Three-Step Four-Point Optimized Hybrid Block Method for Direct Solution of General Third Order Differential Equations

Abstract

This research work considers derivation of three step four point optimized hybrid block method for solving general third order differential equations (odes) without reduction to systems of lower order odes. A combination of power series and exponential function is used as an approximate solution to the general third order ode problems. Continuous linear multistep method is developed by interpolating the basis function at both grid and off-grid points and collocating the differential function at only grid points. The unknown parameters in the system of linear equations arising from the collocation and interpolation functions were determined and the values substituted in the approximate solution to the problem. The required continuous method is obtained after necessary simplification. The derived method is tested and found to be consistent, symmetric and of low error constant. The results obtained showed a better performance than the existing methods in literature under review.

Keywords: Optimal method, Third order, Power series and exponential function, Symmetric, Linear and nonlinear problems

1. Introduction

The use of Mathematics to understand the physical world has been in use for centuries, but the manner and degree to which it can be used has drastically changed in recent years due to the intervention of computer and its ability to perform incredibly complex and computational-intensive tasks. These tasks is especially applicable in the study of rocket launch trajectory analysis, airflow over airplane bodies (aerodynamics), transport and disposition of chemicals through the body, immune-assay chemistry for developing new blood tests, seismic underwater acoustic signal processing, eco-systems, psychology and the likes. The modeling of these physical and biological problems give rise to different forms of ordinary differential equations (odes) of different orders and forms. Most of the time analytic solution of such equations and finding an exact solution cannot be solved, therefore numerical methods is applied for the solution. As a result of our subject which is solving general third order ordinary differential equations, we will refer to some solution methods which have been proposed in recent years by other researchers to solve the equations. Lambert (1973, 1991) discussed extensively the approach of reducing higher order ODEs to system of lower order, specifically order one and then applying various methods available for solving the resulting system of first order IVPs. The direct solution of higher order numerically without reducing to a system of first order initial value problems have been studied by various arthors such as Mohammed and Adeniyi (2014), Kayode and Obarhua (2015), Tumba *et al.* (2018), Jikantoro *et al.* (2018), Allogmany and Ismail, (2020). Kuboye and Omar (2015) proposed seven-step block method for solving third order ODEs. Abdelrahim (2019) developed a one-step hybrid block method for solving third order ODEs. Alabi *et al*. (2008) proposed initial value solvers for second order ODEs using chebyshev polynomial as basis function. Sunday *et al.* (2014) developed numerical solution of stiff and oscillatory first order differential equations, using the combination of power series and exponential function as basis function. Momoh *et al.* (2014) used the same basis function to produce a new numerical integration for the solution of stiff first order ODEs. Moreso, most of the methods mentioned above for solving higher order ODEs which were implemented in block mode was an attempt to overcome very early setback of predictor-corrector method for instance, the combination of predictors of lower order with the correctors in the predictor-corrector method. This work considered the unique properties of hybrid methods that give better accuracy implemented in block mode using the approximate solution method proposed in Sunday *et al.* (2014). In the implementation, it should be noted that the block method is problem independent as against the conventional block methods of problem dependent.

2. Derivation of the Method

In this work, the combination of power series and exponential function of the form

$$y(x) = \sum_{j=0}^{c+i-1} a_j x^j + a_{c+i} \sum_{j=0}^{c+i} \frac{\alpha^j x^j}{j!} \qquad (1)$$

is considered as the basic function for the development of the method, where c and i are the number of collocation and interpolation points respectively. a_j's are the parameters to be determined and $\frac{\alpha^j}{j!}$ is the exponential polynomial.

The differential system arising from equation (1) is as given below

$$y'''(x) = \sum_{j=3}^{c+i-1} j(j-1)(j-2)a_j x^{j-3} + a_{c+i}\sum_{j=3}^{c+i} \frac{\alpha^j x^{j-3}}{(j-3)!} \quad (2)$$

Interpolating the basic function (1) at all the grid points $x = x_{n+i},\, i = 0, r, s, 1, 2, u, v$ except the point of evaluation and collocating the differential system (2) at the four grid points $x = x_{n+i},\ i = 0, 1, 2, 3$ where $0 < r,\ s < 1$ and $2 < u,\ v < 3$ respectively which give rise to a system of equations

$$\sum_{j=0}^{c+i-1} a_j x_{n+i}^j + a_{c+i}\sum_{j=0}^{c+i} \frac{\alpha^j x^j}{j!} = y_{n+j},\ i = 0, r, s, 1, 2, u, v \quad (3)$$

$$\sum_{j=3}^{c+i-1} j(j-1)(j-2)a_j x^{j-3} + a_{c+i}\sum_{j=3}^{c+i} \frac{\alpha^j x^{j-3}}{(j-3)!} = f_{n+i},\quad i = 0, 1, 2, 3 \quad (4)$$

where

$f_{n+i} = f(x_{n+i}, y_{n+i}, y'_{n+i}, y''_{n+i})$ and $y_{n+i} = y(x_{n+i});\, x_{n+i} = x_n + ih$, h is the stepsize.

Solving for a_j's from equations (3) and (4) and substituting the values back into equation (1) gives the continuous hybrid method:

$$y(x) = \sum_{j=0}^{k=1} \alpha_j(x) y_{n+j} + \tau_1(x) y_{n+r} + \tau_2(x) y_{n+s} + \tau_3(x) y_{n+u} + \tau_4(x) y_{n+v} + h^3 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \quad (5)$$

Taking the values of r, s, u, v to be $\frac{1}{3}, \frac{2}{3}, \frac{7}{3}, \frac{8}{3}$ and using the transformation in [13], $t = \frac{1}{h}(x - x_{n+k-1})$, $\frac{dt}{dx} = \frac{1}{h}$, $k = 3$, the continuous coefficients α_j's, τ_i's, β_j's and their respective first and second derivatives as functions of t are respectively obtained as:

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} t^{10} \\ t^9 \\ t^8 \\ t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ t^0 \end{bmatrix}^T \begin{bmatrix}
-\frac{334611}{2754944} & \frac{3035227033647}{3359198781320} & -\frac{5004884019033}{26873590250560} & \frac{1340285954823}{1679599390660} & \frac{39783782166039}{26873590250560} & \frac{412941486483}{3359198781320} & -\frac{1954469720367}{53747180501120} & -\frac{2353347693}{6718397562640} & \frac{126162537039}{6718397562640} & \frac{14069586951}{6718397562640} & \frac{245298393}{6718397562640} \\
-\frac{1673055}{2754944} & \frac{787296360717}{167959939066} & -\frac{5916551292801}{5374718050112} & \frac{689983320033}{1679599390660} & -\frac{41219629879743}{5374718050112} & \frac{74052806337}{83979969533} & -\frac{3224251648143}{10749436100224} & -\frac{1077011595}{671839756264} & \frac{17262127377}{167959939066} & \frac{1625254308}{83979969533} & \frac{27788931}{83979969533} \\
-\frac{10021563}{19284608} & \frac{14778754347861}{3359198781320} & -\frac{48099775873743}{26873590250560} & \frac{44590496925321}{11757195734620} & -\frac{192234397595121}{26873590250560} & \frac{48638951134739}{23514391469240} & -\frac{316146947940567}{376230263507840} & -\frac{1563777871}{1343679512528} & \frac{75150932811}{671839756264} & \frac{77106667533}{1343679512528} & \frac{776908529}{671839756264} \\
\frac{15091029}{9642304} & -\frac{4987185092406}{419899847665} & \frac{10696457023137}{13436795125280} & -\frac{31001673238002}{2939298933655} & \frac{264835394616159}{13436795125280} & \frac{2485005259256}{2939298933655} & -\frac{92536492995207}{188115131753920} & \frac{1395356855}{335919878132} & -\frac{86366916157}{335919878132} & \frac{4299655443}{167959939066} & \frac{614982655}{335919878132} \\
\frac{7026831}{2754944} & -\frac{8456675836833}{419899847665} & \frac{136411676304309}{26873590250560} & -\frac{7566844055121}{419899847665} & \frac{901763606819691}{26873590250560} & -\frac{2119974235344}{419899847665} & \frac{108927741542139}{53747180501120} & \frac{377867362609}{60465578063760} & -\frac{10300161224719}{20155192687920} & -\frac{3406265308241}{20155192687920} & \frac{54683191421}{60465578063760} \\
-\frac{1037367}{2754944} & \frac{921924248364}{419899847665} & \frac{173619955135683}{26873590250560} & \frac{1011186898308}{419899847665} & -\frac{122623307363907}{26873590250560} & -\frac{4352720495832}{419899847665} & \frac{227955062217213}{53747180501120} & -\frac{720600587}{671839756264} & \frac{45008396801}{1007759634396} & -\frac{67728598114}{251939908599} & -\frac{860445169}{1007759634396} \\
-\frac{5052699}{2754944} & \frac{5038457699232}{419899847665} & \frac{141485001281847}{26873590250560} & \frac{5038457699232}{419899847665} & -\frac{624033002785591}{26873590250560} & -\frac{3358397181276}{419899847665} & \frac{146289279781449}{53747180501120} & -\frac{5375057575441}{12093115612752} & \frac{204072154045}{503879817198} & -\frac{85467293263}{4031038537584} & -\frac{1471293169}{1511639451594} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 \\
\frac{2046805}{4821152} & -\frac{482371985992}{419899847665} & -\frac{32775723990193}{6718397562640} & -\frac{9028586979504}{2939298933655} & \frac{32913101005029}{6718397562640} & \frac{12382035379272}{2939298933655} & -\frac{41114147498847}{94057565876960} & \frac{15134577607}{15116394515940} & -\frac{46408224041}{419899847665} & \frac{225734694817}{5038798171980} & \frac{633859037}{3779098628985} \\
-\frac{97925}{1205288} & \frac{655528736}{83979969533} & -\frac{69116481981}{419899847665} & \frac{1753076198592}{2939298933655} & -\frac{73670325552}{83979969533} & \frac{1198363629024}{587859786731} & -\frac{69116481981}{4702878293848} & -\frac{15661771}{83979969533} & \frac{5965219084}{251939908599} & -\frac{2411603503}{83979969533} & -\frac{4064732}{251939908599} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{6}$$

