On the Norms of Toeplitz Matrices with the Generalized Oresme Numbers

Abstract. In this article, we present results on Toeplitz matrices with Oresme numbers components. First, the Toeplitz matrices with Oresme numbers components are created and then the Frobenius(Euclidian), row and column norms of these matrices are found. Furthermore lower and upper bounds are obtained for the spectral norms of these matrices. In addition, the upper bounds for the Frobenius and spectral norms of the Kronecker and Hadamard product matrices of the Toeplitz matrices with the Oresme numbers are calculated.

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1. Introduction

In recent years, many articles have been published on the norms of special matrices with the entry of special cases of Horadam numbers. Solak [18] calculated the spectral norms of Toeplitz matrices with Fibonacci and Lucas numbers. Akbulak and Bozkurt [1] obtained some special norms of Toeplitz matrices given with Fibonacci and Lucas numbers and lower and upper bounds for the spectral norm. Later, Shen [17] obtained some special norms for Toeplitz matrices with k-Fibonacci and k-Lucas numbers components, and bounds for the spectral norms of these matrices, lower and upper bounds for the spectral norms of Hadamard and Kronecker products of these matrices. Eylem G. Karpuz [11] made a study on the norms of Toeplitz matrices whose elements are Pell numbers. Similarly, Daşdemir [6] gave a few special norms of Toeplitz matrices such as Pell, Pell-Lucas and Modified Pell numbers, and lower and upper bounds for spectral norm. Uygun, [25], obtained some special norms of Toeplitz matrices with Jacobsthal and jacobsthal-Lucas numbers, lower and upper bounds for the spectral norm, and the upper bound of the Frobenius norm of the Kronecker and Hadamard products of these matrices. Furthermore, Uygun [26] present a parallel study of the k-jacobsthal and k-jacobsthal-lucas numbers.

Now, in the light of previous articles, we present some special norms of Toeplitz matrices with Oresme numbers and this study in which we obtained the bounds of these norms.

A generalized Oresme sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second -order recurrence relation

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} \tag{1.1}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 4W_{-(n-1)} - 4W_{-(n-2)}$$

for $n = 1, 2, 3, \cdots$. Therefore, recurrence equation (1.1) holds for all integer n.

The first few generalized Oresme numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Oresme numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$4W_0 - 4W_1$
2	$W_1 - \frac{1}{4}W_0$	$12W_0 - 16W_1$
3	$\frac{3}{4}W_1 - \frac{1}{4}W_0$	$32W_0 - 48W_1$
4	$\frac{1}{2}W_1 - \frac{3}{16}W_0$	$80W_0 - 128W_1$
5	$\frac{5}{16}W_1 - \frac{1}{8}W_0$	$192W_0 - 320W_1$
6	$\frac{3}{16}W_1 - \frac{5}{64}W_0$	$448W_0 - 768W_1$
7	$\frac{7}{64}W_1 - \frac{3}{64}W_0$	$1024W_0 - 1792W_1$
8	$\frac{1}{16}W_1 - \frac{7}{256}W_0$	$2304W_0 - 4096W_1$
9	$\frac{9}{256}W_1 - \frac{1}{64}W_0$	$5120W_0 - 9216W_1$
10	$\frac{5}{256}W_1 - \frac{9}{1024}W_0$	$11264W_0 - 20480W_1$

For more information on generalized Oresme numbers, see for example, Soykan [19].

Modified Oresme sequence $\{G_n\}_{n\geq 0}$, Oresme-Lucas sequence $\{H_n\}_{n\geq 0}$ and Oresme sequence $\{O_n\}_{n\geq 0}$ are defined respectively, by the second order recurrence relations;

$$G_{n+2} = G_{n+1} - \frac{1}{4}G_n, \quad G_0 = 0, G_1 = 1, \tag{1.2}$$

$$H_{n+2} = H_{n+1} - \frac{1}{4}H_n, \quad H_0 = 2, H_1 = 1, \tag{1.3}$$

$$O_{n+2} = O_{n+1} - \frac{1}{4}O_n \quad O_0 = 0, O_1 = \frac{1}{2}. \tag{1.4}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{O_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= 4G_{-(n-1)} - 4G_{-(n-2)}, \\ H_{-n} &= 4H_{-(n-1)} - 4H_{-(n-2)}, \\ O_{-n} &= 4O_{-(n-1)} - 4O_{-(n-2)}. \end{aligned}$$

for $n = 1, 2, 3, \cdots$ respectively.

Therefore recurrence Equ. (1.2), Equ. (1.3) and Equ. (1.4) hold for all integer n.

Next, we present the first few values of the modified Oresme, Oresme-Lucas and Oresme numbers with positive and negative subscripts:

Table 2. The first few values of the special second-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11
G_n	0	1	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{5}{16}$	$\frac{3}{16}$	$\frac{7}{64}$	$\frac{1}{16}$	$\frac{9}{256}$	$\frac{5}{256}$	$\frac{11}{1024}$
G_{-n}	....	-4	-16	-48	-128	-320	-768	-1792	-4096	-9216	-20480	-45056
H_n	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$
H_{-n}	....	4	8	16	32	64	128	256	512	1024	2048	4096
O_n	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{5}{32}$	$\frac{3}{32}$	$\frac{7}{128}$	$\frac{1}{32}$	$\frac{9}{512}$	$\frac{5}{512}$	$\frac{11}{2048}$
O_{-n}	....	-2	-8	-24	-64	-160	-384	-896	-2048	-4608	-10240	-22528

Characteristic equation of generalized Oresme sequence $\{W_n\}_{n\geq 0}$ is given as the quadratic equation

$$x^2 - x + \frac{1}{4} = 0,$$

whose roots are α, β and

$$\alpha = \beta = \frac{1}{2}.$$

Binet's formula of Generalized Oresme sequence is given as

$$W_n = \left(nW_1 - \frac{1}{2}(n-1)W_0\right)\left(\frac{1}{2}\right)^{n-1}.$$

Binet's formulas of modified Oresme, Oresme-Lucas and Oresme numbers are

$$\begin{aligned} G_n &= n\alpha^{n-1} = \frac{n}{2^{n-1}}, \\ H_n &= 2\alpha^n = \frac{1}{2^{n-1}}, \\ O_n &= n\alpha^n = \frac{n}{2^n}, \end{aligned}$$

and Binet's formulas of modified Oresme, Oresme-Lucas and Oresme numbers at the negative index are

$$\begin{aligned} G_{-n} &= -4^n G_n = -n \times 2^{n+1}, \\ H_{-n} &= 4^n H_n = 2^{n+1}, \\ O_{-n} &= -4^n O_n = -n \times 2^n. \end{aligned}$$

2. Preliminaries

A matrix $T = [t_{ij}] \in M_n(\mathbb{C})$ is called a Toeplitz matrix if it is of the form $t_{ij} = t_{i-j}$ for

$$T_n = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{1-n} \\ t_1 & t_0 & t_{-1} & \cdots & t_{2-n} \\ t_2 & t_1 & t_0 & \cdots & t_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_0 \end{pmatrix}.$$

Now, we give some preliminaries related to our study. Let $A = (a_{ij})$ be an $m \times n$ matrix. The ℓ_p norm of the matrix A is defined by

$$\|A\|_p = (\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^p)^{\frac{1}{p}} \qquad (1 \le p < \infty).$$

If $p = \infty$, then $\|A\|_\infty = \lim_{p\to\infty} \|A\|_p = \max_{i,j} |a_{ij}|$.

