Original Research Article

THE DAGUM-CAUCHY{EXPONENTIAL} DISTRIBUTION

ABSTRACT

This study proposed a four-parameter continuous distribution, called the Dagum-Cauchy{Exponential} Distribution (DCED) for modelling financial time series returns using the generalized family of Cauchy distribution by Alzaatreh *et al.* (2016). Some structural properties of this new distribution such as quantile function, reliability measures and hazard function, and order statistics are obtained. The method of maximum likelihood estimation was proposed in estimating its parameters.

Keywords: Dagum distribution; Cauchy distribution; Exponential distribution; Quantile function; Reliability analysis.

1. Introduction

Many financial time series are often characterized with fluctuations due to shocks in the system. these shocks are random and most times cannot be explained under normal circumstances. Nolan (2014) notes that modelling volatility using normal distribution-based models which does not truly account for volatility seen in real life financial returns. Stable distributions such as Cauchy and levy distributions has been proposed to handle the different peculiarities of financial time series (Nolan, 2014; Mahdizadeh and Zamanzade, 2017). The Cauchy distribution is one of the widely used stable distribution dues to its properties. It is a bell-shaped symmetric distribution with ‘fat tails’ (a heavy probability mass in the tails) which accounts for extreme or rare events (outliers). It is used in many areas of application such as financial and risk analysis, mechanical and electrical theory, etc. One disadvantage of the Cauchy distribution is that it is not flexible to some stochastic time series with non-Guassian error terms except at lag 1 (Lombardi, 2004).

Dagum distribution is an important distribution widely known for its superiority in modelling income and wealth distribution as well as the size distribution of personal income. Dagum (1977) introduced the Dagum distribution and showed that it can model heavy tails in income and wealth distributions. McDonald (1984) showed that the Dagum distribution is a special case of the generalized beta distribution of the second kind (GB2). It has many applications in finance, economics and actuarial sciences. The summary of the statistical properties of the Dagum distribution was done by Kleiber (2008). Domma *et al.* (2011) obtained the maximum likelihood estimates of the distribution based on censoring.

From the foregoing, it is evident that the Cauchy and Dagum distributions can be a powerful combination to model heavy tailed volatility returns with non-Guassian error term. Many generalizations of the two distributions exist in the literature such as the beta-dagum distribution (Domma and Condino, 2013), The Exponentiated Kumaraswamy-Dagum distribution (Huang and Oluyede, 2014), The gamma-Dagum distribution (Oluyede *et al.*, 2014), the Rayleigh-Cauchy distribution (Ogunsanya *et al.*, 2019), the Gamma-Cauchy{exponential} distribution (Alzaatreh *et al.*, 2016), the Exponentiated-Exponential-Dagum{Lomax} distribution (Ekum *et al.*, 2020) amongst others. Hence, we propose a mixture of these two distributions using the T-R{Y} family framework of distribution proposed by Aljarrah *et al.* (2014) and apply it to the volatility of financial time series returns.

The rest of the paper are organized as follows. Section 2 introduces the well known $T - R\{Y\}$ family of distributions. In Section 3, we introduce the new distribution. In Section 4, we derive some mathematical properties and characteristics of the Dagum-Cauchy{exponential} distribution. Section 5 contains the maximum likelihood estimation of the Dagum-Cauchy{exponential} distribution and its application to volatility and returns time series data. Finally, section 6 concludes the paper.

2. THE $T - R\{Y\}$ FAMILY OF DISTRIBUTIONS

Eugene *et al.* (2002) proposed the well-known beta-G family of distributions using the beta distribution as a generator. Alshawarbeh *et al.* (2013) introduced the beta-Cauchy distribution using the beta-G family of distributions. Alzaatreh *et al.* (2013) extended the beta-G family to the so-called $T - X\{W\}$ family of distribution. The approach of Alzaatreh *et al.* (2013) involves using the function of the CDF of a random variable X, $W(F(x))$ to transform the pdf of another random variable T into a new CDF. Aljarrah *et al.* (2014) further extended this family of distribution by redefining $W(F(x))$ using the quantile function which he called the $T - R\{Y\}$ family of distribution. The c.d.f. of the $T - R\{Y\}$ family is defined as follows.

$$G(x) = \int_a^{Q_Y(F_R(x))} f_T(t)dt = F_T\{Q_Y(F_R(x))\}, \qquad (1)$$

Where $f_T(t)$ is the p.d.f. of a random variable T, $Q_Y(\cdot)$is the quantile function of a random variable Y and $F_R(x)$ is the c.d.f. of a random variable R. $Q_Y[F_R(x)]$ is differentiable and monotonically nondecreasing. The corresponding p.d.f. is given as

$$g(x) = f_R(x)\frac{f_T\{Q_Y(F_R(x))\}}{f_Y\{Q_Y(F_R(x))\}}. \qquad (2)$$

Using, this family of distribution, Alzaatreh *et al.* (2014), Nasir *et al.* (2017), Alzaatreh *et al.* (2016), Jamal *et al.* (2017) and Jamal and Nasir (2019) has proposed different families of distributions in the literature. In this article, we use the generalized family of Cauchy distribution by Alzaatreh *et al.* (2016) to propose a new distribution for modelling financial time series returns.