$$\begin{bmatrix} \alpha_0' \\ \alpha_1' \\ \alpha_2' \\ \tau_1' \\ \tau_2' \\ \tau_3' \\ \tau_4' \\ \beta_0' \\ \beta_1' \\ \beta_2' \\ \beta_3' \end{bmatrix} = \begin{bmatrix} t^9 \\ t^8 \\ t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ t^0 \end{bmatrix}^T \begin{bmatrix}
-\frac{1673055}{1377472} & \frac{3035227033647}{335919878132} & -\frac{500488401903}{2687359025056} & \frac{690142977416}{83979969533} & \frac{39783782166039}{2687359025056} & \frac{412941486483}{335919878132} & -\frac{1954469720367}{5374718050112} & -\frac{23533476693}{671839756264} & \frac{126162537039}{671839756264} & \frac{14069586951}{671839756264} & \frac{245298393}{671839756264} \\
-\frac{15057495}{2754944} & \frac{7085667246453}{167959939066} & -\frac{53248961635209}{5374718050112} & \frac{6209849880297}{1679599390660} & -\frac{370976668917687}{5374718050112} & \frac{666475257033}{83979969533} & -\frac{29018264833287}{10749436100224} & -\frac{9693104355}{671839756264} & \frac{155359146393}{167959939066} & \frac{14627288772}{83979969533} & \frac{250100379}{83979969533} \\
-\frac{10031563}{2410576} & \frac{14778754347861}{419899847665} & -\frac{48099775873743}{3359198781320} & \frac{89180993850643}{2939298933655} & -\frac{192234397595121}{3359198781320} & \frac{49638951134739}{2939298933655} & -\frac{316146947940567}{47028782938480} & -\frac{1563777871}{167959939066} & \frac{75150932811}{83979969533} & \frac{77106667533}{167959939066} & \frac{776908529}{83979969533} \\
\frac{15091029}{1377472} & -\frac{34910295646842}{419899847665} & \frac{74875199161959}{13436795125280} & -\frac{31001673238002}{419899847665} & \frac{1853847762313113}{13436795125280} & \frac{2485005259356}{419899847665} & -\frac{92536492995207}{26873590250560} & \frac{9767497985}{335919878132} & -\frac{604414413099}{335919878132} & \frac{30097588101}{167959939066} & \frac{4304878585}{335919878132} \\
\frac{21080493}{1377472} & -\frac{50740055020998}{419899847665} & \frac{409235028912927}{13436795125280} & -\frac{45401064330726}{419899847665} & \frac{2705290820459073}{13436795125280} & -\frac{12719845412064}{419899847665} & \frac{326783224626417}{26873590250560} & \frac{377867362609}{10077596343960} & -\frac{10300161224719}{3359198781320} & -\frac{3406265308241}{3359198781320} & \frac{54683191421}{10077596343960} \\
-\frac{5186835}{2754944} & \frac{921924248364}{83979969533} & \frac{173619955135683}{5374718050112} & \frac{1011186898308}{83979969533} & -\frac{122623307363907}{5374718050112} & -\frac{4352720495832}{83979969533} & \frac{227955062217213}{10749436100224} & -\frac{3603002935}{671839756264} & -\frac{225041984005}{1007759634396} & -\frac{338642990570}{251939908599} & -\frac{4302225845}{10077596343960} \\
-\frac{5052699}{688736} & \frac{20153830796928}{419899847665} & \frac{141485001281847}{6718397562640} & \frac{20153830796928}{419899847665} & -\frac{624023002785591}{6718397562640} & -\frac{13433588725104}{419899847665} & \frac{146289279781449}{13436795125280} & -\frac{53750575441}{3023278903188} & \frac{408144308090}{251939908599} & -\frac{85467293263}{1007759634396} & -\frac{2942586338}{755819725797} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
\frac{2046805}{2410576} & -\frac{964743971984}{419899847665} & -\frac{32775723990193}{3359198781320} & -\frac{18057173959008}{2939298933655} & \frac{32913101005029}{3359198781320} & \frac{24764070758544}{2939298933655} & -\frac{41114147498847}{47028782938480} & \frac{15134577607}{7558197257970} & -\frac{2816448082}{419899847665} & \frac{225734694817}{2519399085990} & \frac{1267718074}{3779098628985} \\
-\frac{97925}{1205288} & \frac{655528736}{83979969533} & -\frac{69116481981}{419899847665} & \frac{1753076198592}{2939298933655} & -\frac{73670325552}{83979969533} & \frac{1198363629024}{587859786731} & -\frac{69116481981}{4702878293848} & -\frac{15661771}{83979969533} & \frac{5965219084}{251939908599} & -\frac{2411603503}{83979969533} & -\frac{4064732}{251939908599}
\end{bmatrix} \tag{7}$$

$$\begin{bmatrix} \alpha_0'' \\ \alpha_1'' \\ \alpha_2'' \\ \tau_1'' \\ \tau_2'' \\ \tau_3'' \\ \tau_4'' \\ \beta_0'' \\ \beta_1'' \\ \beta_2'' \\ \beta_3'' \end{bmatrix} = \begin{bmatrix} t^8 \\ t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ t^0 \end{bmatrix}^T \begin{bmatrix}
-\frac{15057495}{1377472} & \frac{27317043302823}{335919878132} & -\frac{4504395617127}{2687359025056} & \frac{6031286796744}{83979969533} & \frac{358054039494351}{2687359025056} & \frac{3716473378347}{335919878132} & -\frac{17590227483303}{5374718050112} & -\frac{21180129237}{671839756264} & \frac{1135462833351}{671839756264} & \frac{126626282559}{671839756264} & \frac{2207685537}{671839756264} \\
-\frac{15057495}{344368} & \frac{28342668985812}{83979969533} & -\frac{53248961635209}{671839756264} & \frac{12419699760594}{419899847665} & -\frac{370976668917687}{671839756264} & \frac{5331802056264}{83979969533} & -\frac{29018264833287}{1343679512528} & -\frac{9693104355}{83979969533} & \frac{671436585572}{83979969533} & \frac{117018310176}{83979969533} & \frac{2000803032}{83979969533} \\
-\frac{10031563}{344368} & \frac{103451280435027}{419899847665} & -\frac{336698431116201}{3359198781320} & \frac{89180993850643}{419899847665} & -\frac{1345640783165847}{3359198781320} & \frac{49638951134739}{419899847665} & -\frac{316146947940567}{6718397562640} & -\frac{10946445097}{167959939066} & \frac{526056529677}{83979969533} & \frac{539746672731}{167959939066} & \frac{5438359703}{83979969533} \\
\frac{45273087}{688736} & -\frac{209461773881052}{419899847665} & \frac{224625597485877}{6718397562640} & -\frac{186010039428012}{419899847665} & \frac{5561543286939339}{6718397562640} & \frac{14910031556136}{419899847665} & -\frac{277609478985621}{13436795125280} & \frac{29302493955}{167959939066} & -\frac{1813243239297}{167959939066} & \frac{90292764303}{83979969533} & \frac{12914635755}{167959939066} \\
\frac{105402465}{1377472} & -\frac{50740055020998}{83979969533} & \frac{409235028912927}{2687359025056} & -\frac{45401064330726}{83979969533} & \frac{2705290820459073}{2687359025056} & -\frac{12719845412064}{83979969533} & \frac{326783224626417}{5374718050112} & \frac{377867362609}{2015519268792} & -\frac{10300161224719}{671839756264} & -\frac{3406265308241}{671839756264} & \frac{54683191421}{2015519268792} \\
-\frac{5186835}{688736} & \frac{3687696993456}{83979969533} & \frac{173619955135683}{1343679512528} & \frac{3687696993456}{83979969533} & -\frac{122623307363907}{1343679512528} & -\frac{17410881983328}{83979969533} & \frac{227955062217213}{2687359025056} & -\frac{3603002935}{167959939066} & -\frac{225041984005}{251939908599} & -\frac{1354571962280}{251939908599} & -\frac{4302225845}{251939908599} \\
-\frac{15158097}{688736} & \frac{60461492390784}{419899847665} & \frac{424455003845541}{6718397562640} & \frac{60461492390784}{419899847665} & -\frac{1872069008356773}{6718397562640} & -\frac{40300766175312}{419899847665} & \frac{438867839344347}{13436795125280} & -\frac{53750575441}{1007759634396} & \frac{408144308090}{83979969533} & -\frac{85467293263}{335919878132} & -\frac{2942586338}{251939908599} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\frac{2046805}{2410576} & -\frac{964743971984}{419899847665} & -\frac{32775723990193}{3359198781320} & -\frac{18057173959008}{2939298933655} & \frac{32913101005029}{3359198781320} & \frac{24764070758544}{2939298933655} & -\frac{41114147498847}{47028782938480} & \frac{15134577607}{7558197257970} & -\frac{92816448082}{419899847665} & \frac{225734694817}{2519399085990} & \frac{1267718074}{3779098628985}
\end{bmatrix} \tag{8}$$

