

LOWER AND UPPER BOUNDS OF A JUMP GRAPH USING TOPOLOGICAL INDICES

Abstract

Abstract: The main aim of this paper is to find new bounds of a jump graph using some topological indices like Hyper Zagreb index, Nirmala Index, VL Index and Forgotten topological index. Topological Indices are used to convert molecular structure into a real value.

Keywords: Hyper Zagreb index, Nirmala Index, VL Index and Forgotten topological index.

1 Introduction

Let G be a simple graph connected with vertices and edges. $V(G)$ represents vertices and $E(G)$ represents edges set, the number of edges with u as an end vertex is called degree of u and is denoted as $deg_G(u)$ the minimum and maximum degrees of graph are represented as δ_G and Δ_G respectively.

2 Methodology

Hyper Zagreb Index:

In 2013, Shirdel et al introduced distance based Zagreb indices named Hyper zagreb index as

$$HZ(G) = \sum_{i,j \in E(G)} (d_i + d_j)^2$$

Nirmala Index:

Inspired by the work of Sombor indices, V R Kulli introduces the Nirmala index of a graph G as

$$N(G) = \sum_{i,j \in E(G)} (\sqrt{d_i + d_j})$$

VL Index:

By the work of Zagreb index, Deepika T introduced the VL index of a graph and is defined as

$$VL(G) = \frac{1}{2} \sum_{i,j \in E(G)} [d_i + d_j + d_i * d_j]$$

Forgotten Topological index:

Furtula and Gutman introduced Forgotten topological index and established its some properties . This index is defined as

$$F(G) = \sum_{i,j \in E(G)} [d_i^2 + d_j^2]$$

3 Properties of Jump Graph

The graph has

$$(i). \lambda_1 + \lambda_1 \eta_2 \text{ vertices}$$

$$(ii). \lambda_1(\lambda_2 + \eta_2) + \frac{\lambda_1(\lambda_1 - 1)}{2} - \sum_{i,j \in E(G)} \frac{[deg(i) + deg(j) - 2]}{2} \text{ edges}$$

(iii). The degree of a vertex $v \in v(G)$ is given by

$$deg_G(i) = deg_H(i) + 1, \text{if } i \in V(H)$$

$$deg_{J(G)}(i) + \eta_2, \text{if } i \in V(J(G))$$

. (1)

4 Preliminary results:

Results of Hyper Zagreb Index

$$HZ(G) = \sum_{i,j \in E(G)} (d_i + d_j)^2$$

Theorem 1: Let G and H be two simple connected graphs, then the bounds for the hyper Zagreb index of jump graph given by

$$HZ(G) \geq 4\lambda_1\lambda_2(\Delta + 1)^2 + [\Delta_H - 2\Delta_G + 2 + \lambda_1 + \eta_2]^2 +$$

$$\left[\left(\frac{\lambda_1(\lambda_1 - 1)}{2} - \lambda_1(\Delta_G - 1) \right) [2\lambda_1 - 4\Delta_G - 2 + 2\eta_2] \right]^2$$

and

$$HZ(G) \leq 4\lambda_1\lambda_2(\delta + 1)^2 + [\delta_H - 2\delta_G + 2 + \lambda_1 + \eta_2]^2 +$$

$$\left[\left(\frac{\lambda_1(\lambda_1 - 1)}{2} - \lambda_1(\delta_G - 1) \right) [2\lambda_1 - 4\delta_G - 2 + 2\eta_2] \right]^2$$

Proof:

$$HZ(G) = \lambda_1 \sum_{i,j \in E(G)} [(deg_H(i) + 1) + (deg_H(j) + 1)]^2 +$$

$$\sum_{e \in V(J(G))} \sum_{i \in V(H)} [(deg_H(i) + 1) + (deg_{J(G)}(e) + \eta_2)]^2$$

$$+ \sum_{e,t \in E(J(G))} [(deg_{J(G)}(e) + \eta_2) + (deg_{J(G)}(t) + \eta_2)]^2$$

$$= \lambda_1\lambda_2[(deg_H(i)+1)+(deg_H(j)+1)]^2 + \lambda_1\eta_2[(deg_H(i)+1)+(deg_{J(G)}(e)+\eta_2)]^2 +$$

