# INVESTIGATION OF LOWER AND UPPER BOUNDS OF A JUMP GRAPH USING TOPOLOGICAL INDICES

**Abstract**

**Abstract**:The main aim of this paper is to find new bounds of a jump graph using some topological indices like Hyper Zagreb index, Nirmala Index, VL Index and Forgotten topological index.The Topological indices are mathematical techniques used to mathematically correlate the relationship between the chemical structure and various physical attributes, chemical reactivity, or biological activity.

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**Keywords:** Hyper Zagreb index, Nirmala Index, VL Index , Forgotten topological index and Graph operations.

## 1 Introduction

Let G be a simple graph connected with vertices and edges. V(G) represents vertices and E(G) represents edges set. Clearly, the number of vertices and the number of edges are the two fundamental parameters in topological indices. Numerous topological indices have been developed and used in recent years for a variety of purposes, including chemical documentation, isomer discrimination, molecular complexity research. In any graph,the number of edges with u as an end vertex is called degree of u and is denoted by $deg_G(u)$ the minimum and maximum degrees of graph are represented as $\delta_G$ and $\Delta_G$ respectively.

## 2 Methodology:

Topological indices based on vertex degrees have been used over 40years. From those indices, few are listed below

**Hyper Zagreb Index:**

In 2013, Shirdel et al intoduced distance based Zagreb indices named Hyper zagreb index as

$$HZ(G) = \sum_{i,j \in E(G)} (d_i + d_j)^2$$

**Nirmala Index:**

Inspired by the work of Sombor indices, V R Kulli introduces the Nirmala index of a graph G as

$$N(G) = \sum_{i,j \in E(G)} (\sqrt{d_i + d_j})$$

**VL Index:**

By the work of Zagreb index, Deepika T introduced the VL index of a graph and is defined as

$$VL(G) = \frac{1}{2} \sum_{i,j \in E(G)} [d_i + d_j + d_i \cdot d_j]$$

**Forgotten Topological index:**

Furtula and Gutman introduced Forgotten topological index and established its some properties . This index is defined as

$$F(G) = \sum_{i,j \in E(G)} [d_i^2 + d_j^2]$$

## 3 Properties of Jump Graph

The graph has

$(i).\lambda_1 + \lambda_1\eta_2 \ \ vertices$

$(ii).\lambda_1(\lambda_2 + \eta_2) + \frac{\lambda_1(\lambda_1 - 1)}{2} - \sum_{i,j \in E(G)} \frac{[deg(i) + deg(j) - 2]}{2} \ \ edges$

$(iii).The\ degree\ of\ a\ vertex,$ v$\in v(G)$ is given by

$$deg_G(i) = deg_H(i) + 1, \ \text{if } i \in V(H)$$
$$deg_{J(G)}(i) + \eta_2, \qquad \text{if } i \in V[J(G)]$$

## 4 Preliminary results:

Results of Hyper Zagreb Index

$$HZ(G) = \sum_{i,j} \in E(G)(d_i + d_j)^2$$

**Theorem 1:** Let G and H be two simple connected graphs, then the bounds for the hyper Zagreb index of jump graph given by

$$HZ(G) \geqslant 4\lambda_1\lambda_2(\Delta + 1)^2 + [\Delta_H - 2\Delta_G + 2 + \lambda_1 + \eta_2]^2 +$$

$$[\frac{\lambda_1(\lambda_1 - 1)}{2} - \lambda_1(\Delta_G - 1)][2\lambda_1 - 4\Delta_G - 2 + 2\eta_2]^2$$

and

$$HZ(G) \leqslant 4\lambda_1\lambda_2(\delta + 1)^2 + [\delta_H - 2\delta_G + 2 + \lambda_1 + \eta_2]^2 +$$

$$[\frac{\lambda_1(\lambda_1 - 1)}{2} - \lambda_1(\delta_G - 1)][2\lambda_1 - 4\delta_G - 2 + 2\eta_2]^2$$