Putting $t=1$ in (5) and evaluate its first and second differentials at points $x=x_n,\ x_{n+\frac{1}{3}},\ x_{n+\frac{2}{3}},\ x_{n+1},\ x_{n+2},\ x_{n+\frac{7}{3}},\ x_{n+\frac{8}{3}},\ x_{n+3}$ while the third derivative of (5) is evaluated at points $x=x_{n+\frac{1}{3}},\ x_{n+\frac{2}{3}},\ x_{n+\frac{7}{3}},\ x_{n+\frac{8}{3}}$ to produce the following discrete schemes represented in matrix form: $Y_m = A_i y_i + h^3 b_i f_i$ (9)

$$A^i = \begin{bmatrix}
1 & -\frac{134017173}{19509355} & \frac{257536746}{19509355} & \frac{175049238}{19509355} & \frac{175049238}{19509355} & -\frac{257536746}{19509355} & \frac{134017173}{19509355} \\
-\frac{8401737}{1205288} & \frac{7452814323456}{419899847665} & -\frac{17079998145273}{839799695330} & \frac{4480078025024}{419899847665} & -\frac{1402813301827}{839799695330} & \frac{1603594387008}{2939298933655} & \frac{51494808213}{3359198781320} \\
-\frac{73725}{172184} & -\frac{13716788614478}{2939298933655} & \frac{2871903006483}{335919878132} & -\frac{1287053969019}{335919878132} & \frac{945828983841}{1679599390660} & -\frac{58146772253}{335919878132} & -\frac{61633008219}{4702878293848} \\
\frac{21645}{1205288} & -\frac{308099977152}{587859786731} & -\frac{2248573008103}{839799695330} & \frac{290332979424}{83979969533} & -\frac{68135693643}{167959939066} & \frac{339833458848}{2939298933655} & \frac{77771704901}{4702878293848} \\
\frac{21645}{1205288} & -\frac{308099977152}{587859786731} & -\frac{2248573008103}{839799695330} & \frac{290332979424}{83979969533} & -\frac{68135693643}{167959939066} & \frac{339833458848}{2939298933655} & \frac{77771704901}{4702878293848} \\
-\frac{97925}{1205288} & \frac{1753076198592}{2939298933655} & -\frac{73670325552}{83979969533} & \frac{655528736}{83979969533} & -\frac{701053136481}{419899847665} & \frac{1198363629024}{587859786731} & -\frac{69116481981}{4702878293848} \\
-\frac{21645}{1205288} & \frac{8971225661}{83979969533} & -\frac{592360752711}{1679599390660} & \frac{190399112379}{335919878132} & -\frac{1215459642789}{335919878132} & \frac{1215459642789}{11757195734620} & \frac{269234210757}{671839756264} \\
\frac{73725}{172184} & -\frac{1721368687872}{587859786731} & \frac{978418216069}{167959939066} & -\frac{1849645379904}{419899847665} & \frac{1288802238087}{167959939066} & -\frac{1192648166592}{83979969533} & \frac{178897221471799}{23514391469240} \\
\frac{8401737}{1205288} & -\frac{140792188085874}{2939298933655} & \frac{153637660902267}{1679599390660} & -\frac{20449126961075}{335919878132} & \frac{87130949308933}{1679599390660} & -\frac{842758029830079}{11757195734620} & \frac{141723887783193}{4702878293848} \\
\frac{71845639}{2410576} & -\frac{352454681771712}{2939298933655} & \frac{589130396234811}{3359198781320} & -\frac{40026637807856}{419899847665} & \frac{50996079753593}{3359198781320} & -\frac{14893662351024}{2939298933655} & -\frac{2875799846853}{47028782938480} \\
\frac{17034855}{2410576} & -\frac{35284939011291}{5878597867310} & -\frac{43975991803371}{3359198781320} & \frac{22828178874039}{1679599390660} & -\frac{7949710399893}{3359198781320} & \frac{9468619099701}{11757195734620} & -\frac{163175504397}{47028782938480} \\
-\frac{1739523}{2410576} & \frac{33638118110112}{2939298933655} & -\frac{70456993732131}{3359198781320} & \frac{4352068723056}{419899847665} & -\frac{518693997993}{3359198781320} & -\frac{13814618736}{2939298933655} & \frac{2176047115113}{47028782938480} \\
\frac{2046805}{2410576} & -\frac{39427645910139}{5878597867310} & \frac{65953759395231}{3359198781320} & -\frac{29183992551689}{1679599390660} & \frac{1774309337313}{3359198781320} & -\frac{16586021331531}{11757195734620} & -\frac{14607763561863}{47028782938480} \\
\frac{2046805}{2410576} & -\frac{18057173959008}{2939298933655} & \frac{32913101005029}{3359198781320} & -\frac{964743971984}{419899847665} & -\frac{32775723990193}{3359198781320} & \frac{24764070758544}{2939298933655} & -\frac{41114147498847}{47028782938480} \\
-\frac{1739523}{2410576} & \frac{29412750242949}{5878597867310} & -\frac{32015151922281}{3359198781320} & \frac{10615730702199}{1679599390660} & \frac{13066394382057}{3359198781320} & -\frac{134601704809419}{11757195734620} & \frac{305083934933313}{47028782938480} \\
\frac{17034855}{2410576} & -\frac{142695392816832}{2939298933655} & \frac{316069685834331}{3359198781320} & -\frac{27618163685616}{419899847665} & \frac{258651956833113}{3359198781320} & -\frac{312672813158064}{2939298933655} & \frac{2000683597474587}{47028782938480} \\
\frac{71845639}{2410576} & -\frac{240785630775135}{1175719573462} & \frac{260922764067633}{671839756264} & -\frac{84732708885877}{335919878132} & \frac{260922764067633}{671839756264} & -\frac{512753326753167}{2351439146924} & \frac{797854904562231}{9405756587696} \\
-\frac{2636235}{43046} & \frac{22202498663424}{83979969533} & -\frac{33356998207323}{83979969533} & \frac{18074872486944}{83979969533} & -\frac{2638739151909}{83979969533} & \frac{772610051808}{83979969533} & \frac{177761209797}{167959939066} \\
\frac{255465}{86092} & -\frac{6448326433920}{83979969533} & \frac{31734529634745}{167959939066} & -\frac{11181182987520}{83979969533} & \frac{4480344803655}{167959939066} & -\frac{633316682880}{83979969533} & -\frac{375235934535}{335919878132} \\
-\frac{255465}{86092} & \frac{1805643838956}{83979969533} & -\frac{5312529115929}{167959939066} & -\frac{4224722544}{83979969533} & \frac{17890470616473}{167959939066} & -\frac{12577683576528}{83979969533} & \frac{18945966315591}{335919878132} \\
-\frac{2636235}{43046} & -\frac{35418956811264}{83979969533} & \frac{67120128307323}{83979969533} & -\frac{43508353641984}{83979969533} & \frac{28072220306949}{83979969533} & -\frac{34535740151808}{83979969533} & \frac{26255155085883}{167959939066}
\end{bmatrix}$$

$$
y_i = \begin{bmatrix} y_n \\ y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \\ y_{n+2} \\ y_{n+\frac{7}{3}} \\ y_{n+\frac{8}{3}} \end{bmatrix}, \qquad f_i = \begin{bmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix}
$$