3. THE DAGUM-CAUCHY{EXPONENTIAL} DISTRIBUTION

Theorem I: Given that the quantile function of the exponential distribution is given by $-\log(1 - p)$, let R be a random variable that follows the Cauchy distribution with p.d.f. and c.d.f $f_R(x) =$

$f_C(x) = (\pi\theta[1 + (x/\theta)^2])^{-1}$ and $F_R(x) = F_C(x) = 0.5 + \pi^{-1}\tan^{-1}(x/\theta)$. Then the c.d.f and p.d.f. of the T-Cauchy{exponential} family of distribution is given by

$$F_X(x) = F_T\{-\log(1 - F_C(x))\} \tag{3}$$

$$f_X(x) = \frac{f_C(x)}{1 - F_C(x)} \times f_T(-\log(1 - F_C(x))) \tag{4}$$

Proof: It is clear that (3) and (4) are obtained by direct substitution from (1) and (2) using the p.d.f. and c.d.f. of the Cauchy distribution and the quantile function of the exponential distribution. (see Alzaatreh *et al.*, 2016).

Let T be a random variable that follows the Dagum distribution, then the p.d.f. of the three parameter Dagum distribution defined by Dagum (1977) is given by

$$f_T(x) = \beta\lambda\delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta-1} \tag{5}$$

where $\lambda, \beta, \delta > 0$. λ is the scale parameter and β, δ are the shape parameters. The corresponding c.d.f. is given by

$$F_T(x) = \left(1 + \lambda x^{-\delta}\right)^{-\beta} \tag{6}$$

Substituting (5) and (6) into (3) and (4) we have

$$F_X(x) = \left\{1 + \lambda\left[-\log\left(0.5 - \pi^{-1}\tan^{-1}(x/\theta)\right)\right]^{-\delta}\right\}^{-\beta} \tag{7}$$

$$f_X(x) = \frac{\beta\lambda\delta\left[-\log\left(0.5-\pi^{-1}\tan^{-1}(x/\theta)\right)\right]^{-\delta-1}\left[1+\lambda\left[-\log\left(0.5-\pi^{-1}\tan^{-1}(x/\theta)\right)\right]^{-\delta}\right]^{-\beta-1}}{\pi\theta\left[1+(x/\theta)^2\right]\left[0.5-\pi^{-1}\tan^{-1}(x/\theta)\right]} \tag{8}$$

where $\lambda, \beta, \delta, \theta, \pi > 0$. $\lambda\ and\ \theta$ are scale parameters while β, δ are the shape parameters. Equation (7) and (8) are the c.d.f. and p.d.f. of the new distribution. Hence a random variable X

with c.d.f. (7) and P.d.f. (8) is said to follow the Dagum-Cauchy{exponential} distribution and is denoted by $DC(\lambda, \theta, \beta, \delta)$.

The plots of the pdf of the Dagum-Cauchy{exponential} distribution is shown in the figure 1 below