The well-known Frobenius (Euclidean) and spectral norms of the matrix A are defined respectively by

$$\|A\|_F = (\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}}$$

and

$$\|A\|_2 = \sqrt{\max_{1\le i\le n} |\lambda_i|} \tag{2.1}$$

where the numbers λ_i are the eigenvalues of matrix $A^H A$ and the matrix A^H is the conjugate transpose of the matrix A. The following inequality between the Frobenius and spectral norms of A holds.

$$\frac{1}{\sqrt{n}} \|A\|_F \le \|A\|_2 \le \|A\|_F. \tag{2.2}$$

It follows that

$$\|A\|_2 \le \|A\|_F \le \sqrt{n} \|A\|_2.$$

In literature, there are other types of norms of matrices. The maximum column sum matrix norm of $n \times n$ matrix $A = (a_{ij})$ is

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| \tag{2.3}$$

and the maximum row sum matrix norm is

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|. \tag{2.4}$$

The maximum column lenght norm $c_1(.)$ and maximum row lenght norm $r_1(.)$ of on matrix of order $m \times n$ are defined as follows

$$c_1(A) \equiv \max_{1\le j\le n} \left(\sum_{i=1}^{m} |a_{ij}|^2 \right)^{\frac{1}{2}} = \max_{1\le j\le n} \left\| [a_{ij}]_{i=1}^{m} \right\|_F \tag{2.5}$$

and

$$r_1(A) \equiv \max_{1\le i\le m}\left(\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}} = \max_{1\le i\le m}\left\|[a_{ij}]_{j=1}^{n}\right\|_F \tag{2.6}$$

respectively.

For any $A,B \in M_{mn}(\mathbb{C})$, the Hadamard product of $A=(a_{ij})$ and $B=(b_{ij})$ is entrywise product and defined by $A\circ B=(a_{ij}b_{ij})$ and have the following properties

$$\|A\circ B\|_2 \le r_1(A)\,c_1(B), \tag{2.7}$$

and

$$\|A\circ B\|_2 \le \|A\|_2\,\|B\|_2\,. \tag{2.8}$$

In addition,

$$\|A\circ B\|_F \le \|A\|_F\,\|B\|_F\,. \tag{2.9}$$

Let $A\in M_{mn}(\mathbb{C})$, and $B\in M_{mn}(\mathbb{C})$ be given, then the Kronecker product of A,B is defined by

$$\|A\otimes B\| = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

and have the following properties

$$\begin{aligned} \|A\otimes B\|_2 &= \|A\|_2\,\|B\|_2\,, \\ \|A\otimes B\|_F &= \|A\|_F\,\|B\|_F\,. \end{aligned} \tag{2.10}$$

In the following theorem, we present some sum formulas of generalized Oresme numbers.

Theorem 1. *For generalized Oresme numbers, we have following sum formulas:*

(a): *[19, Proposition 26. a] If $\frac{1}{4}(x-2)^2\neq 0, i.e., x\neq 2,$ then*

$$\sum_{k=0}^{n}x^kW_k = \frac{(x-4)\,x^{n+1}W_n + x^{n+1}W_{n-1} + 4W_0 + 4\,(W_1-W_0)\,x}{(x-2)^2}. \tag{2.11}$$

(b): *[19, Proposition 26. d] If $(2x-1)^2\neq 0, i.e.,\ x\neq \frac{1}{2},$ then*

$$\sum_{k=0}^{n}x^kW_{-k} = \frac{4x^{n+1}W_{-n+1} + 4(x-1)x^{n+1}W_{-n} + W_0 - 4xW_1}{(2x-1)^2}.$$

(c): *[20, Proposition 2.1. a] If $\frac{1}{64}(x-4)^3\neq 0, i.e., x\neq 4,$ then*

$$\sum_{k=0}^{n}x^kW_k^2 = \frac{\Delta}{(x-4)^3}$$

where

$\Delta = (x-4)(x-8)x^{n+1}W_n^2 + (x-4)x^{n+1}W_{n-1}^2 + 16(x-4)W_0^2 - 16x(x-4)(W_0-W_1)^2 - 2^{-2n+5}(W_0-2W_1)^2(2^{2n}-x^n)x.$

(d): *[20, Proposition 2.1. d] If* $(4x-1)^3 \neq 0, i.e., x \neq \frac{1}{4}$ *then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Delta}{(4x-1)^3}$$

where

$\Delta = 16(4x-1)x^{n+1}W_{-n+1}^2 + 8(2x-1)(4x-1)x^{n+1}W_{-n}^2 + (4x-1)W_0^2 - 16(4x-1)xW_1^2 + 8(W_0-2W_1)^2(2^{2n}x^n-1)x.$

If we set $x = 1$ in the last Theorem, we have the following corollary.

Corollary 2. *For generalized Oresme numbers, we have following sum formulas:*

(a):

$$\sum_{k=0}^{n} W_k = -3W_n + W_{n-1} + 4W_1. \tag{2.12}$$

(b):

$$\sum_{k=0}^{n} W_{-k} = 4W_{-n+1} + W_0 - 4W_1. \tag{2.13}$$

(c):

$$\sum_{k=0}^{n} W_k^2 = -\frac{1}{27}(21W_n^2 - 3W_{n-1}^2 - 48W_1(2W_0 - W_1) - 2^{-2n+5}(W_0 - 2W_1)^2(2^{2n}-1)). \tag{2.14}$$

(d):

$$\sum_{k=0}^{n} W_{-k}^2 = \frac{1}{27}(48W_{-n+1}^2 + 24W_{-n}^2 + 3W_0^2 - 48W_1^2 + 8(W_0 - 2W_1)^2(2^{2n}-1)). \tag{2.15}$$

