Original Research Article

COMMON FIXED POINT THEOREM FOR $(\phi, \mathfrak{F})$–MULTI-VALUED MAPPINGS IN CONE *b*-METRIC SPACES OVER BANACH ALGEBRA

Abstract

In the present paper, we introduced the concept of generalized multi-valued contraction mappings, via the class functions Φ and Ψ. Also we proved some fixed point results for $(\phi, \mathfrak{F})$-multi-valued mappings on cone b-metric spaces over Banach algebra $\mathfrak{A}$. The conditions for existence and uniqueness of the fixed point are investigated. We give an example to support our main result.

Keywords: Cone b-Metric space, Multi-valued mappings, Banach algebra.

MSC: 47H10, 54H25.

1 Introduction

In 1922, Stefan Banach [3] proved a fixed point theorem for contractive mappings in complete metric spaces. In 1969, Nadler [12] Introduce the concept of Multivalve function. Later, Czerwik [5,6] initiate the concept of b-metrics which generalized usual metric spaces. After his contribution, many results were presented in β-generalized weak contractive multifunction's and b-metric spaces. In 2007, Huang et al. [10] introduced cone metric space with normal cone, as a generalization of metric space. In 2012, Aydi et al [2]. Reformulate the b-metric space. Many researcher work in this area of research of multivalued function and b-metric spaces [1,4,7,8,9,15]. Liu and Xu [12] introduced the notion of cone metric space over Banach algebras, replacing Banach spaces by Banach algebras as the underlying spaces of cone metric space. They improved the fixed point theorems for $(\phi, \mathfrak{F})$-multi-valued mappings on cone b-metric spaces over Banach algebra, via the class functions Φ and Ψ.

Let $\mathfrak{A}$ be a real Banach algebra, i.e. $\mathfrak{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties:

For all $\varrho, \varsigma, \upsilon \in \mathfrak{A}, \mathfrak{d} \in \mathfrak{R}$

(i) $\varrho(\varsigma\varepsilon) = (\varrho\varsigma)\varepsilon$;
(ii) $\varrho(\varsigma + \varepsilon) = \varrho\varsigma + \varrho\varepsilon \; and \; (\varrho + \varsigma)\varepsilon = \varrho\varepsilon + \varsigma\varepsilon$;
(iii) $\mathfrak{d}(\varrho\varsigma) = (a\varrho)\varsigma = \varrho(\mathfrak{d}\varsigma)$;
(iv) $\| \varrho\varsigma \| \leq \| \varrho \| \| \varsigma \|$.

We shall assume that the Banach algebra $\mathfrak{A}$ has a unit, i.e., a multiplicative identity e such that $e\varrho = \varrho e = \varrho$ for all $\varrho \in \mathfrak{A}$. An element $\varrho \in \mathfrak{A}$ is said to be invertible if there is an inverse element $\varsigma \in \mathfrak{A}$ such that $\varrho\varsigma = \varsigma\varrho = e$. The inverse of ϱ is denoted by ϱ^{-1}.

Let $\mathfrak{A}$ be a real Banach algebra with a unit e and $\varrho \in \mathfrak{A}$. If the spectral radius $\rho(\varrho)$ of ϱ is less than 1, that is

$$\rho(\varrho) = \lim_{\mathfrak{n}\to\infty} \| \varrho^{\mathfrak{n}} \|^{\frac{1}{\mathfrak{n}}} = \inf_{n\geq 1} \| \varrho^{\mathfrak{n}} \|^{\frac{1}{\mathfrak{n}}} < 1$$

then $e - \varrho$ is invertible. Actually,

$$(e-\varrho)^{-1} = \sum_{i=0}^{\infty} \varrho_i.$$

A subset $\mathfrak{P}$ of $\mathfrak{A}$ is called a cone of $\mathfrak{A}$ if

i. $\{\theta, e\} \subset \mathfrak{P}$,
ii. $\mathfrak{P}^2 = \mathfrak{P}\mathfrak{P} \subset \mathfrak{P}, \mathfrak{P} \cap (-\mathfrak{P}) = \{\theta\}$,
iii. $\mathfrak{h}\mathfrak{P} + \beta\mathfrak{P} \subset \mathfrak{P}\ \forall\ \mathfrak{h}, \beta \in \mathfrak{R}$,

For a given cone $\mathfrak{P} \subset \mathfrak{A}$, we define a partial ordering $\preccurlyeq$ with respect to $\mathfrak{P}$ by $\varrho \preccurlyeq \varsigma$ if and only if $\varsigma - \varrho \in \mathfrak{P}$; $\varrho \prec \varsigma$ will stand for $\varrho \preccurlyeq \varsigma$ and $\varrho \neq \varsigma$, while $\varrho \ll \varsigma$ stand for $\varsigma - \varrho \in int\mathfrak{P}$, where $int\mathfrak{P}$ denotes the interior of $\mathfrak{P}$. If $int\mathfrak{P} \neq \emptyset$, then $\mathfrak{P}$ is called a solid cone. Write $\| . \|$ as the norm of $\mathfrak{A}$. A cone $\mathfrak{P}$ is called normal if there is a number $\mathfrak{M} > 0$ such that $\forall\ \varrho, \varsigma \in \mathfrak{A}$, we have

$$\theta \preccurlyeq \varrho \preccurlyeq \varsigma \Longrightarrow \| \varrho \| \leq \mathfrak{M} \| \varsigma \|.$$

The least positive number satisfying above is called the normal constant of $\mathfrak{P}$. Note that, for any normal cone $\mathfrak{P}$ we have $\mathfrak{M} \geq 1$.

In the following we suppose that $\mathfrak{A}$ is a real Banach algebra with a unit $e, \mathfrak{P}$ is a solid cone and $\preccurlyeq$ with respect to $\mathfrak{P}$.

2 Preliminaries

Lemma 2.1

(see [12]) If $\mathfrak{E}$ is a real Banach space with a cone $\mathfrak{P}$ and if $\mathfrak{d} \preccurlyeq \delta\mathfrak{d}$ with $\mathfrak{d} \in \mathfrak{P}$ and $0 \leq \delta < 1$, then $\mathfrak{d} = \theta$.

Lemma 2.2

(see [12]) If $\mathfrak{E}$ is a real Banach space with a solid cone $\mathfrak{P}$ and if $\theta \preccurlyeq \mathfrak{u} \ll \mathfrak{c}$ for each $\theta \ll \mathfrak{c}$, then $\mathfrak{u} = \theta$.