Proof:

$$HZ(G) = \lambda_1 \sum_{i,j \in E(G)} [(deg_H(i) + 1) + (deg_H(j) + 1)]^2 +$$

$$\sum_{e \in V(J(G))} \sum_{i \in V(H)} [(deg_H(i) + 1) + (deg_{J(G)}(e) + \eta_2)]^2$$

$$+ \sum_{e,t \in E(J(G))} [(deg_{J(G)}(e) + \eta_2) + (deg_{J(G)}(t) + \eta_2)]^2$$

$$= \lambda_1\lambda_2[(deg_H(i)+1)+(deg_H(j)+1)]^2 + \lambda_1\eta_2[(deg_H(i)+1)+(deg_{J(G)}(e)+\eta_2]^2 +$$

$$[(\frac{\lambda_1(\lambda_1 - 1)}{2}) - \lambda_1[\frac{deg_G(i) + deg_{J(G)} - 2}{2}]][(deg_{J(G)(e)} + \eta_2) + (deg_{J(G)}(t) + \eta_2)]^2$$

$$= \lambda_1\lambda_2[deg_H(i)+(deg_H(j)+2]^2 + \lambda_1\eta_2[(deg_H(i)+1)+[(\lambda_1-1)-(deg_G(i)+deg_G(j)-2)+\eta_2]^2 +$$

$$[\frac{(\lambda_1(\lambda_1-1)}{2}) - \lambda_1[\frac{deg_G(i)+deg_G(j)-2}{2}]]$$

$$[[(\lambda_1-1)-(deg_G(i)+deg_G(j)-2)+\eta_2]+[(\lambda_1-1)-deg_G(i)+deg_G(j)-2)+\eta_2]]^2$$

$$\geqslant \lambda_1\lambda_2[\Delta_H +_H +2]^2 + \lambda_1\eta_2[(\Delta_H+1)+[(\lambda_1-1)-(\Delta_G+\Delta_G-2)+\eta_2]+$$

$$[(\frac{\lambda_1(\lambda_1-1)}{2})-\lambda 1(\frac{(\Delta_G+\Delta_G-2}{2})][[(\lambda_1-1)-(\Delta_G+\Delta_G-2)+\eta_2]+[(\lambda_1-1)-(\Delta_G+\Delta_G-2)+\eta_2]]^2$$

$$\geqslant \lambda_1\lambda_2 4(\Delta_H+1)^2 + [\Delta_H - 2\Delta_G + 2 + \lambda_1 + \eta_2]^2 +$$

$$[[\frac{\lambda_1(\lambda_1-1)}{2}] - \lambda_1(\Delta_G-1)][\lambda_1 - 2\Delta_G - 3 + \eta_2] + [\lambda_1 - 2\Delta_G + 1 + \eta_2]]^2$$

$$\leqslant \lambda_1\lambda_2 4(\delta_H+1)^2 + [\delta_H - 2\delta_G + 2 + \lambda_1 + \eta_2]^2 +$$

$$[[\frac{\lambda_1(\lambda_1-1)}{2}] - \lambda_1(\delta_G-1)][\lambda_1 - 2\delta_G - 3 + \eta_2] + [\lambda_1 - 2\delta_G + 1 + \eta_2]]^2$$

**Theorem 2:** Let G and H be two simple connected graphs, then the bounds for the Nirmala index of a Jump graph is given by

$$N(G) \geqslant \sqrt{2}\lambda_1\lambda_2(\Delta_H+1) + \lambda_1\lambda_2\sqrt{\Delta_H - 2\Delta_G + \lambda_1 + 2 + \eta_2} +$$

$$[[\frac{\lambda_1(\lambda_1-1)}{2}] - \lambda_1(\Delta_G-1)]\sqrt{2}\sqrt{\lambda_1 - 2\Delta_G + 1 + \eta_2}$$

and

$$N(G) \leqslant \sqrt{2}\lambda_1\lambda_2(\delta_H+1) + \lambda_1\lambda_2\sqrt{\delta_H - 2\delta_G + \lambda_1 + 2 + \eta_2} +$$

$$[[\frac{\lambda_1(\lambda_1-1)}{2}] - \lambda_1(\delta_G-1)]\sqrt{2}\sqrt{\lambda_1 - 2\delta_G + 1 + \eta_2}$$

proof:

$$N(G) = \sum_{i,j \in E(G)} \sqrt{d_i + d_j}$$

$$N(G) = \lambda_1 \sum_{i,j \in E(G)} \sqrt{[(deg_H(i) + 1) + (deg_H(j) + 1)]} +$$

$$\sum_{e \in V(J(G))} \sum_{i \in V(H)} \sqrt{[(deg_H(i) + 1) + (deg_{J(G)}(e) + \eta_2)]}$$

$$+ \sum_{e,t \in E(J(G))} \sqrt{[(deg_{J(G)}(e) + \eta_2) + (deg_{J(G)}(t) + \eta_2)]}$$