3. Main Results

In this paper, we use the notation $A = T(W_0, W_1, \cdots, W_{n-1})$ for the Toeplitz matrix with generalized Oresme numbers, i.e.,

$$A = \begin{pmatrix} W_0 & W_{-1} & W_{-2} & \cdots & W_{1-n} \\ W_1 & W_0 & W_{-1} & \cdots & W_{2-n} \\ W_2 & W_1 & W_0 & \cdots & W_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_{n-1} & W_{n-2} & W_{n-3} & \cdots & W_0 \end{pmatrix}. \tag{3.1}$$

For special cases, we get

$$A = \begin{pmatrix} G_0 & G_{-1} & G_{-2} & \cdots & G_{1-n} \\ G_1 & G_0 & G_{-1} & \cdots & G_{2-n} \\ G_2 & G_1 & G_0 & \cdots & G_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{n-1} & G_{n-2} & G_{n-3} & \cdots & G_0 \end{pmatrix} = \begin{pmatrix} 0 & -4 & -16 & \cdots & G_{1-n} \\ 1 & 0 & -4 & \cdots & G_{2-n} \\ 1 & 1 & 0 & \cdots & G_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{n-1} & G_{n-2} & G_{n-3} & \cdots & 0 \end{pmatrix} \tag{3.2}$$

for the Toeplitz matrix $A = T(G_0, G_1, \cdots, G_{n-1})$ with modified Oresme numbers and

$$A = \begin{pmatrix} H_0 & H_{-1} & H_{-2} & \cdots & H_{1-n} \\ H_1 & H_0 & H_{-1} & \cdots & H_{2-n} \\ H_2 & H_1 & H_0 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & H_{n-2} & H_{n-3} & \cdots & H_0 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 8 & \cdots & H_{1-n} \\ 1 & 2 & 4 & \cdots & H_{2-n} \\ \frac{1}{2} & 1 & 2 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & H_{n-2} & H_{n-3} & \cdots & 2 \end{pmatrix} \tag{3.3}$$

for the Toeplitz matrix $A = T(H_0, H_1, \cdots, H_{n-1})$ with Oresme-Lucas numbers and

$$A = \begin{pmatrix} O_0 & O_{-1} & O_{-2} & \cdots & O_{1-n} \\ O_1 & O_0 & O_{-1} & \cdots & O_{2-n} \\ O_2 & O_1 & O_0 & \cdots & O_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{n-1} & O_{n-2} & O_{n-3} & \cdots & O_0 \end{pmatrix} = \begin{pmatrix} 0 & -2 & -8 & \cdots & O_{1-n} \\ \frac{1}{2} & 0 & -2 & \cdots & O_{2-n} \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & O_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{n-1} & O_{n-2} & O_{n-3} & \cdots & 0 \end{pmatrix} \tag{3.4}$$

for the Toeplitz matrix $A = T(O_0, O_1, \cdots, O_{n-1})$ with Oresme numbers.

In the following theorem, we present the norm value of $\|A\|_1$ and $\|A\|_\infty$ of the largest absolute column sum and the largest absolute row sum of A.

Theorem 3. *Let $A = T(W_0, W_1, \cdots, W_{n-1})$ be a Toeplitz matrix with generalized Oresme numbers then the largest absolute column sum (1-norm) and the largest absolute row sum (∞-norm) of A are*

$$\|A_1\| = \|A\|_\infty = \begin{cases} -4W_{-n+1} - W_0 + 4W_1 + W_{-n} & , \ \text{if } |W_{-k}| \geq |W_k| \ \text{ and } W_{-k} \leq 0 \ , \quad k \in N, -k \in N^- \\ 4W_{-n+1} + W_0 - 4W_1 - W_{-n} & , \ \text{if } |W_{-k}| \geq |W_k| \ \text{ and } W_{-k} \geq 0 \ , \quad k \in N, -k \in N^- \end{cases}$$

where $k = i - j : i, j = 0, 1, \cdots, n-1$.

Proof**.** Consider $A = T(W_0, W_1, \cdots, W_{n-1})$ which is given as in (3.1). By the definitions of $1 - norm$ and $\infty - norm$ and Equ. (2.3) and Equ. (2.4) and Equ. (2.13), we conclude that

(i): If $|W_{-k}| \geq |W_k|, k \in N$ and $W_{-k} \leq 0, k \in N$, then we get

$$\begin{aligned} \|A\|_1 &= \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| = \max\{|a_{1j}| + |a_{2j}| + |a_{3j}| + \cdots + |a_{nj}|\} = \sum_{i=1}^{n} |a_{in}| \\ &= |a_{1n}| + |a_{2n}| + |a_{3n}| + \cdots + |a_{nn}| \quad = \sum_{k=0}^{n-1} |W_{-k}| \\ &= -\sum_{k=0}^{n-1} W_{-k} = -(\sum_{k=0}^{n} W_{-k} - W_{-n}) \\ &= -4W_{-n+1} - W_0 + 4W_1 + W_{-n} \end{aligned}$$

and if $|W_{-k}| \geq |W_k|, k \in N$ and $W_{-k} \geq 0, k \in N$, then we obtain

$$\begin{aligned}
\|A\|_1 &= \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| = \max\{|a_{1j}| + |a_{2j}| + |a_{3j}| + \cdots + |a_{nj}|\} = \sum_{i=1}^{n} |a_{in}| \\
&= |a_{1n}| + |a_{2n}| + |a_{3n}| + \cdots + |a_{nn}| = \sum_{k=0}^{n-1} |W_{-k}| \\
&= \sum_{k=0}^{n-1} W_{-k} = \sum_{k=0}^{n} W_{-k} - W_{-n} \\
&= 4W_{-n+1} + W_0 - 4W_1 - W_{-n}.
\end{aligned}$$

(ii): If $|W_{-k}| \geq |W_k|, k \in N$ and $W_{-k} \leq 0, k \in N$, then it follows that

$$\begin{aligned}
\|A\|_\infty &= \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| = \max\{|a_{i1}| + |a_{i2}| + |a_{i3}| + \cdots + |a_{in}|\} = \sum_{j=1}^{n} |a_{1j}| \\
&= |a_{11}| + |a_{12}| + |a_{13}| + \cdots + |a_{1n}| \\
&= -\sum_{k=0}^{n-1} W_{-k} = -(\sum_{k=0}^{n} W_{-k} - W_{-n}) \\
&= -4W_{-n+1} - W_0 + 4W_1 + W_{-n}
\end{aligned}$$

and if $|W_{-k}| \geq |W_k|, k \in N$ and $W_{-k} \geq 0, k \in N$, then we get

$$\begin{aligned}
\|A\|_\infty &= \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| = \max\{|a_{i1}| + |a_{i2}| + |a_{i3}| + \cdots + |a_{in}|\} = \sum_{j=1}^{n} |a_{1j}| \\
&= |a_{11}| + |a_{12}| + |a_{13}| + \cdots + |a_{1n}| \\
&= \sum_{k=0}^{n-1} W_{-k} = \sum_{k=0}^{n} W_{-k} - W_{-n} \\
&= 4W_{-n+1} + W_0 - 4W_1 - W_{-n}.
\end{aligned}$$

Thus, the proof is completed. □

Remark 4. *In the statement of the theorem 3 the condion on W_n, W_{-n}, $n \in N$ is given to calculade $\|.\|_1$ and $\|.\|_\infty$ norms of modified Oresme, Oresme-Lucas, Oresme numbers. The other cases can be handled similarly.*

From the last Theorem 3, we have the following corollary which present norm values of $\|A\|_1$ and $\|A\|_\infty$ of A with modified Oresme numbers, Oresme-Lucas numbers and Oresme numbers, respectively, (set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

Corollary 5.