Lemma 2.3

(see [12]) Let $\mathfrak{P}$ be a cone in a Banach algebra $\mathfrak{A}$ and $\mathcal{K} \in \mathfrak{P}$ be a given vector. Let $\{\mathfrak{u}_\mathfrak{n}\}$ be a sequence in $\mathfrak{P}$. If $\forall\ \mathfrak{c}_1 \gg \theta$, $\exists\ \mathfrak{N}_1\ \ni \mathfrak{u}_\mathfrak{n} \ll \mathfrak{c}_1\ \forall\ \mathfrak{n} > \mathfrak{N}_1$, then $\forall\ \mathfrak{c}_2 \gg \theta$, $\exists\ \mathfrak{N}_2\ \ni \mathcal{K}\mathfrak{u}_\mathfrak{n} \ll \mathfrak{c}_2\ \forall\ \mathfrak{n} > \mathfrak{N}_2$.

Lemma 2.4.

(see [12]) If $\mathfrak{E}$ is a real Banach space with a solid cone $\mathfrak{P}$ and $\{\varrho_\mathfrak{n}\} \subset \mathfrak{P}$ is a sequence with $\| \varrho_\mathfrak{n} \| \to 0 (\mathfrak{n} \to \infty)$, then $\forall\ \theta \ll \mathfrak{c}$, $\exists\ \mathfrak{N} \in \mathbb{N} \ni \mathfrak{n} > \mathfrak{N}$ we have, $\varrho_\mathfrak{n} \ll \mathfrak{c}$ $i.e.$, $\{\varrho_\mathfrak{n}\}$ is a $\mathfrak{c}$-sequence.

Lemma 2.5

(see [14]) Let $\mathfrak{A}$ be a Banach algebra with a unit $e, \imath, \jmath \in \mathfrak{A}$. If $\imath$ commutes with $\jmath$, then $\rho(\imath + \jmath) \leq \rho(\imath) + \rho(\jmath)$, $\rho(\imath\jmath) \leq \rho(\imath)\rho(\jmath)$.

Remark 2.6

(see [14]) If $\rho(\varrho) < 1$, then $\| \varrho_n \| \to 0$ as $\mathfrak{n} \to \infty$.

Definition 2.7

(see [11]) Let X be a non-empty set, $\omega \geq 1$ be a constant and $\mathfrak{A}$ be a Banach algebra. A function $D_b: X \times X \to \mathfrak{A}$ is said to be a cone $\mathfrak{b}$-metric provide that, for all $\varrho, \varsigma, \varepsilon \in X$,
(d1) $D_b(\varrho, \varsigma) = 0$ *if and onl*ς *if* $\varrho = \varsigma$;
(d2) $D_b(\varrho, \varsigma) = D_b(\varsigma, \varrho)$;
(d3) $D_b(\varrho, \varepsilon) \leq \omega[D_b(\varrho, \varsigma) + D_b(\varsigma, \varepsilon)]$.
A pair (X, D_b) is called a cone $\mathfrak{b}$-metric space over Banach algebra $\mathfrak{A}$.

Example 2.8.

Let $\mathfrak{A} = C[\mathfrak{d}, \mathscr{b}]$ be the set of continuous functions on the interval $[\mathfrak{d}, \mathscr{b}]$ with the supermom norm. Define multiplication in the usual way. Then $\mathfrak{A}$ is a Banach algebra with a unit 1. Set $\mathfrak{P} = \{\varrho \in \mathfrak{A}: \varrho(t) \geq 0, \mathrm{t} \in [\mathfrak{d}, \mathscr{b}]\}$ and $X = \mathfrak{R}$. Defined a mapping $D_b: X \times X \to \mathfrak{A}$ by $D_b(\varrho, \varsigma)(\mathrm{t}) = |\varrho - \varsigma|^{\mathfrak{p}} e^{\mathrm{t}}\ \forall\ \varrho, \varsigma \in X$, where $\mathfrak{p} > 1$ is a constant. This makes (X, D_b) into a cone $\mathfrak{b}$-metric space over Banach algebra $\mathfrak{A}$ with the coefficient $\mathfrak{b} = 2^{\mathfrak{p}-1}$, but it is not a cone metric space over Banach algebra since the triangle inequality is not satisfied.

Definition 2.9

(see [11]) Let (X, D_b) be a cone $\mathfrak{b}$-metric space over Banach algebra $\mathfrak{A}, \varrho \in X$, let $\{\varrho_{\mathfrak{n}}\}$ be a sequence in X. Then

i. $\{\varrho_{\mathfrak{n}}\}$ converges to ϱ whenever for every $\mathfrak{c} \in \mathfrak{A}$ with $\theta \ll \mathfrak{c}$ there is natural number $\mathfrak{n}_0 \ni D_b(\varrho_{\mathfrak{n}}, \varrho) \ll \mathfrak{c}, \forall\ \mathfrak{n} \geq \mathfrak{n}_0$. We denote this by $\lim_{\mathfrak{n}\to\infty} \varrho_{\mathfrak{n}} = \varrho$.
ii. $\{\varrho_{\mathfrak{n}}\}$ is a Cauchy sequence whenever for every $\mathfrak{c} \in \mathfrak{A}$ with $\theta \ll \mathfrak{c}$ there is natural number $\mathfrak{n}_0 \ni D_b(\varrho_{\mathfrak{n}}, \varrho_{\mathfrak{m}}) \ll \mathfrak{c}, \forall\ \mathfrak{n}, \mathfrak{m} \geq \mathfrak{n}_0$.
iii. $\{\varrho, D_b\}$ is complete cone $\mathfrak{b}$-metric if every Cauchy sequence in X is convergent.

Remark 2.10

(see [13]) Let (X, D_b) be a cone $\mathfrak{b}$-metric space over Banach algebra $\mathfrak{A}$ with the coefficient $\omega \geq 1$, denote

$\mathfrak{N}(X) = \{\mathcal{A}: \mathcal{A}$ *is non empt*ς *subset of* $X\}$ and $\mathfrak{s}(\mathfrak{p}) = \{\mathfrak{q} \in \mathfrak{E}: \mathfrak{p} \preccurlyeq \mathfrak{q}\}$ for $\mathfrak{q} \in \mathfrak{A}$,

$\mathfrak{s}(\mathfrak{d}, \mathfrak{B}) = \bigcup_{\mathscr{b}\in\mathfrak{B}} \mathfrak{s}\big(D_b(\mathfrak{d}, \mathscr{b})\big) = \bigcup_{\mathscr{b}\in\mathfrak{B}}\{\varrho \in \mathfrak{A}: D_b(\mathfrak{d}, \mathscr{b}) \preccurlyeq \varrho\}$ *for* $\mathfrak{d} \in X$ and $\mathfrak{B} \in \mathfrak{N}(X)$.