$$= \lambda_1\lambda_2\sqrt{[(deg_H(i) + 1) + (deg_H(j) + 1)]} + \lambda_1\eta_2\sqrt{[(deg_H(i) + 1) + (deg_{J(G)}(e) + \eta_2]} +$$

$$[(\frac{\lambda_1(\lambda_1 - 1)}{2}) - \lambda_1[\frac{deg_G(i) + deg_{J(G)} - 2}{2}]]\sqrt{[(deg_{J(G)(e)} + \eta_2) + (deg_{J(G)}(t) + \eta_2)]}$$

$$= \lambda_1\lambda_2\sqrt{[deg_H(i) + (deg_H(j) + 2]} + \lambda_1\eta_2\sqrt{[(deg_H(i) + 1) + [(\lambda_1 - 1) - (deg_G(i) + deg_G(j) - 2) + \eta_2]}$$

$$+[\frac{(\lambda_1(\lambda_1 - 1)}{2}) - \lambda_1[\frac{deg_G(i) + deg_G(j) - 2}{2}]]$$

$$[\sqrt{[(\lambda_1 - 1) - (deg_G(i) + deg_G(j) - 2) + \eta_2] + [(\lambda_1 - 1) - deg_G(i) + deg_G(j) - 2) + \eta_2]}]$$

$$\geqslant \lambda_1\lambda_2\sqrt{2}(\Delta_H + 1) + \lambda_1\lambda_2\sqrt{\Delta_H - 2\Delta_G + \lambda_1 + 2 + \eta_2} +$$

$$[[\lambda_1(\frac{\lambda_1 - 1)}{2}] - \lambda_1(\Delta_G - 1)][\sqrt{2(\lambda_1 - 1) - 4(\Delta_G - 1) + 2\eta_2}$$

$$\leqslant \lambda_1\lambda_2\sqrt{2}(\delta_H + 1) + \lambda_1\lambda_2\sqrt{\delta_H - 2\delta_G + \lambda_1 + 2 + \eta_2} +$$

$$[[\lambda_1(\frac{\lambda_1-1)}{2}] - \lambda_1(\delta_G - 1)][\sqrt{2(\lambda_1-1) - 4(\delta_G-1) + 2\eta_2}$$

**Theorem 3:** Let G and H be two simple connected graphs , then the bounds for VL index is given by

$$VL(G) \geqslant \frac{1}{2}[[\lambda_1\lambda_2(4\Delta_H + \Delta_H^2 + 3) + \lambda_1\eta_2(\lambda_1 - \Delta_H + \eta_2 + 2)+$$

$$(\Delta_H+1)[\lambda_1-2\Delta_H+1+\eta_2]+[[\frac{\lambda_1(\lambda_1-1)}{2}]-\lambda_1(\Delta_G-1)][2\lambda_1-4\Delta_G+2\eta_2+2]+$$

$$[(\lambda_1-1)^2-4(\lambda_1-1)(\Delta_G-1)+4(\Delta_G-1)^2+2\eta_2(\lambda_1-1)-2\eta_2(\Delta_G-1)+\eta_2^2]]$$

and

$$VL(G) \leqslant \frac{1}{2}[\lambda_1\lambda_2(4\delta_H + \delta_H^2 + 3) + \lambda_1\eta_2(\lambda_1 - \delta_H + \eta_2 + 2)+$$

$$(\delta_H+1)[\lambda_1-2\delta_H+1+\eta_2]+[[\frac{\lambda_1(\lambda_1-1)}{2}]-\lambda_1(\delta_G-1)][2\lambda_1-4\delta_G+2\eta_2+2]+$$

$$[(\lambda_1-1)^2-4(\lambda_1-1)(\delta_G-1)+4(\delta_G-1)^2+2\eta_2(\lambda_1-1)-2\eta_2(\delta_G-1)+\eta_2^2]]$$

Proof:

$$VL(G) = \frac{1}{2}\sum_{i,j\in E(G)}[d_i + d_j + d_i * d_j]$$

$$= \frac{1}{2}[\lambda_1\sum_{i,j\in E(H)}[deg_H(i)+1+(deg_H(j)+1)+(deg_H(j)+1)+(deg_H(i)+1)*(deg_H(i)+1)+$$

$$\sum_{e\in J(G)}\sum_{i\in V(H)}[(deg_H(i)+1)+(deg_{J(G)}(e)+\eta_2)+(deg_{J(G)}(i)+1)(deg_{J(G)}(e)+\eta_2)]+$$