$$
b_i = h^3 \begin{bmatrix}
\frac{10080}{3901871} & -\frac{670544}{3901871} & -\frac{670544}{3901871} & \frac{10080}{3901871} \\
\frac{1255845598}{251939908599} & \frac{37600442656}{251939908599} & -\frac{353308142}{251939908599} & -\frac{556864}{251939908599} \\
-\frac{23796396316}{61221397789557} & -\frac{967463563304}{20407132596519} & \frac{114517461254}{20407132596519} & \frac{403007626}{61221397789557} \\
-\frac{625030261}{61221397789557} & \frac{616176885556}{20407132596519} & -\frac{99180437971}{20407132596519} & -\frac{597717164}{61221397789557} \\
\frac{48814768}{251939908599} & -\frac{3584157453}{83979969533} & \frac{2447557234}{251939908599} & \frac{1964729}{83979969533} \\
-\frac{15661771}{83979969533} & \frac{5965219084}{251939908599} & -\frac{2411603503}{83979969533} & -\frac{4064732}{251939908599} \\
-\frac{3437974064}{61221397789557} & -\frac{36200371081}{20407132596519} & \frac{679156952446}{20407132596519} & -\frac{3465287161}{61221397789557} \\
\frac{68122369126}{61221397789557} & -\frac{1387096676896}{20407132596519} & -\frac{2469077701454}{20407132596519} & \frac{43922965184}{61221397789557} \\
\frac{4536381116}{251939908599} & -\frac{305340277940}{251939908599} & -\frac{264206753858}{251939908599} & \frac{5792783578}{251939908599} \\
-\frac{683823080003}{7558197257970} & -\frac{1739745329176}{1259699542995} & \frac{299428484567}{2519399085990} & -\frac{144708176}{3779098628985} \\
\frac{1677027793646}{306106988947785} & \frac{45777219811523}{204071325965190} & -\frac{1731710321299}{102035662982595} & \frac{8985480863}{612213977895570} \\
-\frac{625113798593}{612213977895570} & -\frac{1879841913686}{102035662982595} & -\frac{1338888923423}{204071325965190} & -\frac{9594174566}{306106988947785} \\
\frac{7021842176}{3779098628985} & -\frac{593361340867}{2519399085990} & \frac{31545340407}{419899847665} & \frac{1444542893}{7558197257970} \\
\frac{15134577607}{612213977895570} & -\frac{92816448082}{419899847665} & \frac{225734694817}{2519399085990} & \frac{1267718074}{3779098628985} \\
-\frac{561056345584}{306106988947785} & \frac{26646182731853}{204071325965190} & \frac{14533488817901}{102035662982595} & -\frac{516187241707}{612213977895570} \\
\frac{11167582884637}{612213977895570} & -\frac{122183317093976}{102035662982595} & -\frac{293607274642073}{204071325965190} & \frac{3911256389104}{306106988947785} \\
\frac{291119546126}{3779098628985} & -\frac{13203623705357}{2519399085990} & -\frac{4712352281219}{1259699542995} & \frac{1265772755903}{7558197257970} \\
\frac{-143595004640}{6802377532173} & \frac{12367833015700}{6802377532173} & \frac{1673964243512}{6802377532173} & -\frac{4161644455}{802377532173} \\
\frac{32601779882}{6802377532173} & -\frac{14033897541888}{6802377532173} & \frac{2190491772567}{6802377532173} & \frac{4547327392}{6802377532173} \\
-\frac{47598188408}{6802377532173} & \frac{5659327362507}{6802377532173} & -\frac{10565061951948}{6802377532173} & -\frac{19543735918}{6802377532173} \\
\frac{1072054931945}{6802377532173} & -\frac{73965282498032}{6802377532173} & -\frac{54224485238820}{6802377532173} & \frac{932621571760}{6802377532173}
\end{bmatrix}, \quad
Y_m = \begin{bmatrix} y_{n+3} \\ hy'_n \\ hy'_{n+\frac{1}{3}} \\ hy'_{n+\frac{2}{3}} \\ hy'_{n+1} \\ hy'_{n+2} \\ hy'_{n+\frac{7}{3}} \\ hy'_{n+\frac{8}{3}} \\ hy'_{n+3} \\ h^2 y''_n \\ h^2 y''_{n+\frac{1}{3}} \\ h^2 y''_{n+\frac{2}{3}} \\ h^2 y''_{n+1} \\ h^2 y''_{n+2} \\ h^2 y''_{n+\frac{7}{3}} \\ h^2 y''_{n+\frac{8}{3}} \\ h^2 y''_{n+3} \\ h^3 y'''_{n+\frac{1}{3}} \\ h^3 y'''_{n+\frac{2}{3}} \\ h^3 y'''_{n+\frac{7}{3}} \\ h^3 y'''_{n+\frac{8}{3}} \end{bmatrix}
$$

Adopting matrix inversion method to solve (9), $y_{n+\frac{1}{3}}, y_{n+\frac{2}{3}}, y_{n+1}, y_{n+2}, y_{n+\frac{7}{3}}, y_{n+\frac{8}{3}}, y_{n+3}, y'_{n+\frac{1}{3}}, y'_{n+\frac{2}{3}}, y'_{n+1}, y'_{n+2}, y'_{n+\frac{7}{3}}, y'_{n+\frac{8}{3}}, y'_{n+3}, y''_{n+\frac{1}{3}}, y''_{n+\frac{2}{3}}, y''_{n+1}, y''_{n+2}, y''_{n+\frac{7}{3}}, y''_{n+\frac{8}{3}}, y''_{n+3}$

are determined and expressed as given below

$$y_{n+\frac{1}{3}} = y_n + \frac{1}{3}hy'_n + \frac{1}{18}h^2 y''_n + h^3\left[\frac{1755457}{493807104}f_n + \frac{267401}{146966400}f_{n+1} - \frac{50639}{58786560}f_{n+2} + \frac{82967}{1234517760}f_{n+3}\right]$$

$$y_{n+\frac{2}{3}} = y_n + \frac{2}{3}hy'_n + \frac{2}{9}h^2 y''_n + h^3\left[\frac{12601}{688905}f_n + \frac{13429}{1148175}f_{n+1} - \frac{2539}{459270}f_{n+2} + \frac{4153}{9644670}f_{n+3}\right]$$

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2 y''_n + h^3\left[\frac{7169}{161280}f_n + \frac{643}{22400}f_{n+1} - \frac{73}{5376}f_{n+2} + \frac{599}{564480}f_{n+3}\right]$$

$$y_{n+2} = y_n + 2hy'_n + 2h^2 y''_n + h^3\left[\frac{83}{441}f_n + \frac{211}{525}f_{n+1} + \frac{1}{70}f_{n+2} + \frac{1}{630}f_{n+3}\right]$$

$$y_{n+\frac{7}{3}} = y_n + \frac{7}{3}hy'_n + \frac{49}{18}h^2 y''_n + h^3\left[\frac{12837461}{50388480}f_n + \frac{15512861}{20995200}f_{n+1} + \frac{1226911}{8398080}f_{n+2} - \frac{45619}{25194240}f_{n+3}\right]$$

$$y_{n+\frac{8}{3}} = y_n + \frac{8}{3}hy'_n + \frac{32}{9}h^2 y''_n + h^3\left[\frac{1592128}{4822335}f_n + \frac{1371136}{1148175}f_{n+1} + \frac{88576}{229635}f_{n+2} - \frac{36352}{4822335}f_{n+3}\right]$$

$$y_{n+3} = y_n + 3hy'_n + \frac{9}{2}h^2 y''_n + h^3\left[\frac{1485}{3584}f_n + \frac{39609}{22400}f_{n+1} + \frac{6561}{8960}f_{n+2} - \frac{963}{62720}f_{n+3}\right]$$

$$y'_{n+\frac{1}{3}} = y'_n + \frac{1}{3}hy''_n + h^2\left[\frac{2193335}{82301184}f_n + \frac{413647}{24494400}f_{n+1} + \frac{387239}{48988800}f_{n+2} + \frac{63233}{102876480}f_{n+3}\right]$$

$$y'_{n+\frac{2}{3}} = y'_n + \frac{2}{3}hy''_n + h^2\left[\frac{394727}{6429780}f_n + \frac{15041}{382725}f_{n+1} - \frac{1043}{54675}f_{n+2} + \frac{4799}{3214890}f_{n+3}\right]$$

$$y'_{n+1} = y'_n + hy''_n + h^2\left[\frac{54041}{564480}f_n + \frac{1207}{16800}f_{n+1} - \frac{2033}{67200}f_{n+2} + \frac{19}{8064}f_{n+3}\right]$$

$$y'_{n+2} = y'_n + 2hy''_n + h^2\left[\frac{47}{252}f_n + \frac{437}{525}f_{n+1} + \frac{128}{525}f_{n+2} - \frac{29}{4410}f_{n+3}\right]$$

$$y'_{n+\frac{7}{3}} = y'_n + \frac{7}{3}hy''_n + h^2\left[\frac{1788157}{8398080}f_n + \frac{1039633}{874800}f_{n+1} + \frac{3892021}{6998400}f_{n+2} - \frac{57281}{4199040}f_{n+3}\right]$$