For $\mathcal{A}, \mathfrak{B} \in \mathfrak{N}(X)$ we denote $\mathfrak{s}(\mathcal{A}, \mathfrak{B}) = \left(\bigcap_{\mathfrak{d}\in\mathcal{A}} \mathfrak{s}(\mathfrak{d}, \mathfrak{B})\right) \cap \left(\bigcap_{\mathscr{b}\in\mathfrak{B}} \mathfrak{s}(\mathscr{b}, \mathcal{A})\right)$.

Remark 2.11

(see [13]) Let (X, D_b) be a cone $\mathfrak{b}$-metric space over Banach algebra $\mathfrak{A}$, with the coefficient $\omega \geq 1$. If $\mathfrak{A} = \mathfrak{R}$ and $\mathfrak{P} = \mathfrak{R}_0^+$ then (X, D_b) is a metric spac. Moreover, for $\mathcal{A}, \mathfrak{B} \in \mathfrak{CB}(X), \mathcal{H}(\mathcal{A}, \mathfrak{B}) = \inf \mathfrak{s}(\mathcal{A}, \mathfrak{B})$ is the Housdorff distance induced by D_b.

Definition 2.12

(see [13]) Let (X, D_b) be a cone $\mathfrak{b}$-metric space over Banach algebra $\mathfrak{A}$. A map $\mathfrak{J}: X \to \mathfrak{CB}(X)$ is said to be multi-valued contraction if $\exists\ 0 \leq \mathfrak{s} < 1$ such that $\mathcal{H}(\mathfrak{J}\varrho, \mathfrak{J}\varsigma) \leq \mathfrak{s}\, D_b(\varrho, \varsigma)$, for all $\varrho, \varsigma \in X$.

Lemma 2.13

(see [13]) If $\mathcal{A}, \mathfrak{B} \in \mathfrak{CB}(X)$ and $\mathfrak{d} \in \mathcal{A}$, then for each $\epsilon > 0$, there exists $\mathscr{b} \in \mathfrak{B}$ such that $D_b(\mathfrak{d}, \mathscr{b}) \leq \mathcal{H}(\mathcal{A}, \mathfrak{B}) + \epsilon$.

Lemma 2.14

(see [11]) Let $\mathfrak{E}$ be a real Banach space with a solid cone $\mathfrak{P}$

1) If $\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3 \in \mathfrak{E}$ and $\mathfrak{d}_1 \preccurlyeq \mathfrak{d}_2 \ll \mathfrak{d}_3$, then $\mathfrak{d}_1 \ll \mathfrak{d}_3$.
2) If $\mathfrak{d}_1 \in \mathfrak{P}$ and $\mathfrak{d}_1 \ll \mathfrak{d}_3$ for each $\mathfrak{d}_3 \gg \theta$, then $\mathfrak{d}_1 = \theta$.

Lemma 2.15

(see [11]) Let $\mathfrak{P}$ be a solid cone in a Banach algebra $\mathfrak{A}$. Suppose that $\mathfrak{h} \in \mathfrak{P}$ and $\{\varrho_\mathfrak{n}\} \subset \mathfrak{P}$ is a $\mathfrak{c}$-sequence. Then $\{\mathfrak{h}\varrho_\mathfrak{n}\}$ is a $\mathfrak{c}$-sequence.

Lemma 2.16

(see [11]) Let $\mathfrak{A}$ be a Banach algebra with a unite $e, \mathfrak{h} \in \mathfrak{A}$, then $\lim_{\mathfrak{n}\to\infty} \| \mathfrak{h}^\mathfrak{n} \|^{\frac{1}{\mathfrak{n}}}$ exists and the spectral radius $\rho(\mathfrak{h})$ satisfies

$$\rho(\mathfrak{h}) = \lim_{\mathfrak{n}\to\infty} \| \mathfrak{h}^\mathfrak{n} \|^{\frac{1}{\mathfrak{n}}} = \inf \| \mathfrak{h}^\mathfrak{n} \|^{\frac{1}{\mathfrak{n}}}.$$

If, then $(\delta e - \mathfrak{h})$ is invertible in $\mathfrak{A}$, moreover,

$$(\delta e - \mathfrak{h})^{-1} = \sum_{i=0}^{\infty} \frac{\mathfrak{h}^i}{\delta^{i+1}},$$

where δ is a complex constant.

Definition 2.18

(see [07]) Let $\mathfrak{J}, \mathfrak{L}: X \to X$ be a mappings on set X.

1) If $\mathfrak{w} = \mathfrak{J}\varrho = \mathfrak{L}\varrho$ for some $\varrho \in X$, then ϱ is called a coincidence point of $\mathfrak{J}$ and $\mathfrak{L}$, and $\mathfrak{w}$ is called a point of coincidence of $\mathfrak{J}$ and $\mathfrak{L}$.
2) The pair $(\mathfrak{J}, \mathfrak{L})$ is called weakly compatible if $\mathfrak{J}$ and $\mathfrak{L}$ commute at all of their coincidence points, that is, $\mathfrak{J}\mathfrak{L}\varrho = \mathfrak{L}\mathfrak{J}\varrho\ \forall\ \varrho \in \mathcal{C}(\mathfrak{J}, \mathfrak{L}) = \{\varrho \in X : \mathfrak{J}\varrho = \mathfrak{L}\varrho\}$.

Lemma 2.19

(see [07]) Let $\mathfrak{J}$ and $\mathfrak{L}$ be weakly compatible self-maps of a set X. If $\mathfrak{J}$ and $\mathfrak{L}$ have a unique point of coincidence $\mathfrak{w} = \mathfrak{J}\varrho = \mathfrak{L}\varrho$, then $\mathfrak{w}$ is the unique common fixed point of $\mathfrak{J}$ and $\mathfrak{L}$.

3 MAIN RESULTS

We prove a unique fixed point for generalized $(\phi,\mathfrak{F})$- multi-valued mappings via the class functions Φ and Ψ.