(a): For $A = T(G_{0,}G_1, \cdots, G_{n-1})$, the values of norms of Toeplitz matrices with modified Oresme numbers have the following property:

$$\|A\|_1 = \|A\|_\infty = -4G_{-n+1} + G_{-n} + 4.$$

(b): For $A = T\left(H_0, H_1, \cdots, H_{n-1}\right)$, the values of norms of Toeplitz matrices with Oresme-Lucas numbers have the following property:

$$\|A\|_1 = \|A\|_\infty = 4H_{-n+1} - H_{-n} - 2.$$

(c): For $A = T(O_0, O_1, \cdots, O_{n-1})$, the values of norms of Toeplitz matrices with Oresme numbers have the following property:

$$\|A\|_1 = \|A\|_\infty = -4O_{-n+1} + O_{-n} + 2.$$

Next theorem presents the Frobenious (Euclidian) norm of a Toeplitz matrix A.

THEOREM 6. *Let $A = T\left(W_0, W_1, \cdots, W_{n-1}\right)$ be a Toeplitz matrix with generalized Oresme numbers components, then the Frobenious (Euclidian) norm of matrix A is*

$$\|A\|_F = \sqrt{\Omega_1}$$

where

$\Omega_1 = \frac{1}{81}(96W_n^2 - 24W_{-n}^2 - 15W_{n-1}^2 + 240W_{-n+1}^2 + (72(2^{2n}) + 192(2^{-2n}) - 96)W_0^2 + (576 - 288(2^{2n}) - 768(2^{-2n}))W_0W_1 + (288(2^{2n}) + 768(2^{-2n}) - 1056)W_1^2).$

Proof. The matrix A is of the form

$$A = \begin{pmatrix} W_0 & W_{-1} & W_{-2} & \cdots & W_{1-n} \\ W_1 & W_0 & W_{-1} & \cdots & W_{2-n} \\ W_2 & W_1 & W_0 & \cdots & W_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_{n-1} & W_{n-2} & W_{n-3} & \cdots & W_0 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \|A\|_F^2 &= nW_0^2 + (n-1)W_{-1}^2 + (n-2)W_{-2}^2 + (n-3)W_{-3}^2 + \cdots + W_{1-n}^2 \\ &\quad +(n-1)W_1^2 + (n-2)W_2^2 + (n-3)W_3^2 + \cdots + W_{n-1}^2 \end{aligned}$$

and so

$$\begin{aligned} \|A\|_F^2 &= (2-n)W_0^2 - \frac{7}{9}\sum_{k=1}^{n-1} W_k^2 + \frac{1}{9}\sum_{k=1}^{n-1} W_{k-1}^2 + \frac{16}{9}\sum_{k=1}^{n-1} W_{-k+1}^2 \\ &\quad +\frac{8}{9}\sum_{k=1}^{n-1} W_{-k}^2 + \frac{16}{9}\sum_{k=1}^{n-1} W_1(2W_0 - W_1) \\ &\quad +\frac{1}{27}\sum_{k=1}^{n-1} 2^{-2k+5}(W_0 - 2W_1)^2(2^{2k} - 1) \\ &\quad +\frac{1}{9}\sum_{k=1}^{n-1} W_0^2 - \frac{16}{9}\sum_{k=1}^{n-1} W_1^2 + \frac{8}{27}\sum_{k=1}^{n-1}(W_0 - 2W_1)^2(2^{2k} - 1). \end{aligned}$$

By using the equalities

$$\begin{aligned} \frac{16}{9}\sum_{k=1}^{n-1} W_1(2W_0-W_1) &= \frac{16}{9}(n-1)W_1(2W_0-W_1), \\ \frac{1}{9}\sum_{k=1}^{n-1} W_0^2 &= \frac{1}{9}(n-1)W_0^2, \\ -\frac{16}{9}\sum_{k=1}^{n-1} W_1^2 &= -\frac{16}{9}(n-1)W_1^2. \end{aligned}$$

$$\begin{aligned} \frac{1}{27}\sum_{k=1}^{n-1} 2^{-2k+5}(W_0-2W_1)^2(2^{2k}-1) &= \frac{(W_0-2W_1)^2}{81}(3n2^5+2^7(2^{-2n}-1)), \\ \frac{8}{27}\sum_{k=1}^{n-1}(W_0-2W_1)^2(2^{2k}-1) &= \frac{8\,(W_0-2W_1)^2}{81}(2^{2n}-3n-1), \end{aligned}$$

we obtain

$$\begin{aligned} P &= \frac{16}{9}\sum_{k=1}^{n-1} W_1(2W_0-W_1)+\frac{1}{9}\sum_{k=1}^{n-1} W_0^2-\frac{16}{9}\sum_{k=1}^{n-1} W_1^2 \\ &\quad +\frac{1}{27}\sum_{k=1}^{n-1} 2^{-2k+5}(W_0-2W_1)^2(2^{2k}-1)+\frac{8}{27}\sum_{k=1}^{n-1}(W_0-2W_1)^2(2^{2k}-1), \end{aligned}$$

and it follows that

$$\begin{aligned} P &= \frac{16}{9}\,(n-1)\,W_1(2W_0-W_1)+\frac{1}{9}(n-1)W_0^2-\frac{16}{9}(n-1)W_1^2 \\ &\quad +\frac{(W_0-2W_1)^2}{81}(3n2^5+2^7(2^{-2n}-1))+\frac{8(W_0-2W_1)^2}{81}(2^{2n}-3n-1). \end{aligned}$$

Moreover, we use equation 2.14 and equation 2.15 in Corollary 2.

Therefore, we get

$\|A\|_F^2=(2-n)\,W_0^2-\frac{7}{9}\sum_{k=1}^{n-1} W_k^2+\frac{1}{9}\sum_{k=1}^{n-1} W_{k-1}^2+\frac{16}{9}\sum_{k=1}^{n-1} W_{-k+1}^2+\frac{8}{9}\sum_{k=1}^{n-1} W_{-k}^2+P$

$=\frac{1}{81}(96W_n^2-24W_{-n}^2-15W_{n-1}^2+240W_{-n+1}^2+(72(2^{2n})+192(2^{-2n})-96)W_0^2+(576-288(2^{2n})-768(2^{-2n}))W_0W_1+(288(2^{2n})+768(2^{-2n})-1056)W_1^2)$.