$$\sum_{e,t\in J(G)}[(deg_{J(G)}(e)+\eta_2)+(deg_{J(G)}(t)+\eta_2)+(deg_{J(G)}(e)+\eta_2)(deg_{J(G)}(t)+\eta_2)]$$

$$= \frac{1}{2}[\lambda_1\lambda_2[deg_H(i)+1+(deg_H(j)+1)+(deg_H(j)+1)+(deg_H(i)+1)(deg_H(i)+1)]]+$$

$$\lambda_1\eta_2[(deg_H(i)+1)+(deg_{J(G)}(e)+\eta_2)+(deg_{J(H)}(i)+1)*(deg_{J(G)}(e)+\eta_2)]+$$

$$[[\frac{\lambda_1(\lambda_1-1)}{2}]-\lambda_1[\frac{deg_G(i)+deg_G(j)-2}{2}[(deg_{J(G)}(e)+\eta_2)+(deg_{J(G)}(t)+\eta_2)+$$

$$(deg_{J(G)}(e)+\eta_2)(deg_{J(G)}(t)+\eta_2)]$$

$$\frac{1}{2}[\lambda_1\lambda_2[deg_H(i)+deg_H(j)+2+deg_H(j)deg_H(i)+deg_H(j)+deg_H(i)+1]$$

$$\lambda_1\eta_2[deg_H(i)+1+[(\lambda_1-1)-(deg_H(i)+deg_H(j)-2]+\eta_2]+$$

$$[[\frac{\lambda_1(\lambda_1-1)}{2}]-\lambda_1[\frac{deg_G(i)+deg_G(j)-2}{2}]][[(\lambda_1-1)-(deg_G(i)+deg_G(j)-2)+\eta_2]+$$

$$[(\lambda_1-1)-(deg_G(i)+deg_G(j)-2)+\eta_2]+[[(\lambda_1-1)-(deg_G(i)+deg_G(j)-2)+\eta_2]$$

$$[(\lambda_1-1)-(deg_G(i)+deg_G(j)-2)+\eta_2]]$$

$$=\frac{1}{2}[\lambda_1\lambda_2[4\Delta_H+\Delta_H^2+3]+\lambda_1\eta_2[(\Delta_H+1)((\lambda_1-1)-2\Delta_H+2)+\eta_2]+$$

$$[[\frac{\lambda_1(\lambda_1-1)}{2}]-\lambda_1(\Delta_G-1)][(\lambda_1-1)-(2\Delta_G-2)+\eta_2]+[(\lambda_1-1)-2\Delta_G+2+\eta_2]+$$

$$[(\lambda_1-1)-2(\Delta_G-1)+\eta_2)((\lambda_1-1)-2(\Delta_G-1)+\eta_2)]$$

$$\geqslant \frac{1}{2}[\lambda_1\lambda_2[4\Delta_H+\Delta_H^2+3]+\lambda_1\eta_2(\lambda_1-\Delta_H+\eta_2)+[(\Delta_H+1)(\lambda_1-2\Delta_H+1)+\eta_2]+$$

$$[[\frac{\lambda_1(\lambda_1-1)}{2}]-\lambda_1(\Delta_G-1)][2\lambda_1-4\Delta_G+2\eta_2+2]+$$

$$[(\lambda_1-1)^2-4(\lambda_1-1)(\Delta_G-1)+4(\Delta_G-1)^2+2\eta_2(\lambda_1-1)-2\eta_2(\Delta_G-1)+\eta_2^2]]$$

$$\leqslant \frac{1}{2}[\lambda_1\lambda_2[4\delta_H+\delta_H^2+3]+\lambda_1\eta_2(\lambda_1-\delta_H+\eta_2)+[(\delta_H+1)(\lambda_1-2\delta_H+1)+\eta_2]+$$

$$[[\frac{\lambda_1(\lambda_1-1)}{2}]-\lambda_1(\delta_G-1)][2\lambda_1-4\delta_G+2\eta_2+2]+$$

$$[(\lambda_1-1)^2-4(\lambda_1-1)(\delta_G-1)+4(\delta_G-1)^2+2\eta_2(\lambda_1-1)-2\eta_2(\delta_G-1)+\eta_2^2]]$$