Lemma 3.1

Let $\mathfrak{A}$ be a Banach algebra with a unite $e, \mathfrak{h} \in \mathfrak{A}$, if δ is a complex constant and $\rho(\mathfrak{h}) < |\delta|$, then $\rho((\delta e - \mathfrak{h})^{-1}) \leq \frac{1}{|\delta| - \rho(\mathfrak{h})}$.

Proof. Since $\rho(\mathfrak{h}) < |\delta|$, it follows by lemma 2.16 that $(\delta e - \mathfrak{h})$ is invertible and

$$(\delta e - \mathfrak{h})^{-1} = \sum_{i=0}^{\infty} \frac{\mathfrak{h}^i}{\delta^{i+1}}.$$

Set $\omega = \sum_{i=0}^{\infty} \frac{\mathfrak{h}^i}{\delta^{i+1}}, \omega_\mathfrak{n} = \sum_{i=0}^{\mathfrak{n}} \frac{\mathfrak{h}^i}{\delta^{i+1}}$, then $\omega_\mathfrak{n} \to \omega (\mathfrak{n} \to \infty)$ and $\omega_\mathfrak{n}$ commutes with $\omega \, \forall \, \mathfrak{n}$. It follows immediately from lemma 2.5 that

$$\rho(\omega_\mathfrak{n}) = \rho(\omega_\mathfrak{n} - \omega + \omega) \leq \rho(\omega - \omega_\mathfrak{n}) + \rho(\omega) \Rightarrow \rho(\omega_\mathfrak{n}) - \rho(\omega) \leq \rho(\omega - \omega_\mathfrak{n}),$$
$$\rho(\omega) = \rho(\omega - \omega_\mathfrak{n} + \omega) \leq \rho(\omega - \omega_\mathfrak{n}) + \rho(\omega_\mathfrak{n}) \Rightarrow \rho(\omega) - \rho(\omega_\mathfrak{n}) \leq \rho(\omega - \omega_\mathfrak{n}),$$

Which imply that

$$|\rho(\omega_\mathfrak{n}) - \rho(\omega)| \leq \rho(\omega - \omega_n) \leq \| \omega - \omega_\mathfrak{n} \| \Rightarrow \rho(\omega_\mathfrak{n}) \to \rho(\omega) (\mathfrak{n} \to \infty).$$

Thus again by lemma 2.5,

$$\rho((\delta e - \mathfrak{h})^{-1}) = \rho\left(\sum_{i=0}^{\infty} \frac{\mathfrak{h}^i}{\delta^{i+1}}\right) = \rho(\omega) = \lim_{\mathfrak{n}\to\infty} \rho(\omega_\mathfrak{n})$$
$$= \lim_{\mathfrak{n}\to\infty} \rho\left(\sum_{i=0}^{\mathfrak{n}} \frac{\mathfrak{h}^i}{\delta^{i+1}}\right)$$
$$\leq \lim_{\mathfrak{n}\to\infty} \sum_{i=0}^{\mathfrak{n}} \frac{[\rho(\mathfrak{h})]^i}{\delta^{i+1}}$$
$$= \lim_{\mathfrak{n}\to\infty} \sum_{i=0}^{\infty} \frac{[\rho(\mathfrak{h})]^i}{\delta^{i+1}} = \frac{1}{|\delta| - \rho(\mathfrak{h})}.$$

Lemma 3.2

Let $\mathfrak{A}$ be a Banach algebra with a unit e and $\mathfrak{P}$ be a solid cone in $\mathfrak{A}$. Let $\gamma \in \mathfrak{A}$ and $\varrho_\mathfrak{n} = \gamma^\mathfrak{n}$. If $\rho(\gamma) < 1$, then $\{\varrho_\mathfrak{n}\}$ is a $\mathfrak{c}$-sequence.

Proof. Since $\rho(\gamma) = \lim_{\mathfrak{n}\to\infty} \| \gamma^\mathfrak{n} \|^{\frac{1}{\mathfrak{n}}} < 1$, then $\exists \, \tau > 0 \ni \lim_{\mathfrak{n}\to\infty} \| \gamma^\mathfrak{n} \|^{\frac{1}{\mathfrak{n}}} < \tau < 1$. Letting $\mathfrak{n}$ be big enough, we obtain $\| \gamma^\mathfrak{n} \|^{\frac{1}{\mathfrak{n}}} \leq \tau$, which implies that $\| \gamma^\mathfrak{n} \|^{\frac{1}{\mathfrak{n}}} \leq \tau^\mathfrak{n} \to 0 (\mathfrak{n} \to \infty)$. So $\| \gamma^\mathfrak{n} \| \to 0$, i.e., $\| \varrho_\mathfrak{n} \| \to 0 (\mathfrak{n} \to \infty)$. Note that $\forall \, \mathfrak{c} \gg \theta$, there is $\beta > 0$ such that

$$U(\mathfrak{c}, \beta) = \{\varrho \in E : \| \varrho - \mathfrak{c} \| < \beta\} \subset \mathfrak{P}.$$

In view of $\| \varrho_n \| \to 0 (\mathfrak{n} \to \infty), \exists \, \mathfrak{N} \ni \| \varrho_\mathfrak{n} \| < \beta \, \forall \, \mathfrak{n} > \mathfrak{N}$. Consequently, $\| (\mathfrak{c} - \varrho_n) - \mathfrak{c} \| = \| \varrho_n \| < \beta$, this loads to $\mathfrak{c} - \varrho_n \in U(\mathfrak{c}, \beta) \subset \mathfrak{P}$, that is, $\mathfrak{c} - \varrho_n \in int\mathfrak{P}$, thus $\varrho_n \ll \mathfrak{c} \, \forall \, \mathfrak{n} > \mathfrak{N}$.

Definition 3.3

Let $\mathfrak{A}$ be a Banach algebra and $\mathfrak{P} = \mathfrak{R}_0^+$ be a cone in $\mathfrak{A}$. A mapping $\mathfrak{F}: \mathfrak{P} \to \mathfrak{P}$ such that

1) $\mathfrak{F}$ is non-decreasing and continuous;
2) $\lim_{\mathfrak{n}\to\infty} \mathfrak{F}^{\mathfrak{n}}(\mathfrak{t}) = \theta$ for all $(\mathfrak{t} \in \mathfrak{P}), \mathfrak{t} \geq 0$, where $\mathfrak{F}^{\mathfrak{n}}$ stands for the $\mathfrak{n}^{th}$ iterate of $\mathfrak{F}$;
3) $\mathfrak{F}(\mathfrak{t}) < \mathfrak{t}$ for each $\mathfrak{t} > 0$;
4) $\mathfrak{F}(\theta) = \theta$.