$$y'_{n+\frac{8}{3}} = y'_n + \frac{8}{3}hy''_n + h^2\left[\frac{384704}{1607445}f_n + \frac{591104}{382725}f_{n+1} + \frac{336256}{382725}f_{n+2} - \frac{6784}{321489}f_{n+3}\right]$$

$$y'_{n+3} = y'_n + 3hy''_n + h^2\left[\frac{16683}{62720}f_n + \frac{21249}{11200}f_{n+1} + \frac{3861}{3200}f_{n+2} - \frac{321}{15680}f_{n+3}\right]$$

$$y''_{n+\frac{1}{3}} = y''_n + h\left[\frac{4913413}{45722880} f_n + \frac{98209}{1088640} f_{n+1} - \frac{45151}{1088640} f_{n+2} + \frac{146693}{45722880} f_{n+3}\right]$$

$$y''_{n+\frac{2}{3}} = y''_n + h\left[\frac{145373}{1428840} f_n + \frac{377}{8505} f_{n+1} - \frac{893}{34020} f_{n+2} + \frac{757}{357210} f_{n+3}\right]$$

$$y''_{n+1} = y''_n + h\left[\frac{6541}{62720} f_n + \frac{2801}{13440} f_{n+1} - \frac{559}{13440} f_{n+2} + \frac{583}{188160} f_{n+3}\right]$$

$$y''_{n+2} = y''_n + h\left[\frac{463}{5880} f_n + \frac{113}{105} f_{n+1} + \frac{347}{420} f_{n+2} - \frac{11}{490} f_{n+3}\right]$$

$$y''_{n+\frac{7}{3}} = y''_n + h\left[\frac{74389}{933120} f_n + \frac{164983}{155520} f_{n+1} + \frac{154007}{155520} f_{n+2} - \frac{18571}{933120} f_{n+3}\right]$$

$$y''_{n+\frac{8}{3}} = y''_n + h\left[\frac{14044}{178605} f_n + \frac{9152}{8505} f_{n+1} + \frac{8032}{8505} f_{n+2} - \frac{4576}{178605} f_{n+3}\right] \quad (10)$$

$$y''_{n+3} = y''_n + h\left[\frac{5133}{62720} f_n + \frac{927}{896} f_{n+1} + \frac{927}{896} f_{n+2} + \frac{5133}{62720} f_{n+3}\right]$$

3. Analysis of the Method

This section examines the proposed main approach in order to determine its validity. The nature of the method's convergence is revealed by these qualities, which include order and error constants, consistency, region of absolute stability, and zero stability.

3.1 Order and Error constant

Consider the linear operator L be associated with the 4-point schemes be defined as

$$L\{y(x), h\} = y\left(x_{n+k}\right) - \sum_{j=0}^{k} \begin{Bmatrix} \alpha_j y(x_{n+j}) + (\tau_1 y(x_{n+r}) + \tau_2 y(x_{n+s}) + \\ \tau_3 y(x_{n+u}) + \tau_4 y(x_{n+v})) + h^3 \beta_j y'''(x_{n+j}) \end{Bmatrix} \quad (11)$$

where α_0 and β_0 are not both zero and $y(x)$ is an arbitrary test function that is continuous and differentiable in the interval $[a, b]$. Expanding y_{n+j} and y'''_{n+j}, $j = 0, 1, \ldots, m$ in Taylor series about x_n and collecting like terms in h and y gives;

$$L[y(x), h] = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y^{(2)}(x) + \ldots + c_p h^p y^{(p)}(x) \quad (12)$$

$$L\{y(x),h\} = \begin{pmatrix} y(x_n) \\ khy'(x_n) \\ \frac{(kh)^2}{2!}y''(x_n) \\ . \\ . \\ . \\ \frac{(kh)^{p+2}}{(p+2)!}y^{p+2}(x_n) \end{pmatrix} - \sum_{j=0}^{k} \left[\begin{array}{l} \alpha_j \begin{pmatrix} y(x_n) \\ (jh)y'(x_n) \\ \frac{(jh)^2}{2!}y''(x_n) \\ . \\ . \\ . \\ \frac{(jh)^{p+2}}{(p+2)!}y^{p+2}(x_n) \end{pmatrix} + \tau_1 \begin{pmatrix} y(x_n) \\ (rh)y'(x_n) \\ \frac{(rh)^2}{2!}y''(x_n) \\ . \\ . \\ . \\ \frac{(rh)^{p+2}}{(p+2)!}y^{p+2}(x_n) \end{pmatrix} + \tau_2 \begin{pmatrix} y(x_n) \\ (sh)y'(x_n) \\ \frac{(sh)^2}{2!}y''(x_n) \\ . \\ . \\ . \\ \frac{(sh)^{p+3}}{(p+3)!}y^{p+3}(x_n) \end{pmatrix} + \\ \tau_3 \begin{pmatrix} y(x_n) \\ (uh)y'(x_n) \\ \frac{(uh)^2}{2!}y''(x_n) \\ . \\ . \\ . \\ \frac{(uh)^{p+3}}{(p+3)!}y^{p+3}(x_n) \end{pmatrix} + \tau_4 \begin{pmatrix} y(x_n) \\ (vh)y'(x_n) \\ \frac{(vh)^2}{2!}y''(x_n) \\ . \\ . \\ . \\ \frac{(vh)^{p+3}}{(p+3)!}y^{p+3}(x_n) \end{pmatrix} + \\ h^3 \sum_{j=0}^{k} \beta_j \begin{pmatrix} 0 \\ 0 \\ 0 \\ y'''(x_n) \\ . \\ . \\ . \\ \frac{(uh)^{p+3}}{(p+3)!}y^{p+3}(x_n) \end{pmatrix} = \begin{pmatrix} C_0 y(x_n) \\ C_1 y'(x_n) \\ C_2 y''(x_n) \\ . \\ . \\ . \\ C_{p+3} y^{p+3}(x_n) \end{pmatrix} \end{array} \right]$$

Therefore, applying the linear operator L (11) to determine the order and error constant of the main method.

$$y_{n+3} = \frac{1}{19509355}\left[\begin{array}{l} 19509355y_n - 134017173y_{n+\frac{1}{3}} + 257536746y_{n+\frac{2}{3}} - 175049238y_{n+1} + 175049238y_{n+2} - \\ 257536746y_{n+\frac{7}{3}} + 134017173y_{n+\frac{8}{3}} \end{array}\right] + \frac{h^3}{3901871}\left[10080f_n - 670544f_{n+1} - 670544f_{n+2} + 10080f_{n+3}\right] \tag{13}$$

Going by [19], the multistep method (13) has order p if

$L\left[y(x),h\right] = 0\left(h^{p+1}\right)$,

$c_0 = c_1 = \cdots = c_p = 0, c_{p+3} \neq 0$. Therefore c_{p+3} is the error constant. The order of the proposed main method is eight while the error constant is -7.0408×10^{-7}.

3.2 Zero Stability

The new block method is zero stable if the first characteristic polynomial

$$\rho(\zeta) = \left|\sum_{i=0}^{k} a^{(i)}\zeta^{k-1}\right| = 0 \tag{14}$$

and satisfies $\left|\zeta_j\right| = 1$, the multiplicity must not exceed the order of the differential equation Omole and Ukpebor (2020).

$$A = \left| \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix} - \begin{bmatrix} \zeta&0&0&0&0&0&1&0 \\ 0&\zeta&0&0&0&0&1&0 \\ 0&0&\zeta&0&0&0&1&0 \\ 0&0&0&\zeta&0&0&1&0 \\ 0&0&0&0&\zeta&0&1&0 \\ 0&0&0&0&0&\zeta&1&0 \\ 0&0&0&0&0&0&\zeta&0 \\ 0&0&0&0&0&0&0&\zeta \end{bmatrix} \right| = \zeta^7(\zeta-1) = 0$$

This implies $A = (1-\zeta)\zeta^7$, $\zeta = 0,0,0,0,0,0,0,0,1$. Therefore, the method is zero-stable.

3.3 Region of Absolute Stability

In this section, the regions of absolute stability of the new methods are determined in order to guide the choice of the stepsize for the methods.

In doing this, let the test problem for the methods be given as

$$y''' + \lambda^3 f = 0 \qquad (15)$$

where $f = f(x, y, y', y'')$ and λ is complex. The stability polynomial of the derived continuous methods (6) given generally by

$$\pi(r, \bar{h}) = \rho(r) - \bar{h}\sigma(r) = 0 \qquad (16)$$

where $\rho(r)$ and $\sigma(r)$ are the first and second characteristic polynomials respectively, $\bar{h} = -\lambda^3 h^3$ and $\lambda = \dfrac{d^3 f}{dy^3}$.

Using the test problem in (17) for the block mode (11) the method yields

$$\bar{h}(r) = -\left(\frac{A^0 Y_m(r) - A^i y_m(r)}{B_i F_m(r)} \right) \qquad (17)$$

since $\bar{h}$ is given as $\bar{h} = h^3\lambda^3$ and $r = e^{i\theta}$, [15].