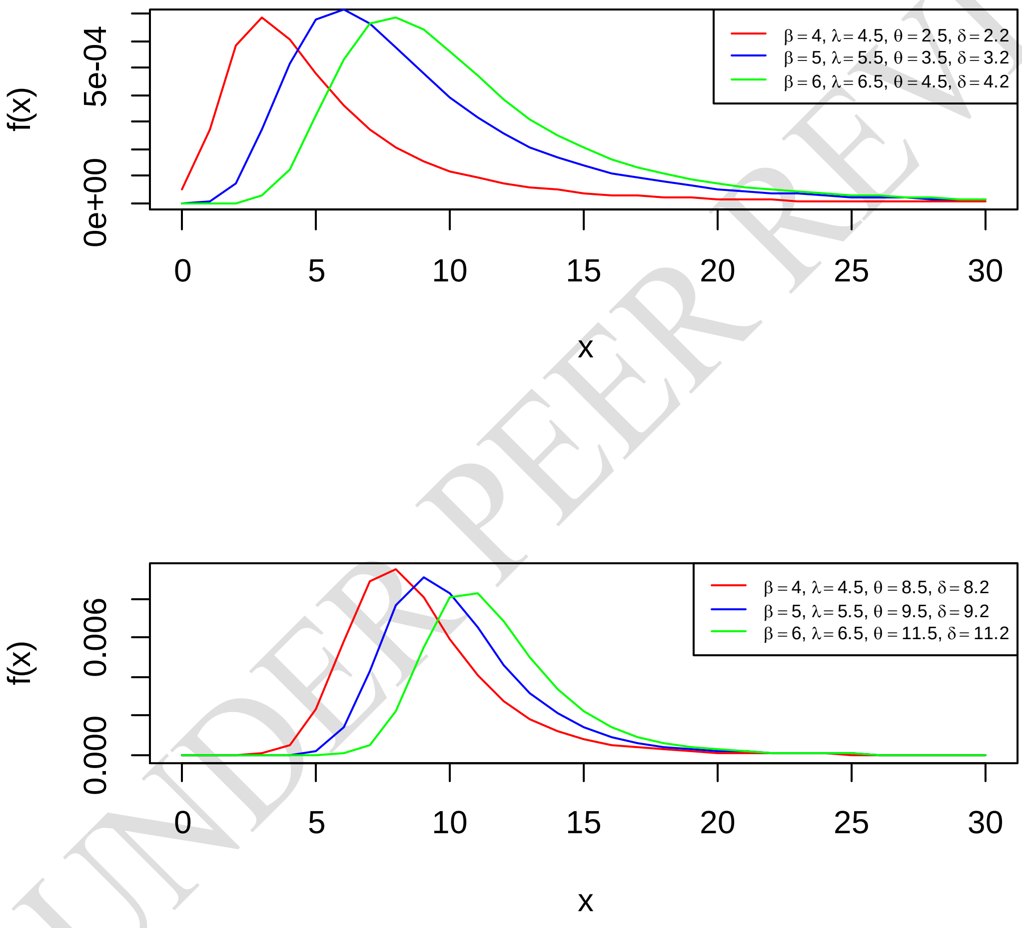

Figure 1: Plot of the p.d.f of the Dagum-Cauchy{exponential} distribution

The plots of the cdf of the Dagum-Cauchy{exponential} distribution is shown in the figure 2 below

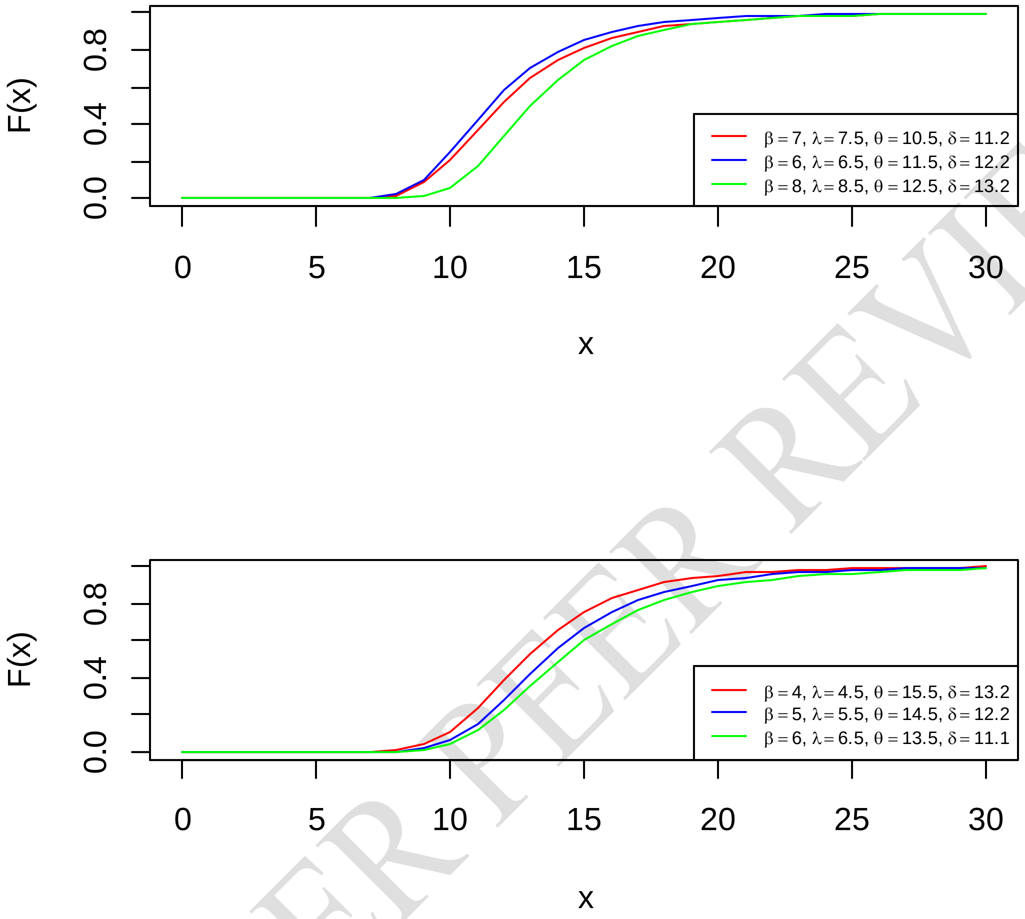

Figure 2: Plot of the c.d.f of the Dagum-Cauchy{exponential} distribution

From figure 1, the p.d.f. of the $DC(\lambda, \theta, \beta, \delta)$ depicts that the distribution can be stable (normal), symmetric, positively skewed or slightly negatively skewed.

4. MATHEMATICAL PROPERTIES

In this section, we derive some of the mathematical properties of the Dagum-Cauchy{exponential} distribution.

4.1 Quantile Function of the Dagum-Cauchy{exponential} Distribution

Remarks: The following remarks follow from Alzaatreh *et al.* (2016)

(i) If a random variable T follows a Dagum distribution with parameters $\lambda, \beta\ and \delta$, then $X = \theta\cot(\pi e^{-T})$ follows the $DC(\lambda, \theta, \beta, \delta)$ distribution.

(ii) The quantile functions for the $T-$Cauchy{exponential} family is $Q_X(p) = \theta\cot(\pi e^{-Q_T(p)})$

Theorem I

The quantile function of the $DC(\lambda, \theta, \beta, \delta)$ for p random variable, uniformly distributed on [0,1], is given by

$$Q_X(p) = \theta\cot\left(\pi e^{-\left[\lambda^{-1}\left(p^{-1/\delta}-1\right)\right]^{-1/\delta}}\right) \qquad (9)$$

Proof: The result follows from the remarks above and the fact that $Q_T(p) = \left[\lambda^{-1}\left(p^{-1/\delta} - 1\right)\right]^{-1/\delta}$

The quantile function in (9) will be used to generate random variates in the simulation study. The median, 1st and 3rd quartiles can be obtained by setting $p = 0.5, 0.25, and\ 0.75$, respectively. Other measures of partitions can also be obtained by setting p appropriately.