Definition 3.4

Let $\mathfrak{A}$ be a Banach algebra and $\mathfrak{P} = \mathfrak{R}_0^+$ be a cone in $\mathfrak{A}$. A mapping $\phi: \mathfrak{P} \to \mathfrak{P}$ such that:

1) ϕ is monotone non-decreasing and continuous $i.e., \theta \leq \mathfrak{t}_1 \leq \mathfrak{t}_2 \Longrightarrow \phi(\mathfrak{t}_1) \leq \phi(\mathfrak{t}_2)$;
2) $\{\phi^{\mathfrak{n}}(\mathfrak{t})\}$ $(\mathfrak{t} > 0)$ is a $\mathfrak{c}$-sequence in $\mathfrak{P}$;
3) If $\{\mathfrak{u}_n\}$ is a $\mathfrak{c}$-sequence in $\mathfrak{P}$, then $\{\phi(\mathfrak{u}_{\mathfrak{n}})\}$ is also a $\mathfrak{c}$-sequence in $\mathfrak{P}$;
4) $\phi(\mathfrak{t}) = \mathcal{K}\mathfrak{t}$, for some $(\mathcal{K} \in \mathfrak{P})$, $\mathcal{K} > 0$.

Theorem 3.5

Let (X, D_b) be a cone b-metric space over Banach algebra $\mathfrak{A}$ and $\mathfrak{P}$ be a solid cone in $\mathfrak{A}$ with the coefficient $\omega \geq 1$. $\mathfrak{h}_i \in \mathfrak{P}$ $(i = 1, 2, \dots 5,)$ be a generalized Lipschitz constant with $2\omega\rho(\mathfrak{h}_5) + (\omega + 1)\rho(\mathfrak{h}_1 + \mathfrak{h}_2 + \omega\, \mathfrak{h}_3 + \omega\mathfrak{h}_4) < 2$. Suppose that $\mathfrak{h}_5$ commutes with $\mathfrak{h}_1 + \mathfrak{h}_2 + \omega\, \mathfrak{h}_3 + \omega\mathfrak{h}_4$ and the mappings $\mathfrak{I}, \mathfrak{L} : X \to \mathfrak{CB}(X)$ be generalized multi-valued $(\phi, \mathfrak{F})$-contraction mapping, satisfies that

$$\phi\big(\mathcal{H}(\mathfrak{I}\varrho, \mathfrak{L}\varsigma)\big) \leq \mathfrak{F}(\phi(\mathcal{M}(\varrho, \varsigma)) \ldots\ldots\ldots\ldots (1.1)$$

where,

$$\mathcal{M}(\varrho, \varsigma) = \mathfrak{h}_1 \frac{D_b(\varrho,\mathfrak{I}\varrho)}{1+D_b(\varrho,\mathfrak{I}\varrho)} + \mathfrak{h}_2 \frac{D_b(\varsigma,\mathfrak{L}\varsigma)}{1+D_b(\varsigma,\mathfrak{L}\varsigma)} + \mathfrak{h}_3 \frac{D_b(\varrho,\mathfrak{L}\varsigma)}{1+D_b(\varrho,\mathfrak{L}\,\varsigma)} + \mathfrak{h}_4 \frac{D_b(\varsigma,\mathfrak{I}\varrho)}{1+D_b(\varsigma,\mathfrak{I}\varrho)} + \mathfrak{h}_5 D_b(\varrho, \varsigma)$$

Where, $\mathfrak{F} \in \Psi$, $\phi \in \Phi$ such that $\forall\, \varrho, \varsigma \in X$. Moreover, if $\mathfrak{I}$ and $\mathfrak{L}$ are weakly compatible, then $\mathfrak{I}$ and $\mathfrak{L}$ have a unique common fixed point.

Proof. Fix any $\varrho \in X$. Define $\varrho_0 = \varrho$ and let $\varrho_1 \in \mathfrak{I}\varrho_0$, $\varrho_2 \in \mathfrak{L}\varrho_1$ such that $\varrho_{2n+1} = \mathfrak{I}\varrho_{2n}$, $\varrho_{2n+2} = \mathfrak{L}\varrho_{2n+1}$, by lemma 2.13, we may choose $\varrho_2 \in \mathfrak{L}\varrho_1$ such that

$$\varphi\big(D_b(\varrho_1, \varrho_2)\big) \leq \varphi\big(H(\mathfrak{I}\varrho_0, \mathfrak{L}\varrho_1)\big) + (\mathfrak{h}_1 + \mathfrak{h}_5 + \omega\mathfrak{h}_3)$$

$$\varphi\big(D_b(\varrho_1, \varrho_2)\big) \leq \left(\begin{matrix}\psi\left(\varphi\left(\begin{matrix}\mathfrak{h}_1 \frac{D_b(\varrho_0,\mathfrak{I}\varrho_0)}{1+D_b(\varrho_0,\mathfrak{I}\varrho_0)} + \mathfrak{h}_2 \frac{D_b(\varrho_1,\mathfrak{L}\varrho_1)}{1+D_b(\varrho_1,\mathfrak{L}\varrho_1)} + \mathfrak{h}_3 \frac{D_b(\varrho_0,\mathfrak{L}\varrho_1)}{1+D_b(\varrho_0,\mathfrak{L}\varrho_1)} \\ +\mathfrak{h}_4 \frac{D_b(\varrho_1,\mathfrak{I}\varrho_0)}{1+D_b(\varrho_1,\mathfrak{I}\varrho_0)} + \mathfrak{h}_5 D_b(\varrho_0, \varrho_1)\end{matrix}\right)\right) \\ +(\mathfrak{h}_1 + \mathfrak{h}_5 + \omega\mathfrak{h}_3)\end{matrix}\right)$$