$$\left[\left(\frac{\lambda_1(\lambda_1 - 1)}{2} - \lambda_1 \left[\frac{deg_G(i) + deg_{J(G)} - 2}{2} \right] \right) [(deg_{J(G)}(e) + \eta_2) + (deg_{J(G)}(t) + \eta_2)] \right]^2$$

$$= \lambda_1\lambda_2[deg_H(i) + (deg_H(j) + 2)]^2 + \lambda_1\eta_2[(deg_H(i) + 1) + [(\lambda_1 - 1) - (deg_G(i) + deg_G(j) - 2) + \eta_2]]^2 +$$

$$\left[\left(\frac{\lambda_1(\lambda_1 - 1)}{2} - \lambda_1 \left[\frac{deg_G(i) + deg_G(j) - 2}{2} \right] \right) \right]$$

$$\left[[(\lambda_1 - 1) - (deg_G(i) + deg_G(j) - 2) + \eta_2] + [(\lambda_1 - 1) - deg_G(i) + deg_G(j) - 2 + \eta_2] \right]^2$$

$$\begin{aligned}
 &\geq \lambda_1 \lambda_2 [\Delta_H + H + 2]^2 + \lambda_1 \eta_2 [(\Delta_H + 1) + [(\lambda_1 - 1) - (\Delta_G + \Delta_G - 2) + \eta_2] + \\
 & [(\frac{\lambda_1(\lambda_1 - 1)}{2}) - \lambda_1(\frac{\Delta_G + \Delta_G - 2}{2})][[(\lambda_1 - 1) - (\Delta_G + \Delta_G - 2) + \eta_2] + [(\lambda_1 - 1) - (\Delta_G + \Delta_G - 2) + \eta_2]]^2 \\
 &\geq \lambda_1 \lambda_2 4(\Delta_H + 1)^2 + [\Delta_H - 2\Delta_G + 2 + \lambda_1 + \eta_2]^2 + \\
 & [[\frac{\lambda_1(\lambda_1 - 1)}{2}] - \lambda_1(\Delta_G - 1)][\lambda_1 - 2\Delta_G - 3 + \eta_2] + [\lambda_1 - 2\Delta_G + 1 + \eta_2]]^2 \\
 &\leq \lambda_1 \lambda_2 4(\delta_H + 1)^2 + [\delta_H - 2\delta_G + 2 + \lambda_1 + \eta_2]^2 + \\
 & [[\frac{\lambda_1(\lambda_1 - 1)}{2}] - \lambda_1(\delta_G - 1)][\lambda_1 - 2\delta_G - 3 + \eta_2] + [\lambda_1 - 2\delta_G + 1 + \eta_2]]^2
 \end{aligned}$$

Theorem 2: Let G and H be two simple connected graphs then the bounds for the Nirmala index of a Jump graph is given by

$$N(G) \geq \sqrt{2} \lambda_1 \lambda_2 (\Delta_H + 1) + \lambda_1 \lambda_2 \sqrt{\Delta_H - 2\Delta_G + \lambda_1 + 2 + \eta_2 +}$$

$$[[\frac{\lambda_1(\lambda_1 - 1)}{2}] - \lambda_1(\Delta_G - 1)] \sqrt{2} \sqrt{\lambda_1 - 2\Delta_G + 1 + \eta_2}$$

and

$$N(G) \leq \sqrt{2} \lambda_1 \lambda_2 (\delta_H + 1) + \lambda_1 \lambda_2 \sqrt{\delta_H - 2\delta_G + \lambda_1 + 2 + \eta_2 +}$$

$$[[\frac{\lambda_1(\lambda_1 - 1)}{2}] - \lambda_1(\delta_G - 1)] \sqrt{2} \sqrt{\lambda_1 - 2\delta_G + 1 + \eta_2}$$

proof:

$$N(G) = \sum_{i,j \in E(G)} \sqrt{d_i + d_j}$$

$$N(G) = \lambda_1 \sum_{i,j \in E(G)} \sqrt{[(deg_H(i) + 1) + (deg_H(j) + 1)] +}$$