Adopting the method of [16], the method is reformulated as

$$\begin{bmatrix} Y \\ . \\ . \\ . \\ Y_{i+1} \end{bmatrix} = \begin{bmatrix} A & & U \\ . & & . \\ . & \cdots & . \\ . & & . \\ B & & V \end{bmatrix} \begin{bmatrix} h^3 f(y) \\ . \\ . \\ . \\ f_{i-1} \end{bmatrix} \qquad (18)$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{162} & \frac{1}{1944} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{4}{81} & \frac{8}{162} & \frac{4}{3645} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{162} & 0 & \frac{6}{162} & \frac{-1}{162} & 0 & 0 & 0 & 0 \\ \frac{23}{485} & 0 & 0 & 1 & \frac{14}{485} & 0 & 0 & 0 \\ \frac{203771}{19830258} & 0 & 0 & -\frac{2514251}{5665788} & -\frac{20891}{2832894} & \frac{15749}{4406724} & 0 & 0 \\ -\frac{240872270}{47893220001} & 0 & 0 & \frac{13309632128}{47893220001} & \frac{2589563160}{47893220001} & 0 & \frac{163532262}{47893220001} & 0 \\ \frac{10080}{3901871} & 0 & 0 & \frac{-670544}{3901871} & -\frac{670544}{3901871} & 0 & 0 & \frac{10080}{3901871} \end{bmatrix},$$

$$B=\begin{bmatrix} \frac{10080}{3901871} & 0 & 0 & \frac{-670544}{3901871} & -\frac{670544}{3901871} & 0 & 0 & \frac{10080}{3901871} \\ \frac{203771}{19830258} & 0 & 0 & -\frac{2514251}{5665788} & -\frac{20891}{2832894} & \frac{15749}{4406724} & 0 & 0 \\ \frac{1}{162} & 0 & \frac{6}{162} & \frac{-1}{162} & 0 & 0 & 0 & 0 \\ \frac{1}{162} & \frac{1}{1944} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$V=\begin{bmatrix} 0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1 \end{bmatrix},\quad f_{i-1}=\begin{bmatrix} f_n\\ f_{n+1}\\ f_{n+2}\\ f_{n+3} \end{bmatrix}, U=\begin{bmatrix} 0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1 \end{bmatrix},\quad f(y)=\begin{bmatrix} f_n\\ f_{n+\frac{1}{3}}\\ f_{n+\frac{2}{3}}\\ f_{n+1}\\ f_{n+2}\\ f_{n+\frac{7}{3}}\\ f_{n+\frac{8}{3}}\\ f_{n+3} \end{bmatrix},\quad Y=\begin{bmatrix} y_n\\ y_{n+\frac{1}{3}}\\ y_{n+\frac{2}{3}}\\ y_{n+1}\\ y_{n+2}\\ y_{n+\frac{7}{3}}\\ y_{n+\frac{8}{3}}\\ y_{n+3} \end{bmatrix}$$

The elements A, B, U, V, M and I are substituted into the stability matrix

$$M(z)=V+zB(M-zA)^{-1}U \qquad (19)$$

where M and I are identity matrix of dimension 8 and 4 respectively, then equation (19) is then substituted into the stability function given as

$$\rho(\eta, z)=\det(\eta I-M(z)) \qquad (20)$$

Computing (21) gives the stability polynomial $f(z)$ and its derivative $f'(z)$ using Maple software. These are then plotted in MATLAB (R2013a) environment to produce the required region of absolute stability of the method.

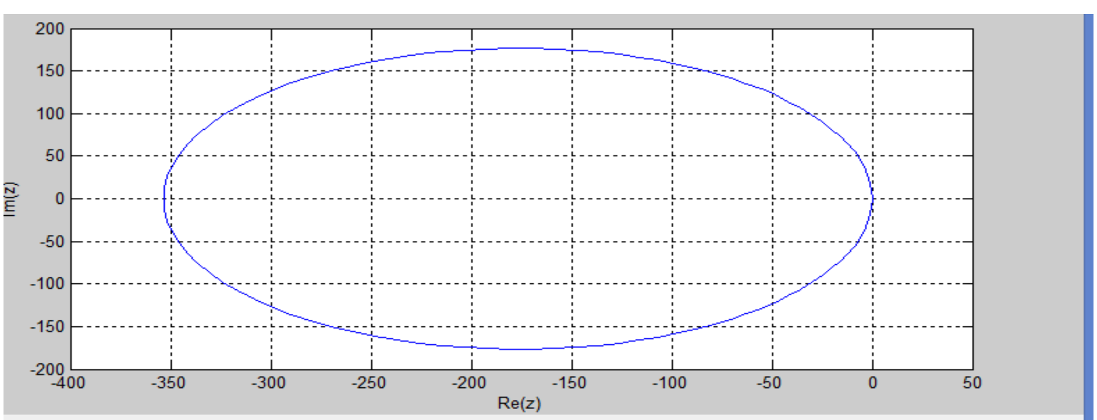

Figure 1: Region of the new, enhanced hybrid method's absolute stability. Figure 1 depicts the area where the approaches are completely stable.

4. Numerical Examples

Tables 1-5 demonstrate the results of using the developed method to solve linear and nonlinear second order ordinary differential problems.

Problem 1.

$$y''' = e^x, \; y(0)=3, \; y'(0)=1, \; y''(0)=5, \; h=0.1$$

Exact solution is

$$y(x) = 2 + 2x^2 + e^x$$

The absolute errors $|y_e - y_c|$ obtained with the method for problem 1 is compared with that of [8], 10-step and [15], 5-step and order of accuracy 8 and 9 respectively.

Table 1: Comparison of results for solving Problem 1 $(h = 0.1)$

x	y_{ex}	y_c	A_e in [8]	A_e in [15]	A_e in New BM,
0.1	3.1251709180756477	3.1251709180756476	2.531308e-14	0.0000e-00	7.26456e-18
0.2	3.3014027581601697	3.3014027581601699	1.612044e-13	2.8422e-13	2.03177e-17
0.3	3.5298588075760033	3.5298588075760031	4.023448e-13	1.6729e-12	2.19073e-17
0.4	3.8118246976412706	3.8118246976412702	7.536194e-13	2.9983e-11	1.10779e-16
0.5	4.1487212707001282	4.1487212707001279	1.212364e-12	3.1673e-11	2.40794e-16
0.6	4.5421188003905097	4.5421188003905085	1.780798e-12	9.1855e-11	4.94384e-16
0.7	4.9937527074704775	4.9937527074704757	2.456702e-12	8.9511e-11	8.59898e-16
0.8	5.5055409284924695	5.5055409284924663	2.212097e-11	1.9168e-10	1.32991e-15
0.9	6.0796031111569526	6.0796031111569476	5.231993e-11	2.1110e-10	2.01568e-15
1.0	6.7182818284590482	6.7182818284590423	8.860113e-11	4.9398e-10	2.90150e-15

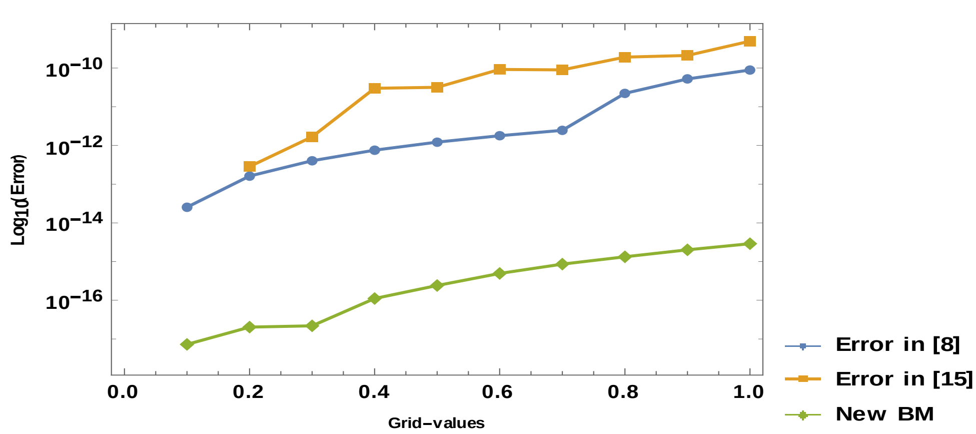

Figure 2. Comparison curve $\log_{10}(error)$ in existing methods with the proposed method in Problem 1 with $h = 10^{-1}, \; x \in (0.1, 1.0)$.

Problem 2. $y''' = 3\sin x, \;\; y(0) = 1, \;\; y'(0) = 0, \;\; y''(0) = -2, h = 0.1$

Exact solution is $y(x) = 3\cos x + \frac{x^2}{2} - 2$

The absolute errors $|y_e - y_c|$ obtained with the method for problem 2 is compared with that of [18] and [19] 2-step and 8-step respectively.