$$= \left(\begin{matrix}\psi\left(\varphi\left(\begin{matrix}\mathfrak{h}_1 \frac{D_b(\varrho_0,\varrho_1)}{1+D_b(\varrho_0,\varrho_1)} + \mathfrak{h}_2 \frac{D_b(\varrho_1,\varrho_2)}{1+D_b(\varrho_1,\varrho_2)} + \mathfrak{h}_3 \frac{D_b(\varrho_0,\varrho_2)}{1+D_b(\varrho_0,\varrho_2)} \\ +\mathfrak{h}_4 \frac{D_b(\varrho_1,\varrho_1)}{1+D_b(\varrho_1,\varrho_1)} + \mathfrak{h}_5 D_b(\varrho_0, \varrho_1)\end{matrix}\right)\right) \\ +(\mathfrak{h}_1 + \mathfrak{h}_5 + \omega\mathfrak{h}_3)\end{matrix}\right)$$

$$\leq \left(\begin{matrix}\psi\left(\varphi\left(\begin{matrix}\mathfrak{h}_1 D_b(\varrho_0, \varrho_1) + \mathfrak{h}_2 D_b(\varrho_1, \varrho_2) + \mathfrak{h}_3 D_b(\varrho_0, \varrho_2) \\ +\mathfrak{h}_4 D_b(\varrho_1, \varrho_1) + \mathfrak{h}_5 D_b(\varrho_0, \varrho_1)\end{matrix}\right)\right) \\ +(\mathfrak{h}_1 + \mathfrak{h}_5 + \omega\mathfrak{h}_3)\end{matrix}\right)$$

$$\leq \left(\psi\left(\varphi\begin{pmatrix}\mathfrak{h}_1 D_b(\varrho_0,\varrho_1)+\mathfrak{h}_2 D_b(\varrho_1,\varrho_2)\\ +\mathfrak{h}_3\omega\,[D_b(\varrho_0,\varrho_1)+D_b(\varrho_1,\varrho_2)]\\ +\mathfrak{h}_5 D_b(\varrho_0,\varrho_1)\end{pmatrix}\right) \atop +(\mathfrak{h}_1+\mathfrak{h}_5+\omega\mathfrak{h}_3)\right)$$

$$\leq \left(\psi\left(\varphi\begin{pmatrix}(\mathfrak{h}_1+\omega\mathfrak{h}_3+\mathfrak{h}_5)D_b(\varrho_0,\varrho_1)\\ +(\mathfrak{h}_2+\omega\mathfrak{h}_3)D_b(\varrho_1,\varrho_2)\end{pmatrix}\right) \atop +(\mathfrak{h}_1+\mathfrak{h}_5+\omega\mathfrak{h}_3)\right)$$

which implies that

$$(e-\mathfrak{h}_2-\omega\mathfrak{h}_3)\varphi\big(D_b(\varrho_1,\varrho_2)\big) \leq \begin{pmatrix}(\mathfrak{h}_1+\mathfrak{h}_5+\omega\,\mathfrak{h}_3)\psi\Big(\varphi\big(D_b(\varrho_0,\varrho_1)\big)\Big)\\ +(\mathfrak{h}_1+\mathfrak{h}_5+\omega\mathfrak{h}_3)\end{pmatrix} \quad (1.2)$$

Then,

$$\varphi(D_b(\varrho_2,\varrho_1)) \leq \varphi\big(H(\mathfrak{L}\varrho_1,\mathfrak{J}\varrho_0)\big)+(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)$$

$$\varphi(D_b(\varrho_2,\varrho_1)) \leq \left(\psi\left(\varphi\begin{pmatrix}\mathfrak{h}_1\frac{D_b(\varrho_1,\mathfrak{L}\varrho_1)}{1+D_b(\varrho_1,\mathfrak{L}\varrho_1)}+\mathfrak{h}_2\frac{D_b(\varrho_0,\mathfrak{J}\varrho_0)}{1+D_b(\varrho_0,\mathfrak{J}\varrho_0)}+\mathfrak{h}_3\frac{D_b(\varrho_1,\mathfrak{J}\varrho_0)}{1+D_b(\varrho_1,\mathfrak{J}\varrho_0)}\\ +\mathfrak{h}_4\frac{D_b(\varrho_0,\mathfrak{L}\varrho_1)}{1+D_b(\varrho_0,\mathfrak{L}\varrho_1)}+\mathfrak{h}_5 D_b(\varrho_1,\varrho_0)\end{pmatrix}\right) \atop +(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)\right)$$

$$= \left(\psi\left(\varphi\begin{pmatrix}\mathfrak{h}_1\frac{D_b(\varrho_1,\varrho_2)}{1+D_b(\varrho_1,\varrho_2)}+\mathfrak{h}_2\frac{D_b(\varrho_0,\varrho_1)}{1+D_b(\varrho_0,\varrho_1)}+\mathfrak{h}_3\frac{D_b(\varrho_1,\varrho_1)}{1+D_b(\varrho_1,\varrho_1)}\\ +\mathfrak{h}_4\frac{D_b(\varrho_0,\varrho_2)}{1+D_b(\varrho_0,\varrho_2)}+\mathfrak{h}_5 D_b(\varrho_0,\varrho_1)\end{pmatrix}\right) \atop +(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)\right)$$

$$\leq \left(\psi\left(\varphi\begin{pmatrix}\mathfrak{h}_1 D_b(\varrho_1,\varrho_2)+\mathfrak{h}_2 D_b(\varrho_0,\varrho_1)+\mathfrak{h}_3 D_b(\varrho_1,\varrho_1)\\ +\mathfrak{h}_4 D_b(\varrho_0,\varrho_2)+\mathfrak{h}_5 D_b(\varrho_0,\varrho_1)\end{pmatrix}\right) \atop +(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)\right)$$

$$\leq \left(\psi\left(\varphi\begin{pmatrix}\mathfrak{h}_1 D_b(\varrho_1,\varrho_2)+\mathfrak{h}_2 D_b(\varrho_0,\varrho_1)\\ +\mathfrak{h}_4\omega[D_b(\varrho_0,\varrho_1)+D_b(\varrho_1,\varrho_2)]\\ +\mathfrak{h}_5 D_b(\varrho_0,\varrho_1)\end{pmatrix}\right) \atop +(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)\right)$$

$$= \left(\psi\left(\varphi\begin{pmatrix}(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)D_b(\varrho_0,\varrho_1)\\ +(\mathfrak{h}_1+\omega\mathfrak{h}_4)D_b(\varrho_2,\varrho_1)\end{pmatrix}\right) \atop +(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)\right)$$

which implies that

$$(e-\mathfrak{h}_1-\omega\mathfrak{h}_4)\varphi\big(D_b(\varrho_2,\varrho_1)\big) \leq \begin{pmatrix}(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)\psi\Big(\varphi\big(D_b(\varrho_0,\varrho_1)\big)\Big)\\ +(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)\end{pmatrix} \quad (1.3)$$