4.2. Moments of the Dagum-Cauchy{exponential} Distribution

We derive the rth moment of the Dagum-Cauchy{exponential} distribution

Theorem II

Let a random variable X follow the $DC(\lambda,\theta,\beta,\delta)$, the rth order moment of $DC(\lambda,\theta,\beta,\delta)$ about origin is given by

$$\mu_r'(\lambda,\theta,\beta,\delta) = \theta^r \lambda^{\frac{j}{\delta}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^j}{j!} C_k B\left(\beta(r-2k)+\frac{j}{\delta}, 1-\frac{j}{\delta}\right) \tag{10}$$

where $B(\cdot,\cdot)$ is the complete beta function and

$$C_0 = \pi^{-1};\ C_m = \pi m^{-1} \sum_{k=1}^{m} (kr-m+k)\, W_k C_{m-k}, m \geq 1, and\ W_k = \frac{(-1)^k 2^{2k} B_{2k} \pi^{2k-1}}{(2k)!}$$

Proof:

Given that the rth moment for the T −Cauchy{exponential} distribution is given by

$$E(X^r) = \theta^r \sum_{k=0}^{\infty} C_k\, M_T(r-2k) \tag{11}$$

where $C_0 = \pi^{-1};\ C_m = \pi m^{-1} \sum_{k=1}^{m} (kr-m+k)\, W_k C_{m-k}, m \geq 1, and\ W_k = \frac{(-1)^k 2^{2k} B_{2k} \pi^{2k-1}}{(2k)!}$, and M_T is the moment generating function of T random variable. (See Alzaatreh *et al.* (2016) for the proof). Let T be a random variable following the Dagum distribution with moment generating function

$$M_T = \lambda^{\frac{j}{\delta}} \sum_{j=0}^{\infty} \frac{t^j}{j!} \frac{\Gamma\left(1-\frac{j}{\delta}\right)\Gamma\left(\beta+\frac{j}{\delta}\right)}{\Gamma(\beta)} \tag{12}$$

Then substituting (12) into (11), we have

$$E(X^r) = \theta^r \sum_{k=0}^{\infty} C_k\, \lambda^{\frac{j}{\delta}} \sum_{j=0}^{\infty} \frac{t^j}{j!} \frac{\Gamma\left(1-\frac{j}{\delta}\right)\Gamma\left(\beta(r-2k)+\frac{j}{\delta}\right)}{\Gamma(\beta(r-2k))} \tag{13}$$

$$\mu_r'(\lambda,\theta,\beta,\delta) = E(X^r) = \theta^r \lambda^{\frac{j}{\delta}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^j}{j!} C_k B\left(\beta(r-2k)+\frac{j}{\delta}, 1-\frac{j}{\delta}\right) \tag{14}$$

where C_k is defined in (11) and $B(\cdot,\cdot)$ is the complete beta function. Hence the mean of $DC(\lambda,\theta,\beta,\delta)$ is given by

$$\mu_1^{'}(\lambda,\theta,\beta,\delta) = \theta\lambda^{\frac{j}{\delta}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^j}{j!} C_k B\left(\beta(1-2k) + \frac{j}{\delta}, 1 - \frac{j}{\delta}\right) \tag{15}$$

where C_k is defined in (11) and $r = 1$. Also, the second order moment is given by

$$\mu_2^{'}(\lambda,\theta,\beta,\delta) = \theta^2\lambda^{\frac{j}{\delta}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^j}{j!} C_k B\left(\beta(2-2k) + \frac{j}{\delta}, 1 - \frac{j}{\delta}\right) \tag{16}$$

where C_k is defined in (11) and $r = 2$.

The third and fourth moments of $DC(\lambda,\theta,\beta,\delta)$ do not always exist. Alternatively, we can define the measure of asymmetry and tail weight based on quantile function. The Galton's skewness S defined by Galton (1883) and the Moors' kurtosis K defined by Moors (1988) are given by

$$S = \frac{Q\left({}^{6}/_{8}\right) - 2Q\left({}^{4}/_{8}\right) + Q\left({}^{2}/_{4}\right)}{Q\left({}^{6}/_{8}\right) - Q\left({}^{2}/_{8}\right)} \tag{17}$$

$$K = \frac{Q\left({}^{7}/_{8}\right) - Q\left({}^{5}/_{8}\right) + Q\left({}^{3}/_{8}\right) - Q\left({}^{1}/_{8}\right)}{Q\left({}^{6}/_{8}\right) - Q\left({}^{2}/_{8}\right)} \tag{18}$$

The When the distribution is symmetric, S= 0 and when the distribution is right (or left) skewed S> 0 (or S< 0). As K increases the tail of the distribution becomes heavier. To investigate the effect of the two shape parameters δ and β on the $DC(\lambda,\theta,\beta,\delta)$ distribution, (17) and (18) are used to obtain the Galtons' skewness and Moors' kurtosis where the quantile function is defined in (9).