Table 2: Comparison of results for solving Problem 2 $(h = 0.1)$

x	y_{ex}	y_c	A_e in [16]	A_e in [17]	A_e in New BM,
0.1	0.9900124958340770	0.9900124958340773	4.1078e-15	2.2204e-16	2.549756e-18
0.2	0.9601997335237251	0.9601997335237249	1.6875e-14	4.4409e-16	6.752039e-18
0.3	0.9110094673768181	0.9110094673768180	5.0848e-14	1.3323e-15	1.093288e-17
0.4	0.8431829820086554	0.8431829820086552	1.1779e-13	3.8858e-15	4.262390e-17
0.5	0.7577476856711178	0.7577476856711181	2.4081e-13	9.2149e-15	8.702609e-17
0.6	0.6560068447290348	0.6560068447290347	4.3709e-13	1.8985e-14	2.126558e-17
0.7	0.5395265618534650	0.5395265618534649	7.3708e-13	3.4084e-14	4.079019e-16
0.8	0.4101201280414957	0.4101201280414956	1.1662e-12	5.7343e-14	6.668432e-16
0.9	0.2698299048119925	0.2698299048119923	1.7587e-12	9.0095e-14	1.096853e-15
1.0	0.1209069176044184	0.1209069176044175	2.5166e-12	1.3678e-13	1.683625e-15

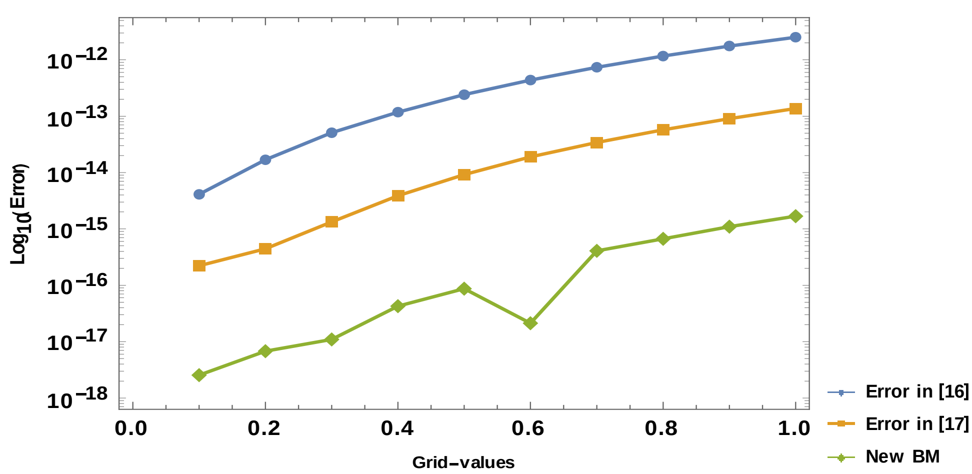

Figure 3. Comparison curve $\log_{10}(error)$ in existing methods with the proposed method in Problem 2 with $h = 10^{-1}$, $x \in (0.1, 1.0)$.

Problem 3. $y''' + 4y' = x,\;\; y(0) = 0,\;\; y'(0) = 0,\;\; y''(0) = 1,\;\; h = 0.1$

Exact solution is $y(x) = \dfrac{3}{16}(1 - \cos 2x) + \dfrac{1}{8}x^2$

In this example, the results of the new method of order 8 are compared with those of [18].

Table 3: Comparison of results for solving Problem 3 $(h = 0.1)$

b	TS	A_e in [18]	b	TS	A_e in the New BM
5.0	46 56 88	1.20e-10 3.69e-11 2.44e-12	5.0	46 56 88	6.44e-16 2.19e-14 2.35e-13
10.0	61 91 136	5.54e-09 5.04e-10 4.53e-11	10.0	61 91 136	1.68e-15 5.10e-14 3.44e-13
15.0	76 91 180	2.67e-08 2.91e-09 1.52e-10	15.0	76 91 180	2.85e-15 9.50e-14 1.24e-13
20.0	91 129 204	5.29e-08 6.54e-09 4.19e-10	20.0	91 129 204	1.10e-14 1.50e-13 3.26e-12

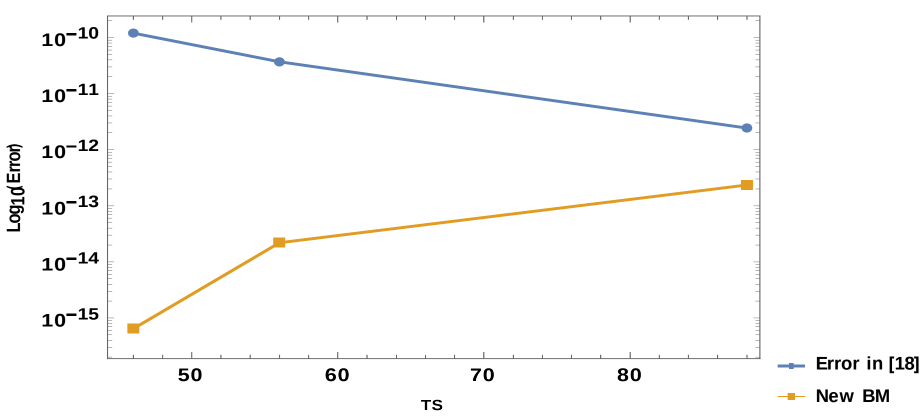

Figure 4. Comparison curve $\log_{10}(error)$ in existing method with the proposed method in Problem 3 with $h = 10^{-1}$.

Problem 4. $y''' + 2y'' - y' - 2y = e^x, \quad y(0) = 1, \quad y'(0) = 2, \; y''(0) = 0, \; h = 0.1$

Exact solution is $y(x) = \frac{1}{36}\left(43e^x + 9e^{-x} - 16e^{-2x} + 6xe^x\right)$

This example is solved using the new method of order 8. This can be seen in Table 4.

Table 4: Numerical solution for problem 4, $k = 3, \; p = 8, \; h = 0.1$

x	y_{ex}	y_c	A_e
0.1	1.2008137983659488	1.2008137983659530	4.06850e-15
0.2	1.4063738319947532	1.4063738319947635	1.02746e-14
0.3	1.6211125663343329	1.6211125663343186	1.41049e-14
0.4	1.8492349517044135	1.8492349517043517	6.16797e-14
0.5	2.0948300925243477	2.0948300925242221	1.25670e-13
0.6	2.3619703731235764	2.3619703731233539	2.21776e-13
0.7	2.6548012251017639	2.6548012251014190	3.44204e-13
0.8	2.9776242436411247	2.9776242436406358	4.88114-e13
0.9	3.3349759807254564	3.3349759807247930	6.61704e-13
1.0	3.7317044453680683	3.7317044453672050	8.61546e-13

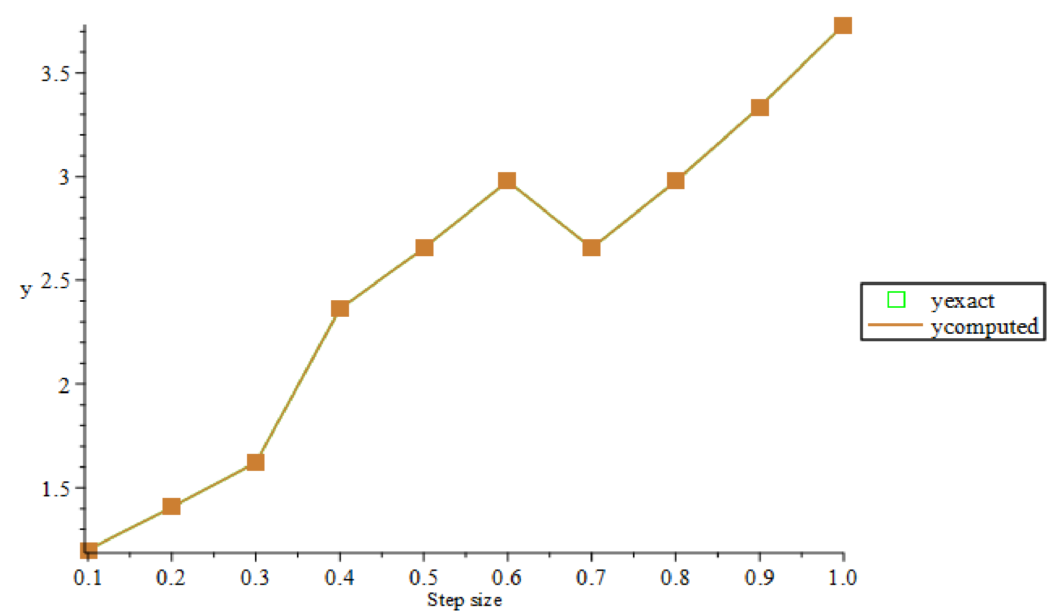

Figure 5. Numerical finding of the new method on problem 4 with $h = 10^{-1}, \, x \in [0, 1]$.