Adding inequalities (1.2) and (1.3), we obtain $\varphi(D_b(\varrho_1,\varrho_2))$ where,

$$(2e-\mathfrak{h}_1-\mathfrak{h}_2-\omega\mathfrak{h}_3-\omega\mathfrak{h}_4)\varphi\big(D_b(\varrho_1,\varrho_2)\big) \leq \begin{pmatrix}\begin{pmatrix}2\mathfrak{h}_5+\mathfrak{h}_1+\mathfrak{h}_2\\ +\omega\,\mathfrak{h}_3+\omega\mathfrak{h}_4\end{pmatrix}\psi\Big(\varphi\big(D_b(\varrho_0,\varrho_1)\big)\Big)\\ +(2\mathfrak{h}_5+\mathfrak{h}_1+\mathfrak{h}_2+\omega\,\mathfrak{h}_3+\omega\mathfrak{h}_4)\end{pmatrix} (1.4)$$

Denote $\mathfrak{h}_1+\mathfrak{h}_2+\omega\,\mathfrak{h}_3+\omega\mathfrak{h}_4=\mathfrak{h}$, then (1.4) yields that

$$(2e-\mathfrak{h})\varphi\big(D_b(\varrho_1,\varrho_2)\big) \leq (2\mathfrak{h}_5+\mathfrak{h}\,)\psi\Big(\varphi\big(D_b(\varrho_0,\varrho_1)\big)\Big)+(2\mathfrak{h}_5+\mathfrak{h}) \qquad (1.5)$$

Similarly, it can be shown that, there exists $\varrho_2\in\mathfrak{J}\varrho_1,\ \varrho_3\in\mathfrak{L}\varrho_2$ such that

$$\varphi\big(D_b(\varrho_2,\varrho_3)\big) \leq \varphi\big(H(\mathfrak{I}\varrho_1,\mathfrak{L}\varrho_2)\big) + (\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)^2$$

$$\varphi\big(D_b(\varrho_2,\varrho_3)\big) \leq \left(\begin{array}{c}\psi\left(\varphi\left(\begin{array}{c}\mathfrak{h}_1\frac{D_b(\varrho_1,\mathfrak{I}\varrho_1)}{1+D_b(\varrho_1,\mathfrak{I}\varrho_1)}+\mathfrak{h}_2\frac{D_b(\varrho_2,\mathfrak{L}\varrho_2)}{1+D_b(\varrho_2,\mathfrak{L}\varrho_2)}+\mathfrak{h}_3\frac{D_b(\varrho_1,\mathfrak{L}\varrho_2)}{1+D_b(\varrho_1,\mathfrak{L}\varrho_2)}\\ +\mathfrak{h}_4\frac{D_b(\varrho_2,\mathfrak{I}\varrho_1)}{1+D_b(\varrho_2,\mathfrak{I}\varrho_1)}+\mathfrak{h}_5D_b(\varrho_1,\varrho_2)\end{array}\right)\right)\\ +(\mathfrak{h}_1+\mathfrak{h}_5+\omega\mathfrak{h}_3)^2\end{array}\right)$$

$$= \left(\begin{array}{c}\psi\left(\varphi\left(\begin{array}{c}\mathfrak{h}_1\frac{D_b(\varrho_1,\varrho_2)}{1+D_b(\varrho_1,\varrho_2)}+\mathfrak{h}_2\frac{D_b(\varrho_2,\varrho_3)}{1+D_b(\varrho_2,\varrho_3)}+\mathfrak{h}_3\frac{D_b(\varrho_1,\varrho_3)}{1+D_b(\varrho_1,\varrho_3)}\\ +\mathfrak{h}_4\frac{D_b(\varrho_2,\varrho_2)}{1+D_b(\varrho_2,\varrho_2)}+\mathfrak{h}_5D_b(\varrho_1,\varrho_2)\end{array}\right)\right)\\ +(\mathfrak{h}_1+\mathfrak{h}_5+\omega\mathfrak{h}_3)^2\end{array}\right)$$

$$\leq \left(\begin{array}{c}\psi\left(\varphi\left(\begin{array}{c}\mathfrak{h}_1D_b(\varrho_1,\varrho_2)+\mathfrak{h}_2D_b(\varrho_2,\varrho_3)+\mathfrak{h}_3D_b(\varrho_1,\varrho_3)\\ +\mathfrak{h}_4D_b(\varrho_2,\varrho_2)+\mathfrak{h}_5D_b(\varrho_1,\varrho_2)\end{array}\right)\right)\\ +(\mathfrak{h}_1+\mathfrak{h}_5+\omega\mathfrak{h}_3)^2\end{array}\right)$$

$$\leq \left(\begin{array}{c}\psi\left(\varphi\left(\begin{array}{c}\mathfrak{h}_1D_b(\varrho_1,\varrho_2)+\mathfrak{h}_2D_b(\varrho_2,\varrho_3)\\ +\mathfrak{h}_3\omega\,[D_b(\varrho_1,\varrho_2)+D_b(\varrho_2,\varrho_3)]\\ +\mathfrak{h}_5D_b(\varrho_1,\varrho_2)\end{array}\right)\right)\\ +(\mathfrak{h}_1+\mathfrak{h}_5+\omega\mathfrak{h}_3)^2\end{array}\right)$$

$$\leq \left(\begin{array}{c}\psi\left(\varphi\left(\begin{array}{c}(\mathfrak{h}_1+\omega\mathfrak{h}_3+\mathfrak{h}_5)D_b(\varrho_1,\varrho_2)\\ +(\mathfrak{h}_2+\omega\mathfrak{h}_3)D_b(\varrho_2,\varrho_3)\end{array}\right)\right)\\ +(\mathfrak{h}_1+\mathfrak{h}_5+\omega\mathfrak{h}_3)^2\end{array}\right)$$

which implies that

$$(e-\mathfrak{h}_2-\omega\mathfrak{h}_3)\varphi\big(D_b(\varrho_1,\varrho_2)\big) \leq \left(\begin{array}{c}(\mathfrak{h}_1+\mathfrak{h}_5+\omega\,\mathfrak{h}_3)^2\psi\Big(\varphi\big(D_b(\varrho_0,\varrho_1)\big)\Big)\\ +2(\mathfrak{h}_1+\mathfrak{h}_5+\omega\mathfrak{h}_3)^2\end{array}\right) \quad (1.6)$$