4.3 Reliability analysis

Given any probability distribution, the reliability analysis is always considered based on the survival function and the hazard rate function of the distribution. Hence, for the Dagum-Cauchy{exponential} distribution, the survival and hazard rate function is given below;

4.3.1 Survival function

The survival function is defined as the probability that an item does not fail prior to some time t. It is given by

$$S_X(x) = 1 - F_X(x)$$

$$S_X(x) = 1 - \left\{1 + \lambda\left[-\log\left(0.5 - \pi^{-1}\tan^{-1}\left({}^{x}/_{\theta}\right)\right)\right]^{-\delta}\right\}^{-\beta} \tag{19}$$

4.3.2 Hazard rate function

The hazard rate function on the other hand can be seen as the conditional probability of failure, given it has survived to the time t. It is given by

$$h_X(x) = \frac{f_X(x)}{1 - F_X(x)}$$

$$h_X(x)$$

$$= \frac{\beta\lambda\delta\left[-\log\left(0.5 - \pi^{-1}\tan^{-1}\left({}^{x}/_{\theta}\right)\right)\right]^{-\delta-1}\left[1 + \lambda\left[-\log\left(0.5 - \pi^{-1}\tan^{-1}\left({}^{x}/_{\theta}\right)\right)\right]^{-\delta}\right]^{-\beta-1}}{\pi\theta\left[1 + \left({}^{x}/_{\theta}\right)\right]\left[0.5 - \pi^{-1}\tan^{-1}\left({}^{x}/_{\theta}\right)\right]}$$

$$\cdot\frac{1}{1 - \left\{1 + \lambda\left[-\log\left(0.5 - \pi^{-1}\tan^{-1}\left({}^{x}/_{\theta}\right)\right)\right]^{-\delta}\right\}^{-\beta}}$$

$$h_X(x) =$$

$$\frac{\left[1+\lambda\left[-\log\left(0.5-\pi^{-1}\tan^{-1}\left({}^{x}/_{\theta}\right)\right)\right]^{-\delta}\right]^{-\beta-1}\left[\beta\lambda\delta\left[-\log\left(0.5-\pi^{-1}\tan^{-1}\left({}^{x}/_{\theta}\right)\right)\right]^{-\delta-1}-\left\{1+\lambda\left[-\log\left(0.5-\pi^{-1}\tan^{-1}\left({}^{x}/_{\theta}\right)\right)\right]^{-\delta}\right\}^{-\beta}\right]}{\pi\theta\left[1+\left({}^{x}/_{\theta}\right)\right]\left[0.5-\pi^{-1}\tan^{-1}\left({}^{x}/_{\theta}\right)\right]}.$$

(20)

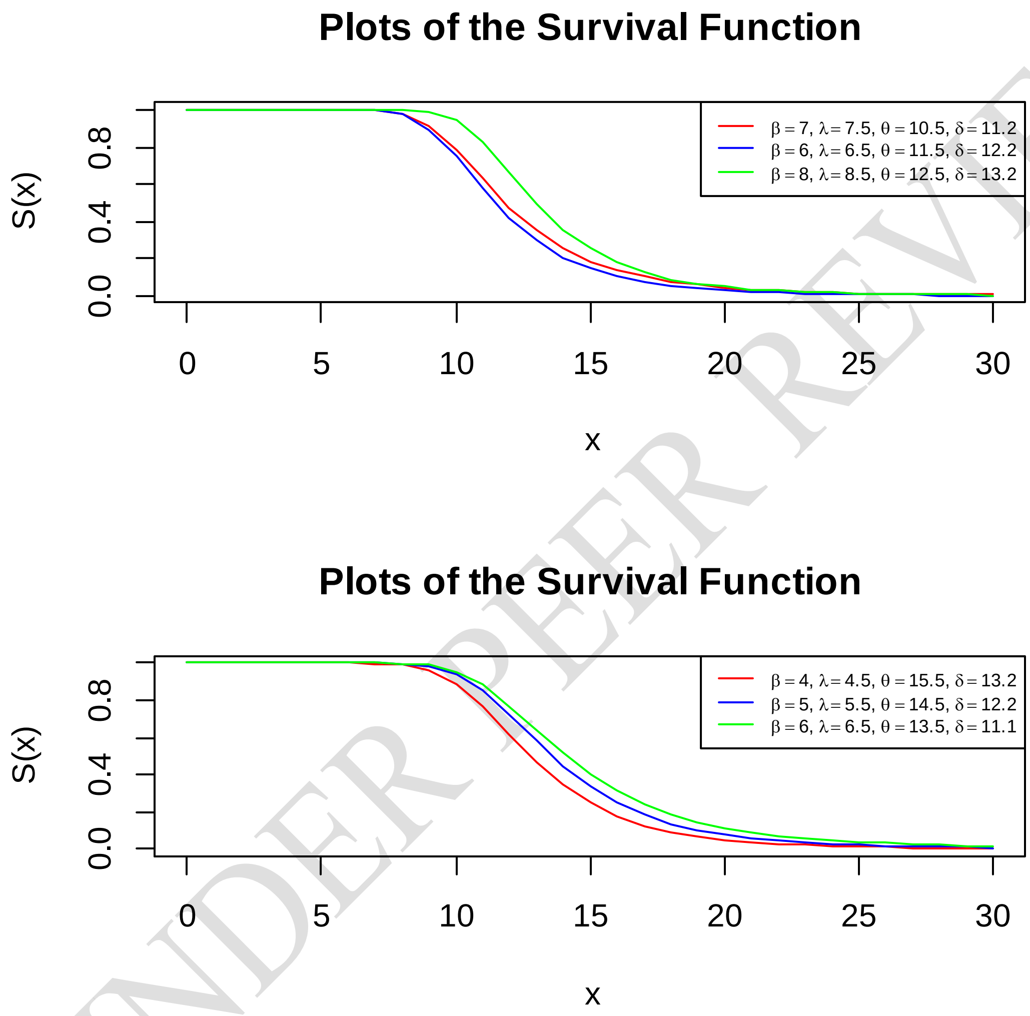

Figure 3: Plot of the survival function of the Dagum-Cauchy{exponential} distribution