Problem 5. $y''' = -6(y)^4,\ \ y(1) = -1,\ \ y'(1) = -1,\ y''(1) = -2,\ \ h = 0.05$

Exact solution is $y(x) = \dfrac{1}{(x-2)}$

This example is solved using the new method of order 8. This can be seen in Table 5.

Table 5: Numerical solution for Problem 5, $k = 3,\ p = 8,\ h = 0.1$

x	y_{ex}	y_c	A_e	$t_e(s)$
1.05	-1.0526315789473684	-1.0526315789467432	6.20520e-12	0.021
1.10	-1.1111111111111112	-1.1111111111532876	4.21764e-11	0.025
1.15	-1.1764705882352944	-1.1764705886745383	4.39244e-10	0.029
1.20	-1.2500000000000002	-1.2500000003728135	3.72813e-10	0.030
1.25	-1.3333333333333337	-1.3333333337923525	4.59019e-10	0.030
1.30	-1.4285714285714290	-1.4285714243045637	7.33135e-10	0.033
1.35	-1.5384615384615392	-1.5384615361267500	2.33479e-09	0.033
1.40	-1.6666666666666676	-1.6666666635672367	3.09943e-09	0.033
1.45	-1.8181818181818195	-1.8181818196378920	1.45607e-09	0.034
1.50	-2.0000000000000018	-2.0000000047281456	4.72814e-09	0.034

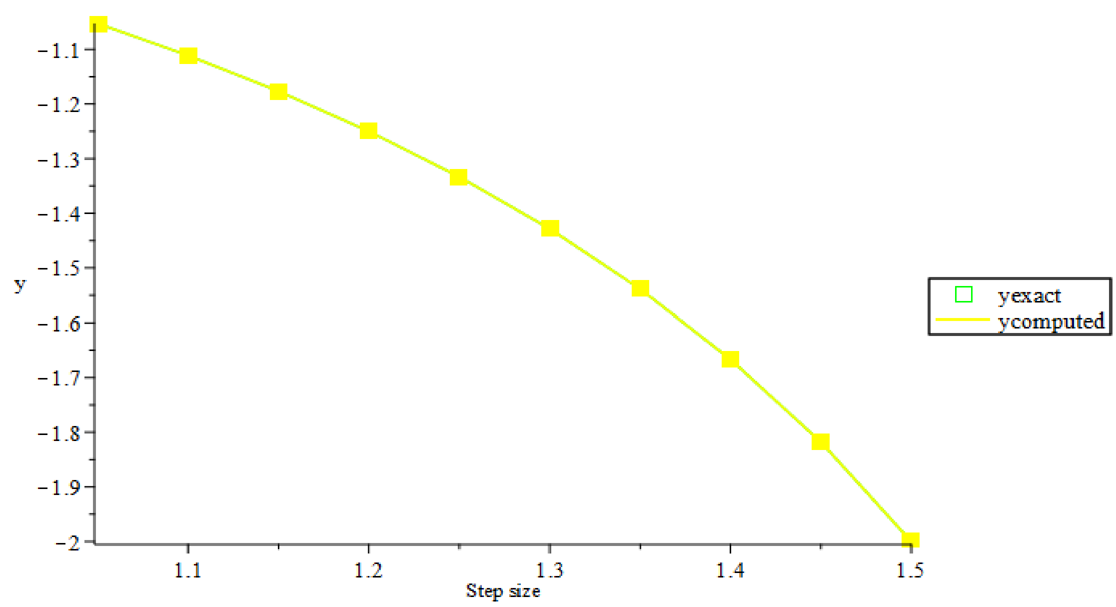

Figure 6. Solution obtained for problem 5 using the proposed method on Problem 5 with $h = 10^{-1},\ x \in [0, 1.5]$.

5. Conclusions

A new three-step four-point block method for solving general third-order ordinary differential equations directly has been presented in this paper. To acquire the hybrid points at $y-function$, the collocation and interpolation points were chosen. The inclusion of several offstep locations permitted the use of a linear multistep technique to avoid the "zero stability barrier" and the "problem dependent barrier," and as a result improved the method's order of accuracy.

In comparison to Kuboye and Omar (2015), Awoyemi *et al.* (2014), Adoghe and Omole (2019), and Adeyeye and Omar (2019), the hybrid block technique has shown improved accuracy with fewer steps.

Furthermore, when compared to past higher-order techniques, the unique hybrid block strategy outperforms them. For further comparisons, Tables 4 and 5 show the utility of the new hybrid block strategy. When compared to the existing approaches under consideration, the results show that the method is superior. As a result, this new problem-independent method can be used to numerically integrate general third-order initial value problems involving ordinary differential equations.

References

[1] Lambert, J.D. (1973): Computational Methods in Ordinary Differential Equations, *John Wiley & Sons Inc., New-York.*

[2] Lambert, J.D. (1991): Numerical Methods for ordinary differential Equations, *John Wiley & Sons Inc., New-York.*

[3] Mohammed, U., and Adeniyi, R.B. (2014): A Three Step Implicit Hybrid Linear Multistep Method for the Solution of Third order Ordinary Differential Equations. *General Mathematics Notes, 25(1): 62-74.*

[4] Kayode S.J. and Obarhua, F. O. (2015): 3-Step y-function Hybrid Methods for Direct Numerical Integration of Second Order IVPs in Ordinary Differential Equations. *Theo. Math & Appl. 5(1), 39-51.*

[5] Tumba P., Sabo J. & Hamadina M. (2018): Uniformly Order Eight Implicit Second Derivative Method for Solving Second Order Stiff Ordinary Differential Equations, *Academic Journal of Applied Mathematical Sciences, 4: 43-48.*

[6] Jikantoro, Y.; Ismail, F.; Senu, N.; Ibrahim, Z. (2018) A New Integrator for Special Third Order Differential Equations With Application to Thin Film Flow Problem. *Indian J. Pure Appl. Math. 49, 151–167.*

[7] Allogmany R.,Ismail F., (2020): Implicit Three-Point Block Numerical Algorithm for Solving Third Order Initial Value Problem Directly With Applications. *Mathematics, 8(10), p. 1771.*

[8] Kuboye J.O., Omar Z., (2015): Numerical Solution of Third Order Ordinary Differential Equations Using A Seven-Step Block Method, *Int. J. of Mathematical analysis, 9(15), pp. 743-745.*

[9] Abdelrahim R. (2019): Numerical Solution of Third Order Boundary Value Problems using One-Step Hybrid Block Method, *Ain Shams Engineering Journal, 10: 179-183.*

[10] Alabi, M. O., Oladipo, A. T, and Adesanya, A. O. (2008): Initial value solvers for second order Ordinary Differential Equations using Chebyshev Polynomial as Bases Functions. *Journal of Modern Maths & Stats 2(1), 18-27*

[11] Sunday, J., Odekunle, M.R., James, A.A. and Adesanya, A.O., (2014): Numerical Solution of Stiff and Oscillatory differential Equations using a block Integrator, *British J. of Mathematics & Computer science, 4:2471-2481.*

[12] Momoh, A.A., Adesanya, A.O., Fasasi, K. M. and Tahir, A. (2014), A New Numerical Integrator for the Solution of Stiff First Order Ordinary Differential Equations, *Engineering and Mathematics Letters, 4: 1-10.*

[13] Obarhua F.O., Adegboro J.O., (2021): An Order Four Continuous Numerical Method for Solving General Second Order Ordinary Differential Equations. *J. Nig. Soc. Phys. Sci. 3, 42-47.*

[14] James, A.A., Adesanya, A.O., Sunday, J. and Yakubu, D.G. (2013): Half-Step Continuous Block Method for the Solutions of Modeled Problems of Ordinary Differential Equations, *American Journal of computational Mathematics, (3), 261-269.*

[15] Awoyemi, D.O., Kayode S.J., Adoghe L.O. (2014): A Five-Step P-Stable Method for the Numerical Integration of Third Order Ordinary Differential Equations. *American J. of Comptnal Maths., 9, 119-126.*

[16] Kashkari, K.S.H. and Alqarni, S., (2019): Optimization of Two-Step Block Method with Three Hybrid Points for Solving Third Order Initial Value Problems, *Journal of Nonlinear Science and Application, 12 450-469.*

[17] Adoghe, L.O. and Omole, E.O., (2019): A Fifth-Fourth Continuous Block Implicit Hybrid Method for the Solution of Third Order Initial Value Problems in Ordinary Differential Equations, *Applied & Computational Mathematics, 8, 50-57.*

[18] Adeyeye O.,Omar Z.,(2019): Solving Third Order Ordinary Differential Equations Using One-Step Block Method with Four Equidistant Generalized Hybrid Points. *IAENG International Journal of Applied Mathematics. 49(2).*

[19] Omole, E.O. and Ukpebor, L.A. (2020): A Step by Step Guide on Derivative and Analysis of a New Numerical Method for Solving Fourth-Order Ordinary Differential Equations. *Journal of Mathematics letters. Vol. 6, No. 2, pp. 13-31, doi: 10. 11648/j.ml.20200602. 12.*