$$\begin{aligned}
& \sum_{e \in V(J(G))} \sum_{i \in V(H)} \sqrt{[(deg_H(i) + 1) + (deg_{J(G)}(e) + \eta_2)]} \\
& + \sum_{e, t \in E(J(G))} \sqrt{[(deg_{J(G)}(e) + \eta_2) + (deg_{J(G)}(t) + \eta_2)]} \\
& = \lambda_1 \lambda_2 \sqrt{[(deg_H(i) + 1) + (deg_H(j) + 1)]} + \lambda_1 \eta_2 \sqrt{[(deg_H(i) + 1) + (deg_{J(G)}(e) + \eta_2)]} + \\
& [(\frac{\lambda_1(\lambda_1 - 1)}{2}) - \lambda_1 [\frac{deg_G(i) + deg_{J(G)} - 2}{2}]] \sqrt{[(deg_{J(G)}(e) + \eta_2) + (deg_{J(G)}(t) + \eta_2)]} \\
& = \lambda_1 \lambda_2 \sqrt{[deg_H(i) + (deg_H(j) + 2)]} + \lambda_1 \eta_2 \sqrt{[(deg_H(i) + 1) + [(\lambda_1 - 1) - (deg_G(i) + deg_G(j) - 2) + \eta_2]]} \\
& \quad + [(\frac{\lambda_1(\lambda_1 - 1)}{2}) - \lambda_1 [\frac{deg_G(i) + deg_G(j) - 2}{2}]] \\
& [\sqrt{[(\lambda_1 - 1) - (deg_G(i) + deg_G(j) - 2) + \eta_2]} + [(\lambda_1 - 1) - deg_G(i) + deg_G(j) - 2] + \eta_2] \\
& \geq \lambda_1 \lambda_2 \sqrt{2}(\Delta_H + 1) + \lambda_1 \lambda_2 \sqrt{\Delta_H - 2\Delta_G + \lambda_1 + 2 + \eta_2} + \\
& [[\lambda_1(\frac{\lambda_1 - 1}{2})] - \lambda_1(\Delta_G - 1)][\sqrt{2(\lambda_1 - 1) - 4(\Delta_G - 1) + 2\eta_2}] \\
& \leq \lambda_1 \lambda_2 \sqrt{2}(\delta_H + 1) + \lambda_1 \lambda_2 \sqrt{\delta_H - 2\delta_G + \lambda_1 + 2 + \eta_2} + \\
& [[\lambda_1(\frac{\lambda_1 - 1}{2})] - \lambda_1(\delta_G - 1)][\sqrt{2(\lambda_1 - 1) - 4(\delta_G - 1) + 2\eta_2}]
\end{aligned}$$

Theorem 3: Let G and H be two simple connected graphs then the bounds for VL index is given by

$$VL(G) \geq \frac{1}{2} [[\lambda_1 \lambda_2 (4\Delta_H + \Delta_H^2 + 3) + \lambda_1 \eta_2 (\lambda_1 - \Delta_H + \eta_2 + 2) +$$

$$(\Delta_H + 1)[\lambda_1 - 2\Delta_H + 1] + \eta_2 + \left[\left[\frac{\lambda_1(\lambda_1 - 1)}{2} \right] - \lambda_1(\Delta_G - 1) \right] [2\lambda_1 - 4\Delta_G + 2\eta_2 + 2] +$$

$$[(\lambda_1 - 1)^2 - 4(\lambda_1 - 1)(\Delta_G - 1) + 4(\Delta_G - 1)^2 + 2\eta_2(\lambda_1 - 1) - 2\eta_2(\Delta_G - 1) + \eta_2^2]$$

and

$$VL(G) \leq \frac{1}{2} [\lambda_1 \lambda_2 (4\delta_H + \delta_H^2 + 3) + \lambda_1 \eta_2 (\lambda_1 - \delta_H + \eta_2 + 2) +$$

$$(\delta_H + 1)[\lambda_1 - 2\delta_H + 1] + \eta_2 + \left[\left[\frac{\lambda_1(\lambda_1 - 1)}{2} \right] - \lambda_1(\delta_G - 1) \right] [2\lambda_1 - 4\delta_G + 2\eta_2 + 2] +$$

$$[(\lambda_1 - 1)^2 - 4(\lambda_1 - 1)(\delta_G - 1) + 4(\delta_G - 1)^2 + 2\eta_2(\lambda_1 - 1) - 2\eta_2(\delta_G - 1) + \eta_2^2]$$