5. The maximum likelihood estimation of the Dagum-Cauchy{exponential} distribution

Let $X_1, X_2, \dots, X_n$ be a random sample of size n drawn from the $DC(\lambda, \theta, \beta, \delta)$. The log-likelihood function is given by

$$ll = -n\ln(\pi\theta) - \ln\left(1 + \left({}^{x_i}/_{\theta}\right)^2\right) - \ln P_i + n\ln(\beta\lambda\delta) + (-\delta - 1)\sum_{i=0}^{n}\ln(-\log P_i) + (-\beta - 1)\sum_{i=0}^{n}\ln(G_i(\lambda)) \tag{21}$$

where $P_i = 0.5 - \pi^{-1}\tan^{-1}\left({}^{x_i}/_{\theta}\right)$ and $G_i(\lambda) = 1 + \lambda\left[-\log\left(0.5 - \pi^{-1}\tan^{-1}\left({}^{x_i}/_{\theta}\right)\right)\right]^{-\delta}$.

$$\frac{\partial ll}{\partial \beta} = \frac{n}{\beta} + \sum_{i=0}^{n}\ln(G_i(\lambda)) \tag{22}$$

$$\frac{\partial ll}{\partial \lambda} = \frac{n}{\lambda} + (-\beta - 1)\sum_{i=0}^{n}\frac{1}{G_i(\lambda)}\ln[-\log P_i]^{-\delta} \tag{23}$$

$$\frac{\partial ll}{\partial \lambda} = -\frac{n}{\theta} + \sum_{i=0}^{n}\frac{x_i}{\theta(\theta + x_i)} + \sum_{i=0}^{n}\frac{x_i}{\pi(\theta^2 + x_i{}^2)P_i} + [(-\delta - 1)(\log P_i)^{-1}]\sum_{i=0}^{n}\frac{x_i}{\pi(\theta^2 + x_i{}^2)P_i} + (-\beta - 1)\sum_{i=0}^{n}\frac{-x_i\lambda\delta}{G_i(\lambda)\pi(\theta^2 + x_i{}^2)P_i} \tag{24}$$

$$\frac{\partial ll}{\partial \lambda} = \frac{n}{\delta} + \sum_{i=0}^{n}\ln(-\log P_i) \tag{25}$$

In order to obtain the estimates of the parameters, we use the Adequacy Model package in R software.

6. Applications

In this section, we fit the Dagum-Cauchy{exponential} distribution to two data sets. The first dataset is the exchange rate return series of Nigeria from 2000-2017. While the second data is the exchange rate volatility series of Nigeria from 2000-2017.

We also show in this section the probability fit of the Dagum-Cauchy{exponential} distribution to the two datasets.

The result of the data estimation is shown in Table 1

Table 1: Log-likelihood,AIC, BIC,HQIC, and K-S pvalue of the two datasets

Distribution	**Data 1**	**Data 2**
Parameter Estimates	$\boldsymbol{\beta}$**= 0.3238** **(0.0431)** $\boldsymbol{\lambda}$**=0.9203** **(0.3250)** $\boldsymbol{\theta}$**=0.1007** **(0.0157)** $\boldsymbol{\delta}$**=6.7586** **(0.9029)**	$\boldsymbol{\beta}$**= 0.2448** **(0.0184)** $\boldsymbol{\lambda}$**=3.4146** **(0.5343)** $\boldsymbol{\theta}$**=0.2777** **(0.0001)** $\boldsymbol{\delta}$**=0.1980** **(0.0136)**
Log-likelihood	**69.8327**	**813.2703**
AIC	**147.6654**	**1634.541**
BIC	**161.0728**	**1647.948**
HQIC	**153.0849**	**1639.96**
K-S	**0.27702**	**0.6756**
K-S p-value	**0.0000172**	**0.0000022**