**Theorem 4:** Let G and H be two simple connected graphs the bounds for the forgotten index of a Jump graph is given by

$$F(G) \geqslant 2\lambda_1\lambda_2(\Delta_H+1)^2+\lambda_1\eta_2[\Delta_H^2+2\Delta_H-2\Delta_G+3\lambda_1+\eta_2]^2+$$

$$[[\frac{\lambda_1(\lambda_1-1)}{2}]-\lambda_1(\Delta_G-1)][\lambda_1-2\Delta_G+\eta_2+1]^2+[\lambda_1-2\Delta_G+\eta_2+1]^2$$

$$F(G) \leqslant 2\lambda_1\lambda_2(\delta_H+1)^2+\lambda_1\eta_2[\delta_H^2+2\delta_H-2\delta_G+3\lambda_1+\eta_2]^2+$$

$$[[\frac{\lambda_1(\lambda_1-1)}{2}]-\lambda_1(\delta_G-1)][\lambda_1-2\delta_G+\eta_2+1]^2+[\lambda_1-2\delta_G+\eta_2+1]^2$$

Proof:

$$F(G) = \sum_{i,j \in E(G)} [d_i^2 + d_j^2]$$

$$F(G) = \lambda_1 \sum_{i,j \in E(G)} [(deg_H(i) + 1)^2 + (deg_H(j) + 1)^2] +$$

$$\sum_{e,t \in E(G)} [(deg_H(i) + 1)^2 + (deg_{J(G)}(t) + \eta_2)^2] +$$

$$\sum_{e \in v(i,j)} \sum_{i \in v(G)} [(deg_H(i) + 1)^2 + (deg_{J(G)}(e) + \eta_2)^2]$$

$$= \lambda_1 \lambda_2 [(\deg_H(i) + 1)^2 + (deg_H(j) + 1)^2] +$$

$$\lambda_1 \eta_2 [(deg_H(i) + 1)^2 + (deg_{J(G}(e) + \eta_2)^2] +$$

$$[[\frac{\lambda_1(\lambda_1 - 1)}{2}] - \lambda_2 [\frac{deg_G(i) + deg_G(j) - 2]}{2}]][deg_{(J(G)}(e) + \eta_2)^2 + (deg_{J(G)}(t) + \eta_2)^2]$$

$$= \lambda_1 \lambda 2 [(deg_G(i) + 1)^2 + (deg_H(j) + 1)^2] +$$

$$\lambda_1 \eta_2 [(deg_H(i) + 1)^2 + [(\lambda_1 - 1) - (deg_G(i) + deg_G(j) - 2) + \eta_2]^2 +$$

$$[[\frac{\lambda_1(\lambda_1 - 1)}{2}] - \lambda_1 [\frac{deg_G(i) + deg_G(j) - 2}{2}]]$$

$$[(\lambda_1 - 1) - [deg_G(i) + deg_G(j) - 2] + \eta_2]^2 + [(\lambda_1 - 1) - [deg_G(i) + deg_G(j) - 2] + \eta_2]^2$$

$$\geqslant \lambda_! \lambda_2 [(\Delta_H + 1)^2 + (\Delta_H + 1)^2] + \lambda_1 \eta_2 [(\Delta_H + 1)^2 + ((\lambda_1 - 1) - (\Delta_G + \Delta_G - 2) + \eta_2]^2 +$$

$$[[\frac{\lambda_1(\lambda_1-1)}{2}]-\lambda_1[\frac{\Delta_G+\Delta_G-2}{2}]][(\lambda_1-1)-(\Delta_G+\Delta_G-2)+\eta_2]^2+$$

$$[(\lambda_1-1)-(\Delta_G+\Delta_G-2)+\eta_2]^2$$

$$\geqslant 2\lambda_1\lambda_2(\Delta_H+1)^2+\lambda_1\eta_2[\Delta_H^2+2\Delta_H+3\lambda_1-2\Delta_G+\eta_2]^2+$$

$$[[\frac{\lambda_1(\lambda_1-1)}{2}]-\lambda_1(\Delta_G-1)][\lambda_1-2\Delta_G-1)][\lambda_1-2\Delta_G+\eta_2+1]^2+[\lambda_1-2\Delta_G+\eta_2+1]^2$$

$$\leqslant 2\lambda_1\lambda_2(\delta_H+1)^2+\lambda_1\eta_2[\delta_H^2+2\delta_H+3\lambda_1-2\delta_G+\eta_2]^2+$$

$$[[\frac{\lambda_1(\lambda_1-1)}{2}]-\lambda_1(\delta_G-1)][\lambda_1-2\delta_G-1)][\lambda_1-2\delta_G+\eta_2+1]^2+[\lambda_1-2\delta_G+\eta_2+1]^2$$

## 5 Conclusion:

In this article we considered four topological indices to determine the lower and upper bounds. In the same way researchers can consider other topological indices and determine their bounds for the graph.

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