Proof:

$$VL(G) = \frac{1}{2} \sum_{i,j \in E(G)} [d_i + d_j + d_i * d_j]$$

$$= \frac{1}{2} [\lambda_1 \sum_{i,j \in E(H)} [deg_H(i) + 1] + (deg_H(j) + 1) + (deg_H(j) + 1) + (deg_H(i) + 1) * (deg_H(i) + 1) +$$

$$\sum_{e \in J(G)} \sum_{i \in V(H)} [(deg_H(i) + 1) + (deg_{J(G)}(e) + \eta_2) + (deg_{J(G)}(i) + 1)(deg_{J(G)}(e) + \eta_2)] +$$

$$\sum_{e,t \in J(G)} [(deg_{J(G)}(e) + \eta_2) + (deg_{J(G)}(t) + \eta_2) + (deg_{J(G)}(e) + \eta_2)(deg_{J(G)}(t) + \eta_2)]$$

$$= \frac{1}{2} [\lambda_1 \lambda_2 [deg_H(i) + 1] + (deg_H(j) + 1) + (deg_H(j) + 1) + (deg_H(i) + 1)(deg_H(i) + 1)] +$$

$$\lambda_1 \eta_2 [(deg_H(i) + 1) + (deg_{J(G)}(e) + \eta_2) + (deg_{J(H)}(i) + 1) * (deg_{J(G)}(e) + \eta_2)] +$$

$$\left[\left[\frac{\lambda_1(\lambda_1 - 1)}{2} \right] - \lambda_1 \left[\frac{deg_G(i) + deg_G(j) - 2}{2} \right] [(deg_{J(G)}(e) + \eta_2) + (deg_{J(G)}(t) + \eta_2) +$$

$$(deg_{J(G)}(e) + \eta_2)(deg_{J(G)}(t) + \eta_2)$$

$$\begin{aligned} & \frac{1}{2}[\lambda_1\lambda_2[deg_H(i) + deg_H(j) + 2 + deg_H(j)deg_H(i) + deg_H(j) + deg_H(i) + 1] \\ & \quad \lambda_1\eta_2[deg_H(i) + 1 + [(\lambda_1 - 1) - (deg_H(i) + deg_H(j) - 2) + \eta_2] + \\ & \quad [[\frac{\lambda_1(\lambda_1 - 1)}{2}] - \lambda_1[\frac{deg_G(i) + deg_G(j) - 2}{2}]]][[(\lambda_1 - 1) - (deg_G(i) + deg_G(j) - 2) + \eta_2] + \\ & \quad [(\lambda_1 - 1) - (deg_G(i) + deg_G(j) - 2) + \eta_2] + [[(\lambda_1 - 1) - (deg_G(i) + deg_G(j) - 2) + \eta_2]] \\ & = \frac{1}{2}[\lambda_1\lambda_2[4\Delta_H + \Delta_H^2 + 3] + \lambda_1\eta_2[(\Delta_H + 1)((\lambda_1 - 1) - 2\Delta_H + 2) + \eta_2] + \\ & \quad [[\frac{\lambda_1(\lambda_1 - 1)}{2}] - \lambda_1(\Delta_G - 1)][(\lambda_1 - 1) - (2\Delta_G - 2) + \eta_2] + [(\lambda_1 - 1) - 2\Delta_G + 2 + \eta_2] + \\ & \quad [(\lambda_1 - 1) - 2(\Delta_G - 1) + \eta_2)((\lambda_1 - 1) - 2(\Delta_G - 1) + \eta_2)] \\ & \geq \frac{1}{2}[\lambda_1\lambda_2[4\Delta_H + \Delta_H^2 + 3] + \lambda_1\eta_2(\lambda_1 - \Delta_H + \eta_2) + [(\Delta_H + 1)(\lambda_1 - 2\Delta_H + 1) + \eta_2] + \\ & \quad [[\frac{\lambda_1(\lambda_1 - 1)}{2}] - \lambda_1(\Delta_G - 1)][2\lambda_1 - 4\Delta_G + 2\eta_2 + 2] + \end{aligned}$$