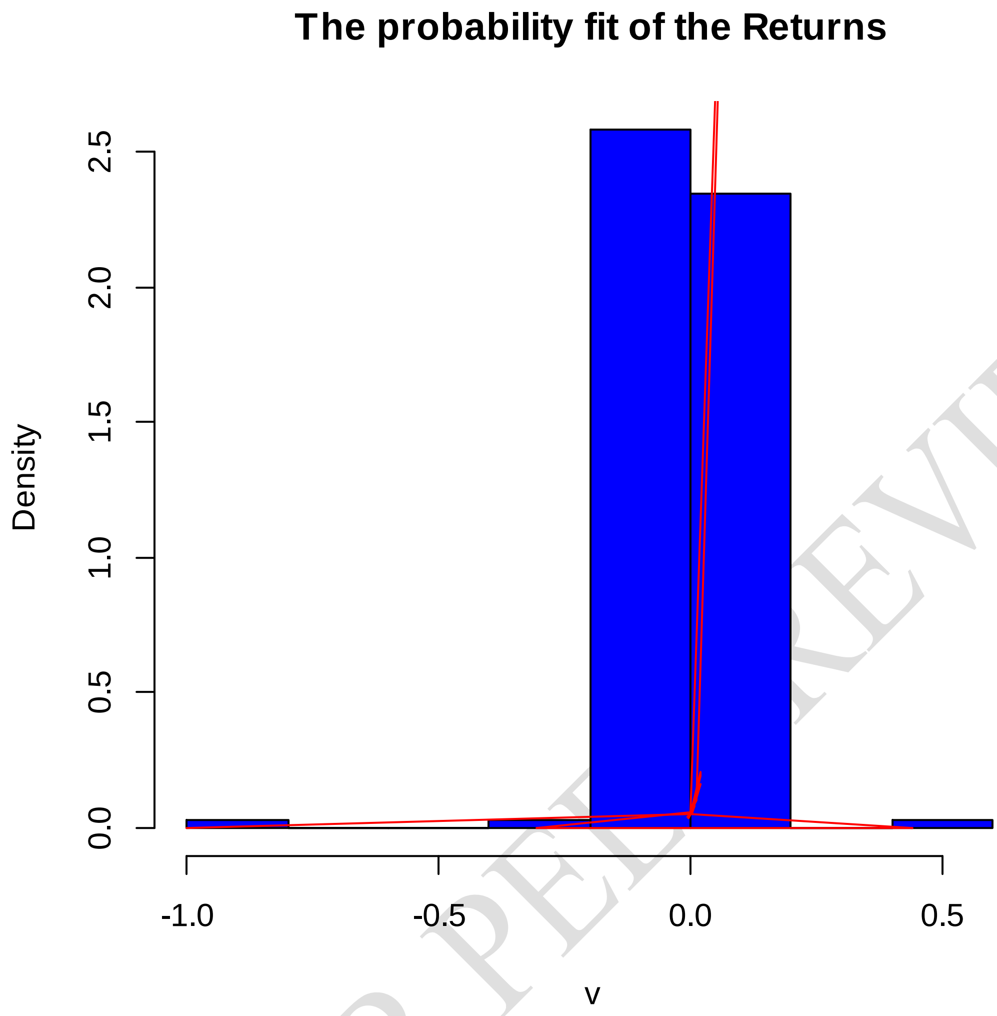

Figure 4: Density Plot of the histogram for the returns series

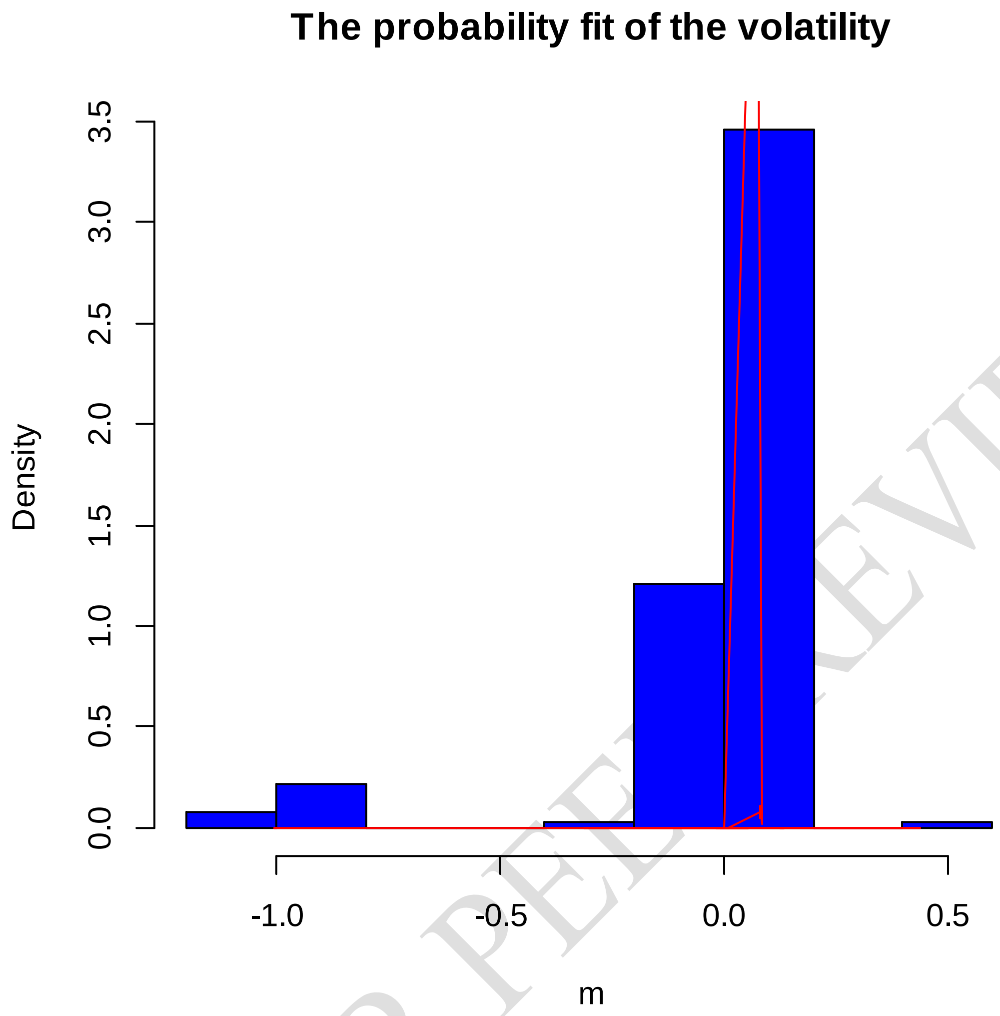

Figure 5: Density Plot of the histogram for the Volatility series

From Figures 4 and 5, the Dagum-Cauchy{exponential} distribution gave a good fit to the returns and the volatility datasets.

7. Conclusion

This paper proposed a new distribution known as the Dagum-Cauchy{exponential} distribution using the so-called using the T-X{Y} family framework of distribution proposed by Aljarrah *et al.* (2014). The mathematical properties of the newly developed distribution including the quantile function, Moments and Moment generating function and reliability analysis was also proposed and derived. Furthermore, the maximum likelihood estimation was discussed. The distribution was fitted to two datasets and proved to be a good fit to the datasets.

References

Aljarrah, M.A., Lee, C. and Famoye, F. (2014). On generating T-X family of distributions using quantile functions. Journal of Statistical Distributions and Applications 2014, 1:2.

Alshawarbeh, E., Famoye, F., & Lee, C. (2013). Beta-Cauchy distribution: some properties and applications. *Journal of Statistical Theory and Applications*, *12*(4), 378-391.

Alzaatreh, A., Lee, C., & Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, *71*(1), 63-79.

Alzaatreh, A., Lee, C., & Famoye, F. (2014). T-normal family of distributions: A new approach to generalize the normal distribution. *Journal of Statistical Distributions and Applications*, *1*(1), 16.

Alzaatreh, A., Lee, C., Famoye, F. and Ghosh, I. (2016). The generalized Cauchy family of distributions with applications. *Journal of Statistical Distributions and Applications* 2014, 1:16

Dagum, C. (1977). A New Model for Personal Income Distribution: Specification and Estimation. *Economic Applique*, 30(3), 413–437.

Domma, F., & Condino, F. (2013). The beta-dagum distribution: definition and properties. *Communications in Statistics-Theory and Methods*, *42*(22), 4070-4090.

Domma, F., Giordano, S., & Zenga, M. (2011). Maximum likelihood estimation in dagum distribution with censored samples. *Journal of Applied Statistics*, *38*(12), 2971-2985.

