F_R(q)))^{-(1/\beta)}} v^{-\beta k-c-1}e^{-bv^{-c}}\,dv\,. \tag{48}$$

Making the substitution $w=bv^{-c}$, simplifying and applying $\Gamma(\alpha,z)=\int_0^z w^{\alpha-1}e^{-w}dw$, we obtain

$$S_{\exp(-(1/\alpha)(v^{-\beta}))}(q,0,i+1)=\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{i+1}{\alpha}\right)^k}{b^{\frac{\beta k}{c}}k!}\Gamma\left(\left(\tfrac{\beta k}{c}+1\right),\left(-\ln(F_R(q))\right)^{-(1/\beta)}\right) \tag{49}$$

Then putting (49) in (27) and applying to $MAD_{(\mu)}=2\mu F(\mu)-2\mathrm{I}_{\mu}$ and $MAD_{(M)}=\mu-2\mathrm{I}_M$, we obtain the results in (45) and (46) respectively.

**Proposition.** The Shannon's entropy of a random variable following the GumTII-EE{F} distribution is as defined below

$$\eta_X=\eta_T-\log(\alpha)+\mu_X+\log(\beta)-(\beta+1)\psi(\log T,\,f_T(t))-(\alpha^{-1})(b^{-\frac{\beta}{c}})\Gamma(\tfrac{\beta}{c}+1) \tag{50}$$

where $\psi(\log T,\,f_T(t))=\int_0^{\infty}\log(T)\,f_T(t)dt=\mathrm{E}(\log(t))$.

Proof. Recall (28). With $T$ following the Gumbel Type-II distribution, we have

$\mathrm{E}\{\log(T)\}=\int_0^{\infty}(\log T)\,f_T(t)dt=\psi(\log T,\,f_T(t))$ and

$$\mathrm{E}(T^{-\beta})=bc\int_0^{\infty}t^{-\beta-c-1}e^{-bt^{-c}}\,dt\,. \tag{51}$$

Making the substitution $w=bt^{-c}$ in (51) and simplifying, we obtain $\mathrm{E}(T^{-\beta})=b^{-\frac{\beta}{c}}\Gamma\left(\frac{\beta}{c}+1\right)$. Then substituting accordingly for $\mathrm{E}\{\log(T)\}$ and $\mathrm{E}(T^{-\beta})$ in (28) we obtain the result in (50).

**Estimation**

If we take $X_1,X_2,X_3,\ldots,X_n$ to be a random sample of size $n$ from the GumbelTypeII-ExponentiatedExponential{Frechet} distribution with P.D.F. as given in equation (36), then the likelihood function and the corresponding log-likelihood function for the GumTII-EE{F} distribution, denoted by $L$ and $\ln\mathrm{L}$ respectively, are

$$L=\prod_{i=1}^{n} f_X(x_i)=\alpha^n\lambda^n b^n e^{-\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left(1-e^{-x_i}\right)^{-n}\prod_{i=1}^{n}\left(-\ln\left(1-e^{-x_i}\right)^{\alpha}\right)^{\lambda-1} e^{-b\sum_{i=1}^{n}\left(-\ln\left(1-e^{-x_i}\right)^{\alpha}\right)^{\lambda}} \tag{52}$$

$$\ln\mathrm{L}=n\ln\alpha+n\ln\lambda+n\ln b-\sum_{i=1}^{n}x_i-n\sum_{i=1}^{n}\left(1-e^{-x_i}\right)+(\lambda-1)\sum_{i=1}^{n}\left(-\ln\left(1-e^{-x_i}\right)^{\alpha}\right)$$
$$-b\sum_{i=1}^{n}\left(-\ln\left(1-e^{-x_i}\right)^{\alpha}\right)^{\lambda}. \tag{53}$$

Taking the partial derivative of (53) w.r.t. each of the parameters gives us the likelihood equations for the GumTII-EE{F} distribution below:

$$\frac{\partial\ln L}{\partial\alpha}=\frac{n}{\alpha}-(\lambda-1)\sum_{i=1}^{n}\left(-\ln\left(1-e^{-x_i}\right)\right)+b\lambda\sum_{i=1}^{n}\left(-\ln\left(1-e^{-x_i}\right)^{\alpha}\right)^{\lambda-1}\ln\left(1-e^{-x_i}\right) \tag{54}$$

$$\frac{\partial\ln L}{\partial\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\left(-\ln\left(1-e^{-x_i}\right)^{\alpha}\right)-b\sum_{i=1}^{n}\left(-\ln\left(1-e^{-x_i}\right)^{\alpha}\right)^{\lambda}\ln\left(-\ln\left(1-e^{-x_i}\right)^{\alpha}\right) \tag{55}$$

$$\frac{\partial\ln L}{\partial b}=\sum_{i=1}^{n}\left(-\ln\left(1-e^{-x_i}\right)^{\alpha}\right)^{\lambda}. \tag{56}$$

Then the maximum likelihood estimates, $\hat{\xi}$ of $\xi$ (the vector of the parameters) is the simultaneous solution of the equations $\frac{\partial\ln L}{\partial\alpha}=0$, $\frac{\partial\ln L}{\partial\lambda}=0$ and $\frac{\partial\ln L}{\partial b}=0$. However, the equations are not in closed-form and cannot be solved analytically. Therefore, iterative numerical methods are used for solving the equations in R software.