This complates the proof. □

From the last Theorem 6, we have the following corollary which gives Frobenius norm formulas of modified Oreme numbers, Oresme-Lucas numbers and Oresme numbers, respectively, (take $W_n=G_n$, with $G_0=0, G_1=1$ and $W_n=H_n$, with $H_0=2, H_1=1$ and $W_n=O_n$, with $O_0=0, O_1=\frac{1}{2}$, respectively).

Corollary 7. *For $n\geq 0$, Toeplitz matrices with the modified Oresme, Oresme-Lucas and Oresme numbers, respectively have the following properties:*

(a): $\|A\|_F=\sqrt{\Omega_2}$

where A is given as in (3.2)

$$\Omega_2=\frac{1}{81}(96G_n^2-24G_{-n}^2-15G_{-n+1}^2+288(2^{2n})+768(2^{-2n})-1056).$$

(b): $\|A\|_F = \sqrt{\Omega_3}$

where A is given as in (3.3)

$$\Omega_3 = \frac{1}{81}(96H_n^2 - 24H_{-n}^2 - 15H_{n-1}^2 + 240H_{-n+1}^2 - 288).$$

(c): $\|A\|_F = \sqrt{\Omega_4}$

where A is given as in (3.4)

$$\Omega_4 = \frac{1}{81}(96O_n^2 - 24O_{-n}^2 - 15O_{n-1}^2 + 240O_{-n+1}^2 + 72(2^{2n}) + 192(2^{-2n}) - 264).$$

In the following theorem, we present the lower and upper bounds of the spectral norms of the Toeplitz matrices with the modified Oresme numbers, Oresme-Lucas numbers and Oresme numbers, respectively, (set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

Theorem 8.

(a): *Consider $A = T(G_0, G_1, \cdots, G_{n-1})$ which is given as in (3.2). Let*

$$C = \begin{pmatrix} 1 & G_{-1} & G_{-2} & \cdots & G_{1-n} \\ 1 & G_0 & G_{-1} & \cdots & G_{2-n} \\ 1 & G_1 & G_0 & \cdots & G_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & G_{n-2} & G_{n-3} & \cdots & G_0 \end{pmatrix} = \begin{pmatrix} 1 & -4 & -16 & \cdots & G_{1-n} \\ 1 & 0 & -4 & \cdots & G_{2-n} \\ 1 & 1 & 0 & \cdots & G_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & G_{n-2} & G_{n-3} & \cdots & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} G_0 & 1 & 1 & \cdots & 1 \\ G_1 & 1 & 1 & \cdots & 1 \\ G_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}$$

such that $A = C \circ D$ (Hadamart Product of C and D).

(i):

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Omega_2}$$

where Ω_2 is as in Corollary 7.

(ii):

$$\|A\|_2 \leq \Omega_5$$

where

$$\Omega_5 = (\frac{1}{27}(48G_{-n+1}^2 - 3G_{-n}^2 + 32(2^{2n} - 1) - 21))^{\frac{1}{2}} \times \sqrt[2]{n}.$$

(b): *Consider $A = T(H_0, H_1, \cdots, H_{n-1})$ which is given as in (3.3). Let*

$$C = \begin{pmatrix} 1 & H_{-1} & H_{-2} & \cdots & H_{1-n} \\ 1 & H_0 & H_{-1} & \cdots & H_{2-n} \\ 1 & H_1 & H_0 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & H_{n-2} & H_{n-3} & \cdots & H_0 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 8 & \cdots & H_{1-n} \\ 1 & 2 & 4 & \cdots & H_{2-n} \\ 1 & 1 & 2 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & H_{n-2} & H_{n-3} & \cdots & 2 \end{pmatrix}$$

and

$$D = \begin{pmatrix} H_0 & 1 & 1 & \cdots & 1 \\ H_1 & 1 & 1 & \cdots & 1 \\ H_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{2} & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

such that $A = C \circ D$ (Hadamart Product of C and D).

(i):

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Omega_3}$$

where Ω_3 is as in Corollary 7.

(ii):

$$\|A\|_2 \leq \Omega_6$$

where

$$\Omega_6 = \begin{cases} (\frac{1}{27}(48H_{-n+1}^2 - 3H_{-n}^2 - 117))^{\frac{1}{2}} \times \sqrt{n} & , \quad n \geq 6 \\ (-\frac{1}{27}(48H_n^2 - 3H_{n-1}^2 - 144))^{\frac{1}{2}} \times (\frac{1}{27}(48H_{-n+1}^2 - 3H_{-n}^2 - 117))^{\frac{1}{2}} & , \quad 0 \leq n < 6 \end{cases}.$$

(c): *Consider $A = T(O_0, O_1, \cdots, O_{n-1})$ which is given as in (3.4). Let*

$$C = \begin{pmatrix} 1 & O_{-1} & O_{-2} & \cdots & O_{1-n} \\ 1 & O_0 & O_{-1} & \cdots & O_{2-n} \\ 1 & O_1 & O_0 & \cdots & O_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & O_{n-2} & O_{n-3} & \cdots & O_0 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -8 & \cdots & O_{1-n} \\ 1 & 0 & -2 & \cdots & O_{2-n} \\ 1 & \frac{1}{2} & 0 & \cdots & O_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & O_{n-2} & O_{n-3} & \cdots & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} O_0 & 1 & 1 & \cdots & 1 \\ O_1 & 1 & 1 & \cdots & 1 \\ O_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ \frac{1}{2} & 1 & 1 & \cdots & 1 \\ \frac{1}{2} & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}$$

such that $A = C \circ D$ (Hadamart Product of C and D).

(i):

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Omega_4}$$

where Ω_4 *is as in Corollary 7.*

(ii):

$$\|A\|_2 \leq \Omega_7$$

where

$$\Omega_7 = \left(\frac{1}{27}(48O_{-n+1}^2 - 3O_{-n}^2 + 15 + 8(2^{2n} - 1))\right)^{\frac{1}{2}} \times \sqrt{n}.$$

Proof.

(a):

(i): We use equation (2.2).