Ekum, M. I., Adamu, M. O., & Akarawak, E. E. (2020). T-Dagum: A Way of Generalizing Dagum Distribution Using Lomax Quantile Function. *Journal of Probability and Statistics*, *2020*.

Eugene, N., Lee, C. and Famoye, F. (2012). Beta-normal distribution and its application, *Commun. Statist. Theory Meth*., 31(4), 497–512 (2002).

Galton, F. (1883). *Inquiries into human faculty and its development*. Macmillan.

Huang, S., & Oluyede, B. O. (2014). Exponentiated Kumaraswamy-Dagum distribution with applications to income and lifetime data. *Journal of Statistical Distributions and Applications*, *1*(1), 8.

Jamal, F., & Nasir, M. (2019, January). Some new members of the TX family of distributions.

Jamal, F., Nasir, M. A., Tahir, M. H., & Montazeri, N. H. (2017). The odd Burr-III family of distributions. *Journal of Statistics Applications and Probability*, *6*(1), 105-122.

Kleiber, C. (2008). A guide to the Dagum distributions. In *Modeling income distributions and Lorenz curves* (pp. 97-117). Springer, New York, NY.

Lombardi, M. (2004). Simulation-based Estimation Methods for α-Stable Distributions and Processes. Available at https://pdfs.semanticscholar.org/4ddc/4a153e0ee355a4b6dbed6b3de29d4bb833ab.pdf.

Mahdizadeh, M., & Zamanzade, E. (2017). New goodness of fit tests for the Cauchy distribution. *Journal of Applied Statistics*, *44*(6), 1106-1121.

McDonald, J. B. (1984). Some generalized functions for the size distribution of income. In *Modeling income distributions and Lorenz curves* (pp. 37-55). Springer, New York, NY

Moors, J. J. A. (1988). A quantile alternative for kurtosis. *Journal of the Royal Statistical Society: Series D (The Statistician)*, *37*(1), 25-32.

Nasir, M. A., Aljarrah, M., Jamal, F., & Tahir, M. H. (2017). A new generalized Burr family of distributions based on quantile function. *Journal of Statistics Applications and Probability*, *6*(3), 1-14.

Nolan, J. P. (2014). Financial modeling with heavy- tailed stable distributions. *Wiley Interdisciplinary Reviews: Computational Statistics*, *6*(1), 45-55.

Ogunsanya, A. S., Akarawak, E. E., & Ekum, M. I. On Some Properties of Rayleigh-Cauchy Distribution.

Oluyede, B. O., Huang, S., & Pararai, M. (2014). A new class of generalized dagum distribution with applications to income and lifetime data. *Journal of Statistical and Econometric Methods*, *3*(2), 125.