## Application

We present in this section some applications of the GumTII-EE{F} distribution by fitting the distribution to two real life data sets. The method of maximum likelihood discussed above is used for the estimation of the parameters of the distribution. For the purpose of comparing the distribution with some existing distributions, we report the values of the goodness-of-fit indices, namely the Akaike Information Criterion and the Bayesian Information Criterion, for the fitted distributions. The smaller the value of these indices, the better the fit of the distribution to the data.

**Data Set 1:** The first data set pertains to relief times (in minutes) of twenty patients receiving an analgesic. The data was reported in [28] and has been widely applied by many authors to illustrate the modelling capabilities of their proposed probability distributions. Some of the existing distributions fitted to the data include the Odd FrechetInverseWeibull (OFIW) distribution [29], the Extended Topp Leone Exponentiated Generalized Exponential (ETLGenExEx) distribution [30], Extended Exponentiated Chen (EE-C) distribution [31] and Burr-XII Exponentiated Exponential (BrXIIEE) distribution [32]. The results of fitting the GumTII-EE{F} distribution and these existing distributions are presented in Table 1. As shown in Table 1, the best fit to the data was provided by the GumTII-EE{F} distribution.

**Table 1: The MLEs and the goodness-of-fit indices for the Relief Times Data – Data Set 1**

| Models | Parameter Estimates | AIC | BIC |
|---|---|---|---|
| GumTII-EE{F} | $\hat{\alpha}=1.722768$<br>$\hat{\lambda}=2.069364$<br>$\hat{\text{b}}=7.206328$ | 38.00 | 40.99 |
| OFIW | $\hat{\alpha}=25.815$<br>$\hat{\beta}=13.215$<br>$\hat{\theta}=0.208$ | 44.25 | 42.16 |
| ETLGenExEx | $\hat{\alpha}=4.8070$<br>$\hat{\beta}=0.8105$<br>$\hat{\delta}=0.6391$<br>$\hat{\lambda}=1.8022$<br>$\hat{\theta}=3.6225$ | 44.49 | 49.47 |
| EE-C | $\hat{\alpha}=43:94$<br>$\hat{\beta}=4:44$<br>$\hat{a}=1:30$<br>$\hat{b}=0:39$ | 39.41 | 43.39 |
| BrXIIEE | $\hat{\alpha}=3.911$<br>$\hat{\beta}=0.273$<br>$\hat{a}=3.777$<br>$\hat{b}=1.298$ | 38.9 | 42.9 |

**The maximum likelihood estimates and the goodness-of-fit indices for the competing distributions were respectively obtained from their various authors.*

**Data Set 2:** The second data was sourced from [33] and represents the times between failures for repairable items. It is a reliability data from the engineering discipline and many distributions have been fitted to the data in the literature. Among the distributions fitted to the data include Gamma Generalized Pareto (GGP) distribution [33], Exponentiated Generalized Gumbel (EGGu) distribution [34], Exponentiated Generalized Fréchet Geometric (EGFG) distribution [35] and Exponentiated Weibull Power Function (EWPF) Distribution [36]. In Table 2, the results of fitting these distributions and the GumTII-EE{F} distribution to the data are presented. According to the results, the GumTII-EE{F} distribution provided the best fit to the data when compared with the other distributions.

**Table 2: The MLEs and the goodness-of-fit indices for the Times between Failure of Repairable Items' Data – Data Set 2**

| Models | Parameter Estimates | AIC | BIC |
|---|---|---|---|
| GumTII-EE{F} | $\hat{\alpha} = 0.7749828$<br>$\hat{\lambda} = 1.0012793$<br>$\hat{b} = 2.7325119$ | 85.23 | 89.43 |
| GGP | $\hat{\alpha} = 2.100$<br>$\hat{\sigma} = 0.698$<br>$\hat{\xi} = 0.028$ | 85.25 | 89.45 |
| EGGu | $\hat{\alpha} = 0.2914$<br>$\hat{\beta} = 1.3294$<br>$\hat{\mu} = 0.3146$<br>$\hat{\sigma} = 0.3004$ | 87.55 | 93.16 |
| EGFG | $\hat{\kappa} = 42.5002$<br>$\hat{\delta} = 0.2211$<br>$\hat{\phi} = 0.6997$<br>$\hat{\Gamma} = 18.6780$<br>$\hat{\tau} = 0.1020$ | 91.31 | 98.32 |
| EWPF | $a = 39.68$<br>$\alpha = 5.629$<br>$\beta = 0.248$<br>$\lambda = 4.73$<br>$\theta = 1.033$ | 89.25 | 96.25 |

**The maximum likelihood estimates and the goodness-of-fit indices for the competing distributions were respectively obtained from their various authors.*

## CONCLUSION

This article set out to generalize the Exponentiated Exponential distribution using the T-R{Y} framework. The T-Exponentiated Exponential{Frechet} family of distributions was derived and the GumbelTypeII-Exponentiated Exponential{Frechet} distribution was subsequently defined as a new generalized Exponentiated Exponential distribution. Various properties of the T-Exponentiated Exponential{Frechet} family of distributions were derived which are applicable to prospective members of the family. Some properties which are specific to the GumbelTypeII-ExponentiatedExponential{Frechet} distribution were also derived. These include quantile function, hazard rate function, moments, mode, Shannon's entropy and mean absolute deviations due to mean and median.

The new distribution was fitted to two real life data sets from the medical and engineering disciplines for comparison with some existing distributions. The results obtained from fitting to the two data sets showed that, on the basis of AIC and BIC goodness-of-fit indices, the GumbelTypeII-ExponentiatedExponential{Frechet} distribution provided better fits to the two data sets than the distributions compared with it.

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