$$[(\lambda_1 - 1)^2 - 4(\lambda_1 - 1)(\Delta_G - 1) + 4(\Delta_G - 1)^2 + 2\eta_2(\lambda_1 - 1) - 2\eta_2(\Delta_G - 1) + \eta_2^2]$$

$$\leq \frac{1}{2}[\lambda_1\lambda_2[4\delta_H + \delta_H^2 + 3] + \lambda_1\eta_2(\lambda_1 - \delta_H + \eta_2) + [(\delta_H + 1)(\lambda_1 - 2\delta_H + 1) + \eta_2] +$$

$$[[\frac{\lambda_1(\lambda_1 - 1)}{2}] - \lambda_1(\delta_G - 1)][2\lambda_1 - 4\delta_G + 2\eta_2 + 2] +$$

$$[(\lambda_1 - 1)^2 - 4(\lambda_1 - 1)(\delta_G - 1) + 4(\delta_G - 1)^2 + 2\eta_2(\lambda_1 - 1) - 2\eta_2(\delta_G - 1) + \eta_2^2]$$

Results of Forgotten index:

Theorem: Let G and H be two simple connected graphs the the bounds for the forgotten index of a Jump graph is given by

$$F(G) \geq 2\lambda_1\lambda_2(\Delta_H + 1)^2 + \lambda_1\eta_2[\Delta_H^2 + 2\Delta_H - 2\Delta_G + 3\lambda_1 + \eta_2]^2 +$$

$$[[\frac{\lambda_1(\lambda_1 - 1)}{2}] - \lambda_1(\Delta_G - 1)][\lambda_1 - 2\Delta_G + \eta_2 + 1]^2 + [\lambda_1 - 2\Delta_G + \eta_2 + 1]^2$$

$$F(G) \leq 2\lambda_1\lambda_2(\delta_H + 1)^2 + \lambda_1\eta_2[\delta_H^2 + 2\delta_H - 2\delta_G + 3\lambda_1 + \eta_2]^2 +$$

$$[[\frac{\lambda_1(\lambda_1 - 1)}{2}] - \lambda_1(\delta_G - 1)][\lambda_1 - 2\delta_G + \eta_2 + 1]^2 + [\lambda_1 - 2\delta_G + \eta_2 + 1]^2$$

Proof:

$$F(G) = \sum_{i,j \in E(G)} [d_i^2 + d_j^2]$$

$$F(G) = \lambda_1 \sum_{i,j \in E(G)} [(deg_H(i) + 1)^2 + (deg_H(j) + 1)^2] +$$

$$\sum_{e,t \in E(G)} [(deg_H(i) + 1)^2 + (deg_{J(G)}(t) + \eta_2)^2] +$$

$$\begin{aligned}
 & \sum_{e \in v(i,j)} \sum_{i \in v(G)} [(deg_H(i) + 1)^2 + (deg_{J(G)}(e) + \eta_2)^2] \\
 &= \lambda_1 \lambda_2 [(deg_H(i) + 1)^2 + (deg_H(j) + 1)^2] + \\
 & \quad \lambda_1 \eta_2 [(deg_H(i) + 1)^2 + (deg_{J(G)}(e) + \eta_2)^2] + \\
 & \quad \left[\left[\frac{\lambda_1(\lambda_1 - 1)}{2} \right] - \lambda_2 \left[\frac{deg_G(i) + deg_G(j) - 2}{2} \right] \right] [(deg_{J(G)}(e) + \eta_2)^2 + (deg_{J(G)}(t) + \eta_2)^2] \\
 &= \lambda_1 \lambda_2 [(deg_G(i) + 1)^2 + (deg_H(j) + 1)^2] + \\
 & \quad \lambda_1 \eta_2 [(deg_H(i) + 1)^2 + [(\lambda_1 - 1) - (deg_G(i) + deg_G(j) - 2) + \eta_2]^2] + \\
 & \quad \left[\left[\frac{\lambda_1(\lambda_1 - 1)}{2} \right] - \lambda_1 \left[\frac{deg_G(i) + deg_G(j) - 2}{2} \right] \right] \\
 & \quad [(\lambda_1 - 1) - [deg_G(i) + deg_G(j) - 2] + \eta_2]^2 + [(\lambda_1 - 1) - [deg_G(i) + deg_G(j) - 2] + \eta_2]^2 \\
 & \geq \lambda_1 \lambda_2 [(\Delta_H + 1)^2 + (\Delta_H + 1)^2] + \lambda_1 \eta_2 [(\Delta_H + 1)^2 + ((\lambda_1 - 1) - (\Delta_G + \Delta_G - 2) + \eta_2)^2] + \\
 & \quad \left[\left[\frac{\lambda_1(\lambda_1 - 1)}{2} \right] - \lambda_1 \left[\frac{\Delta_G + \Delta_G - 2}{2} \right] \right] [(\lambda_1 - 1) - (\Delta_G + \Delta_G - 2) + \eta_2]^2 + \\
 & \quad [(\lambda_1 - 1) - (\Delta_G + \Delta_G - 2) + \eta_2]^2 \\
 & \geq 2\lambda_1 \lambda_2 (\Delta_H + 1)^2 + \lambda_1 \eta_2 [\Delta_H^2 + 2\Delta_H + 3\lambda_1 - 2\Delta_G + \eta_2]^2 +
 \end{aligned}$$