(ii): By definition, we get

$$\begin{aligned} r_1(C) &= \max_i (\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{1j}|^2)^{\frac{1}{2}} = (1 + \sum_{k=1}^{n-1} W_{-k}^2)^{\frac{1}{2}} \\ &= (\frac{1}{27}(48G_{-n+1}^2 - 3G_{-n}^2 - 24G_0^2 - 48G_1^2 + 8(G_0 - 2G_1)^2(2^{2n} - 1)) + 1)^{\frac{1}{2}} \\ &= (\frac{1}{27}(48G_{-n+1}^2 - 3G_{-n}^2 + 32(2^{2n} - 1) - 21))^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} c_1(D) &= \max_j (\sum_i |d_{ij}|^2)^{\frac{1}{2}} \\ &= \sqrt{n}\ (0 \leq G_i \leq 1, \quad \text{for} \quad (i \geq 0 \text{ and } \quad n \geq 0)). \end{aligned}$$

So, from inequality (2.7),

$$\|A\|_2 \leq r_1(C)c_1(D) = \Omega_5 = (\frac{1}{27}(48G_{-n+1}^2 - 3G_{-n}^2 + 32(2^{2n} - 1) - 21))^{\frac{1}{2}} \times \sqrt{n}$$

(b):

(i): We use equation (2.2).

(ii): We get

$$\begin{aligned} r_1(C) &= \max_i (\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{1j}|^2)^{\frac{1}{2}} = (1 + \sum_{k=1}^{n-1} H_{-k}^2)^{\frac{1}{2}} \\ &= (\frac{1}{27}(48H_{-n+1}^2 - 3H_{-n}^2 - 24H_0^2 - 48H_1^2 + 8(H_0 - 2H_1)^2(2^{2n} - 1)) + 1)^{\frac{1}{2}} \\ &= (\frac{1}{27}(48H_{-n+1}^2 - 3H_{-n}^2 - 117))^{\frac{1}{2}} \end{aligned}$$

and

$$c_1(D) = \max_j (\sum_i |d_{ij}|^2)^{\frac{1}{2}} = \begin{cases} \sqrt{n} & , \quad n \geq 6 \\ (-\frac{1}{27}(48H_n^2 - 3H_{n-1}^2 - 144))^{\frac{1}{2}} & , \quad 0 \leq n < 6 \end{cases}$$

so by defination of Hadamard product and from inequality (2.7)

$$\|A\|_2 \leq r_1(C)c_1(D) = \Omega_6 \begin{cases} (\frac{1}{27}(48H_{-n+1}^2 - 3H_{-n}^2 - 117))^{\frac{1}{2}} \times \sqrt{n} & , \quad n \geq 6 \\ (-\frac{1}{27}(48H_n^2 - 3H_{n-1}^2 - 144))^{\frac{1}{2}} \\ \times(\frac{1}{27}(48H_{-n+1}^2 - 3H_{-n}^2 - 117))^{\frac{1}{2}} & , \quad 0 \leq n < 6 \end{cases}.$$

(c):

(i): We use equation (2.2).

(ii): We get

$$\begin{aligned} r_1(C) &= \max_i (\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{1j}|^2)^{\frac{1}{2}} = (1 + \sum_{k=1}^{n-1} O_{-k}^2)^{\frac{1}{2}} \\ &= (\frac{1}{27}(48O_{-n+1}^2 - 3O_{-n}^2 - 24O_0^2 - 48O_1^2 + 8(O_0 - 2O_1)^2(2^{2n} - 1)) + 1)^{\frac{1}{2}} \\ &= (\frac{1}{27}(48O_{-n+1}^2 - 3O_{-n}^2 + 15 + 8(2^{2n} - 1)))^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} c_1(D) &= \max_j (\sum_i |d_{ij}|^2)^{\frac{1}{2}} \\ &= \sqrt{n} \ (0 \leq O_i \leq 1, \quad \text{for} \quad (i \geq 0 \quad \text{and} \quad n \geq 0)). \end{aligned}$$

so, from inequality (2.7)

$$\|A\|_2 \leq r_1(C)c_1(D) = \Omega_7 = \ (\frac{1}{27}(48O_{-n+1}^2 - 3O_{-n}^2 + 15 + 8(2^{2n} - 1)))^{\frac{1}{2}} \times \sqrt{n}.$$

This complates the proof. □

From the equation (2.10) and Corollary 7, we have the following corollary which gives the Frobenius norms of the Kronecker products of the Toeplitz matrices with special cases of generalized Oresme numbers.

Corollary 9.

(a): Let $A = T(G_0, G_1, \cdots, G_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with modified Oresme numbers and Oresme-Lucas numbers, respectively, then we have the following property.

$$\begin{aligned} \|A \otimes B\|_F &= \|A\|_F \|B\|_F \\ &= \sqrt{\Omega_2}\sqrt{\Omega_3} \end{aligned}$$

where Ω_2 and Ω_3 are as in Corollary 7 (a) and (b),

(set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 1$, respectively).

(b): Suppose that $A = T(G_0, G_1, \cdots, G_{n-1})$ and $B = T(O_0, O_1, \cdots, O_{n-1})$ be Toeplitz matrices with modified Oresme numbers and Oresme numbers, respectively, then we obtain the following property:

$$\begin{aligned} \|A \otimes B\|_F &= \|A\|_F \|B\|_F \\ &= \sqrt{\Omega_2}\sqrt{\Omega_4} \end{aligned}$$

where Ω_2 and Ω_4 are as in Corollary 7 (a) and (c),

(set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

(c): Given $A = T(H_0, H_1, \cdots, H_{n-1})$ and $B = T(O_0, O_1, \cdots, O_{n-1})$ be Toeplitz matrices with Oresme-Lucas numbers and Oresme numbers, respectively, then we get the following property:

$$\begin{aligned} \|A \otimes B\|_F &= \|A\|_F \|B\|_F \\ &= \sqrt{\Omega_3}\sqrt{\Omega_4} \end{aligned}$$

where Ω_3 and Ω_4 are as in Corallary 7 (b) and (c),

(set $W_n = H_n$ with $H_0 = 2, H_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

Proof. (a), (b) and (c) follows from equation (2.10) and Theorem 6 and Corollary 7. □

From the above inequality (2.9) and Theorem 6 and Corollary 7, we have the following result, which gives an upper bound for the Frobenius norm of Hadamard products of Toeplitz matrices by exclusive cases of generalized Oresme numbers.

Corollary 10.

(a): Let $A = T(G_0, G_1, \cdots, G_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with modified Oresme numbers and Oresme-Lucas numbers, respectively, then we have the following property:

$$\begin{aligned} \|A \circ B\|_F &\leq \|A\|_F \|B\|_F \\ &\leq \sqrt{\Omega_2}\sqrt{\Omega_3} \end{aligned}$$

where Ω_2 and Ω_3 are as in Corollary 7 (a) and (b),

(set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 1$, respectively).