$$\begin{aligned} & \left[\left[\frac{\lambda_1(\lambda_1 - 1)}{2} \right] - \lambda_1(\Delta_G - 1) \right] \left[\lambda_1 - 2\Delta_G - 1 \right] \left[\lambda_1 - 2\Delta_G + \eta_2 + 1 \right]^2 + \left[\lambda_1 - 2\Delta_G + \eta_2 + 1 \right]^2 \\ & \leq 2\lambda_1\lambda_2(\delta_H + 1)^2 + \lambda_1\eta_2[\delta_H^2 + 2\delta_H + 3\lambda_1 - 2\delta_G + \eta_2]^2 + \end{aligned}$$

$$\left[\left[\frac{\lambda_1(\lambda_1 - 1)}{2} \right] - \lambda_1(\delta_G - 1) \right] \left[\lambda_1 - 2\delta_G - 1 \right] \left[\lambda_1 - 2\delta_G + \eta_2 + 1 \right]^2 + \left[\lambda_1 - 2\delta_G + \eta_2 + 1 \right]^2$$

5 Conclusion:

In this article we considered four topological indices to determine the lower and upper bounds. In the same way researchers can consider other topological indices and determine their bounds for the graph.

References

- [1] V Loksha "Certain topological indices and related polynomials for polysaccharides", TWMS J.App and eng.Math. v.13, N.3, 2023, pp.990-997
- [2] Kulli V. R ., (2021), Nirmala index, International Journal of Mathematics trends and technology, 8-12
- [3] Deepika T., (2021), "An Emphatic study on VL index and their bounds on tensor product of F sum graph. TWMS Journal oa Applied and Engineering Mathematics", 11(2), 374-385
- [4] B.Furtula, and Gutman, I.A forgotten topological index, J.Math.Chem. 53(4), 1184-1190, 2015
- [5] Abdul Aqid Alameri., "Zagreb indices, Hyper Zagreb indices and Redefined Zageb indices of conical graph ", Advances in Mathematics: Scientific journal 9(2020) no, 6, 3643-3652 ISSN:1857-8365 (printed); 1857-8438(electronic) [https:// doi.org/10.37418/amsj.9.6.42](https://doi.org/10.37418/amsj.9.6.42)
- [6] Furtula .B. and Gutman I.,(2015), A forgotten topological index,

J.Math chem., 53pp.1184-1190.

- [7] V. Loksha, K.Zeba Yasmeenn (2019). S K Indices Forgotten topological indices and Hyper Zagreb index of Q operator of carbon nanocone, TWMS.J.App and Eng, Math. V.9, N.3, pp.675-680.

- [8] Mohammed Imran on the "bounds of degree based topological indices of the cartesian product of F-Sum of connected graphs" Journal of Inequalities and Applications 2017, Artical number: 305(2017).

- [9] B. Furtula, I . Gutman, A forgotten topological index, J. Math. Chem. 53(2015) 1184-1190.

- [10] Zhongyuan Che, Zhibo Chen "Lower and Upper Bounds of the Forgotten Topological Index", MATCH Commun. Math. Comput. Chem. 76(2016) 635-648 ISSN 0340-6253.