(b): Suppose that $A = T(G_0, G_1, \cdots, G_{n-1})$ and $B = T(O_0, O_1, \cdots, O_{n-1})$ be Toeplitz matrices with modified Oresme numbers and Oresme numbers, respectively, then we obtain the following property:

$$\begin{aligned} \|A \circ B\|_F &\leq \|A\|_F \|B\|_F \\ &\leq \sqrt{\Omega_2}\sqrt{\Omega_4} \end{aligned}$$

where Ω_2 and Ω_4 are as in Corollary 7 (a) and (c),

(set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

(c): Assume that $A = T(H_0, H_1, \cdots, H_{n-1})$ and $B = T(O_0, O_1, \cdots, O_{n-1})$ be Toeplitz matrices with Oresme-Lucas numbers and Oresme numbers, respectively, then we have the following property:

$$\begin{aligned} \|A \circ B\|_F &\leq \|A\|_F \|B\|_F \\ &\leq \sqrt{\Omega_3}\sqrt{\Omega_4} \end{aligned}$$

where Ω_3 and Ω_4 are as in Corollary 7 (b) and (c),

(set $W_n = H_n$ with $H_0 = 2, H_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

In the last inequality (2.8) and Theorem 8, we have the following Corollary, which gives an upper bound for the spectral norm of Hadamard products of Toeplitz matrices with special cases of generalized Oresme numbers.

Corollary 11.

(a): Given $A = T(G_0, G_1, \cdots, G_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with modified Oresme numbers and Oresme-Lucas numbers, respectively, then we have following property:

$$\|A \circ B\|_2 \leq \Omega_5 \times \Omega_6$$

where Ω_5 and Ω_6 are as in Theorem 8,

(take $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 1$, respectively).

(b): Let $A = T(G_0, G_1, \cdots, G_{n-1})$ and $B = T(O_0, O_1, \cdots, O_{n-1})$ be Toeplitz matrices with modified Oresme numbers and Oresme numbers respectively, then we have the following property:

$$\|A \circ B\|_2 \leq \Omega_5 \times \Omega_7$$

where Ω_5 and Ω_7 are as in Theorem 8,

(set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

(c): Suppose that $A = T(H_0, H_1, \cdots, H_{n-1})$ and $B = T(O_0, O_1, \cdots, O_{n-1})$ be Toeplitz matrices with Oresme-Lucas numbers and Oresme numbers respectively, then we get the following property:

$$\|A \circ B\|_2 \leq \Omega_6 \times \Omega_7$$

where Ω_6 and Ω_7 are as in Theorem 8,

(set $W_n = H_n$ with $H_0 = 2, H_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

Proof. For (a), (b) and (c) see inequality (2.8) and Theorem 8.□

From the related equation (2.10) and Theorem 8, we have the following Corollary which gives an upper bound for the spectral norm of Kronocker products of Toeplitz matrices with special cases of generalized Oresme numbers.

Corollary 12.

(a): Let $A = T(G_0, G_1, \cdots, G_{n-1})$ and $B = (H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with modified Oresme numbers and Oresme-Lucas numbers, respectively, then we have the following property:

$$\|A \otimes B\|_2 \leq \Omega_5 \times \Omega_6$$

where Ω_5 and Ω_6 are as in Theorem 8,

(set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 1$, respectively).

(b): Let $A = T(G_0, G_1, \cdots, G_{n-1})$ and $B = T(O_0, O_1, \cdots, O_{n-1})$ be Toeplitz matrices with modified Oresme numbers and Oresme numbers respectively, then we get the following property:

$$\|A \otimes B\|_2 \leq \Omega_5 \times \Omega_7$$

where Ω_5 and Ω_7 are as in Theorem 8,

(set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

(c): Let $A = T(H_0, H_1, \cdots, H_{n-1})$ and $B = T(O_0, O_1, \cdots, O_{n-1})$ be Toeplitz matrices with Oresme-Lucas numbers and Oresme numbers respectively, then we obtain the following property:

$$\|A \otimes B\|_2 \leq \Omega_6 \times \Omega_7$$

where Ω_6 and Ω_7 are as in Theorem 8,

(set $W_n = H_n$ with $H_0 = 2, H_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

Proof. For (a), (b) and (c) see equation (2.10) and Theorem 8.□

4. Conclusions

The sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. Recently, there have been so many studies of the sequences of numbers in the literature that concern about subsequences of the Horadam numbers which have second order recurrence relations. Generalized Oresme numbers are special cases of Horadam numbers.

In this paper, we obtain results on Toeplitz matrices with Oresme numbers components.

- In chapter 1, We present some known results on Oresme numbers such as recurrence relation, characteristic equation and Binet's formulas.
- In chapter 2, We give some basic definitions and result of special norms of the Toeplitz matrices and find sum formulas of Toeplitz matrices with Oresme numbers.
- In chapter 3, We obtain special norms of Toeplitz matrices with Oresme numbers and find upper and lower bounds for spectral norms of Toeplitz matrices with Oresme numbers components.

Linear recurrence relations (sequences) have many applications. Now, we present some applications of second order sequences.

- For the applications of Gaussian Fibonacci and Gaussian Lucas numbers to Pauli Fibonacci and Pauli Lucas quaternions, see [2].

- For the application of Pell Numbers to the solutions of three-dimensional difference equation systems, see [4].
- For the application of Jacobsthal numbers to special matrices, see [29].
- For the application of generalized k-order Fibonacci numbers to hybrid quaternions, see [10].
- For the applications of Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [27].
- For the applications of generalized bivariate Fibonacci and Lucas polynomials to matrix polynomials, see [28].
- For the applications of generalized Fibonacci numbers to binomial sums, see [24].
- For the application of generalized Jacobsthal numbers to hyperbolic numbers, see [21].
- For the application of generalized Fibonacci numbers to dual hyperbolic numbers, see [22].
- For the application of Laplace transform and various matrix operations to the characteristic polynomial of the Fibonacci numbers, see [7].
- For the application of Generalized Fibonacci Matrices to Cryptography, see [15].
- For the application of higher order Jacobsthal numbers to quaternions, see [16].
- For the application of Fibonacci and Lucas Identities to Toeplitz-Hessenberg matrices, see [8].
- For the applications of Fibonacci numbers to lacunary statistical convergence, see [3].
- For the applications of Fibonacci numbers to lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, see [12].
- For the applications of Fibonacci numbers to ideal convergence on intuitionistic fuzzy normed linear spaces, see [13].
- For the applications of k-Fibonacci and $k-$Lucas numbers to spinors, see [14].
- For the application of dual-generalized complex Fibonacci and Lucas numbers to Quaternions, see [23].
- For the application of special cases of Horadam numbers to Neutrosophic analysis see [9].
- For the application of Hyperbolic Fibonacci numbers to Quaternions, see [5].

References

[1] Akbulak, M., Bozkurt, D., On the Norms of Toeplitz Matrices Involving Fibonacci and Lucas Numbers, Hacettepe Journal of Mathematics and Statistics, 37(2), 89-95, 2008.

[2] Azak, A.Z., Pauli Gaussian Fibonacci and Pauli Gaussian Lucas Quaternions. Mathematics, 2022, 10, 4655. https://doi.org/10.3390/math10244655

[3] Bilgin, N.G., Fibonacci Lacunary Statistical Convergence of Order γ in IFNLS, International Journal of Advances in Applied Mathematics and Mechanics, 8(4), 28-36, 2021.

[4] Büyük, H., Taşkara, N., On The Solutions of Three-Dimensional Difference Equation Systems Via Pell Numbers, European Journal of Science and Technology, Special Issue 34, 433-440, 2022.

[5] Daşdemir, A., On Recursive Hyperbolic Fibonacci Quaternions, Communications in Advanced Mathematical Sciences, 4(4), 198-207, 2021. DOI:10.33434/cams.997824

[6] Daşdemir, A., On the Norms of Toeplitz Matrices with the Pell, Pell-Lucas and Modfied Pell Numbers, Journal of Engineering Technology and Applied Sciences, 1(2), 51-57, 2016.

[7] Deveci, Ö., Shannon, A.G., On Recurrence Results From Matrix Transforms, Notes on Number Theory and Discrete Mathematics, 28(4), 589–592, 2022. DOI: 10.7546/nntdm.2022.28.4.589-592

[8] Goy, T.,Shattuck, M., Fibonacci and Lucas Identities from Toeplitz-Hessenberg Matrices, Appl. Appl. Math, 14(2), 699–715, 2019.

[9] Gökbaş, H., Topal, S., Smarandache, F., Neutrosophic Number Sequences: An Introductory Study, International Journal of Neutrosophic Science (IJNS), 20(01), 27-48, 2023. https://doi.org/10.54216/IJNS.200103

[10] Gül, K., Generalized k-Order Fibonacci Hybrid Quaternions, Erzincan University Journal of Science and Technology, 15(2), 670-683, 2022. DOI: 10.18185/erzifbed.1132164

[11] Karpuz, G., E., Ateş, F., Güngör, A.D., Cangül, I. N., and Çevik, A.S., On the Norms of Toeplitz and Hankel Matrices with Pell Numbers, CP1281, ICNAAM, Numerical Analysis and Applied Mathematics, International Conference, 2010.

[12] Kişi, Ö., Tuzcuoglu, I., Fibonacci Lacunary Statistical Convergence in Intuitionistic Fuzzy Normed Linear Spaces, Journal of Progressive Research in Mathematics 16(3), 3001-3007, 2020.

[13] Kişi, Ö., Debnathb, P., Fibonacci Ideal Convergence on Intuitionistic Fuzzy Normed Linear Spaces, Fuzzy Information and Engineering, 1-13, 2022. https://doi.org/10.1080/16168658.2022.2160226

[14] Kumari, M., Prasad, K., Frontczak, R., On the k-Fibonacci and $k-$Lucas Spinors, Notes on Number Theory and Discrete Mathematics, 29(2), 322-335, 2023. DOI: 10.7546/nntdm.2023.29.2.322-335

[15] Prasad, K., Mahato, H., Cryptography Using Generalized Fibonacci Matrices with Affine-Hill Cipher, Journal of Discrete Mathematical Sciences & Cryptography, 25(8-A), 2341–2352, 2022. DOI : 10.1080/09720529.2020.1838744

[16] Özkan, E., Uysal, M., On Quaternions with Higher Order Jacobsthal Numbers Components, Gazi University Journal of Science, 36(1), 336-347, 2023. DOI: 10.35378/gujs. 1002454

[17] Shen, S., On the Norms of Toeplitz Matrices Involving $k-$Fibonacci and $k-$Lucas Numbers, Int. J. Contemp. Math. Sciences, 7(8), 363-368, 2012.

[18] Solak, S., Bahşi, M., On the Spectral Norms of Toeplitz Matrices with Fibonacci and Lucas Numbers, Hacettepe Journal of Mathematics and Statistics, 42(1), 15-19, 2013.

[19] Soykan, Y., Generalized Oresme Numbers, Eartline Journal of Mathematical Sciences, 7(2), 333-367, 2021.

[20] Soykan, Y., A Study on the Sum of the Squares of Generalized Oresme Numbers:The Sum Formula $\sum_{k=0}^{n} x^k W_{mk+j}^2$, Asian Journal of Pure and Applied Mathematics 4(1), 16-27, 2022.

[21] Soykan, Y., Taşdemir, E., A Study On Hyperbolic Numbers With Generalized Jacobsthal Numbers Components, International Journal of Nonlinear Analysis and Applications, 13(2), 1965–1981, 2022. http://dx.doi.org/10.22075/ijnaa.2021.22113.2328

[22] Soykan, Y., On Dual Hyperbolic Generalized Fibonacci Numbers, Indian J Pure Appl Math, 2021. https://doi.org/10.1007/s13226-021-00128-2

[23] Şentürk, G.Y., Gürses, N., Yüce, S., Construction of Dual-Generalized Complex Fibonacci and Lucas Quaternions, Carpathian Math. Publ. 2022, 14 (2), 406-418, 2022. doi:10.15330/cmp.14.2.406-418

[24] Ulutaş, Y.T., Toy, D., Some Equalities and Binomial Sums about the Generalized Fibonacci Number u_n, Notes on Number Theory and Discrete Mathematics, 28(2), 252–260, 2022. DOI: 10.7546/nntdm.2022.28.2.252-260

[25] Uygun, Ş., On the Bounds for the Norms of Toeplitz Matrices with the Jacobsthal and Jacobsthal-Lucas Numbers, Journal of Engineering Technology and Applied Sciences 4(3), 105-114, 2019.

[26] Uygun, Ş., Aytar, H., Bounds for the Norms of Toeplitz Matrices with $k-$Jacobsthal and $k-$Jacobsthal-Lucas Numbers, Journal of Scientific Reports-A, 45, 90-100, 2020.

[27] Yılmaz, N., Split Complex Bi-Periodic Fibonacci and Lucas Numbers, Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat. 71(1), 153–164, 2022. DOI:10.31801/cfsuasmas.704435

[28] Yılmaz, N., The Generalized Bivariate Fibonacci and Lucas Matrix Polynomials, Mathematica Montisnigri, Vol LIII, 33-44, 2022. DOI: 10.20948/mathmontis-2022-53-5

[29] Vasanthi, S., Sivakumar, B., Jacobsthal Matrices and their Properties. Indian Journal of Science and Technology 15(5): 207-215, 2022, https://doi.org/10.17485/IJST/v15i5.1948