Then,

$$\varphi\big(D_b(\varrho_3,\varrho_2)\big) \leq \varphi\big(H(\mathfrak{L}\varrho_2,\mathfrak{I}\varrho_1)\big) + (\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)^2$$

$$\varphi(D_b(\varrho_3,\varrho_2)) \leq \left(\begin{array}{c}\psi\left(\varphi\left(\begin{array}{c}\mathfrak{h}_1\frac{D_b(\varrho_2,\mathfrak{L}\varrho_2)}{1+D_b(\varrho_2,\mathfrak{L}\varrho_2)}+\mathfrak{h}_2\frac{D_b(\varrho_1,\mathfrak{I}\varrho_1)}{1+D_b(\varrho_1,\mathfrak{I}\varrho_1)}+\mathfrak{h}_3\frac{D_b(\varrho_2,\mathfrak{I}\varrho_1)}{1+D_b(\varrho_2,\mathfrak{I}\varrho_1)}\\ +\mathfrak{h}_4\frac{D_b(\varrho_1,\mathfrak{L}\varrho_2)}{1+D_b(\varrho_1,\mathfrak{L}\varrho_2)}+\mathfrak{h}_5D_b(\varrho_2,\varrho_1)\end{array}\right)\right)\\ +(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)^2\end{array}\right)$$

$$= \left(\begin{array}{c}\psi\left(\varphi\left(\begin{array}{c}\mathfrak{h}_1\frac{D_b(\varrho_2,\varrho_3)}{1+D_b(\varrho_2,\varrho_3)}+\mathfrak{h}_2\frac{D_b(\varrho_1,\varrho_2)}{1+D_b(\varrho_1,\varrho_2)}+\mathfrak{h}_3\frac{D_b(\varrho_2,\varrho_2)}{1+D_b(\varrho_2,\varrho_2)}\\ +\mathfrak{h}_4\frac{D_b(\varrho_1,\varrho_3)}{1+D_b(\varrho_1,\varrho_3)}+\mathfrak{h}_5D_b(\varrho_2,\varrho_1)\end{array}\right)\right)\\ +(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)^2\end{array}\right)$$

$$\leq \left(\begin{array}{c}\psi\left(\varphi\left(\begin{array}{c}\mathfrak{h}_1D_b(\varrho_2,\varrho_3)+\mathfrak{h}_2D_b(\varrho_1,\varrho_2)+\mathfrak{h}_3D_b(\varrho_2,\varrho_2)\\ +\mathfrak{h}_4D_b(\varrho_1,\varrho_3)+\mathfrak{h}_5D_b(\varrho_2,\varrho_1)\end{array}\right)\right)\\ +(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)^2\end{array}\right)$$

$$\leq \left(\begin{array}{c}\psi\left(\varphi\left(\begin{array}{c}\mathfrak{h}_1D_b(\varrho_2,\varrho_3)+\mathfrak{h}_2D_b(\varrho_1,\varrho_2)\\ +\mathfrak{h}_4\omega[D_b(\varrho_1,\varrho_2)+D_b(\varrho_2,\varrho_3)]\\ +\mathfrak{h}_5D_b(\varrho_2,\varrho_1)\end{array}\right)\right)\\ +(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)^2\end{array}\right)$$

$$= \left(\psi\left(\varphi\left(\begin{array}{c} (\mathfrak{h}_2 + \mathfrak{h}_5 + \omega\mathfrak{h}_4) D_b(\varrho_1, \varrho_2) \\ +(\mathfrak{h}_1 + \omega\mathfrak{h}_4) D_b(\varrho_2, \varrho_3) \end{array} \right)\right) \atop +(\mathfrak{h}_2 + \mathfrak{h}_5 + \omega\mathfrak{h}_4)^2 \right)$$

which implies that

$$(e - \mathfrak{h}_1 - \omega\mathfrak{h}_4)\varphi\big(D_b(\varrho_3, \varrho_2)\big) \le \left(\begin{array}{c} (\mathfrak{h}_2 + \mathfrak{h}_5 + \omega\mathfrak{h}_4)^2 \psi\Big(\varphi\big(D_b(\varrho_0, \varrho_1)\big)\Big) \\ +2(\mathfrak{h}_2 + \mathfrak{h}_5 + \omega\mathfrak{h}_4)^2 \end{array} \right) \quad (1.7)$$

Adding inequalities (1.6) and (1.3), we obtain $\varphi(D_b(\varrho_1, \varrho_2))$ where,

$$\begin{pmatrix} 2e - \mathfrak{h}_1 - \mathfrak{h}_2 \\ -\omega\mathfrak{h}_3 - \omega\mathfrak{h}_4 \end{pmatrix} \varphi\big(D_b(\varrho_2, \varrho_3)\big) \le \left(\begin{array}{c} \begin{pmatrix} 2\mathfrak{h}_5 + \mathfrak{h}_1 + \mathfrak{h}_2 \\ +\omega\, \mathfrak{h}_3 + \omega\mathfrak{h}_4 \end{pmatrix}^2 \psi\Big(\varphi\big(D_b(\varrho_0, \varrho_1)\big)\Big) \\ +2(2\mathfrak{h}_5 + \mathfrak{h}_1 + \mathfrak{h}_2 + \omega\mathfrak{h}_3 + \omega\mathfrak{h}_4)^2 \end{array} \right) \quad (1.8)$$

Denote $\mathfrak{h}_1 + \mathfrak{h}_2 + \omega\, \mathfrak{h}_3 + \omega\mathfrak{h}_4 = \mathfrak{h}$, then (1.8) yields that

$$(2e - \mathfrak{h})\varphi\big(D_b(\varrho_1, \varrho_2)\big) \le (2\mathfrak{h}_5 + \mathfrak{h}\,)^2 \psi\Big(\varphi\big(D_b(\varrho_0, \varrho_1)\big)\Big) + 2(2\mathfrak{h}_5 + \mathfrak{h})^2 \quad (1.9)$$

Continuing this process, we obtain by induction a sequence $\{\varrho_n\}$ such that $\varrho_{2n+1} \in \mathfrak{I}\varrho_{2n}$, $\varrho_{2n+2} \in \mathfrak{L}\varrho_{2n+1}$ such that

$$\varphi\big(D_b(\varrho_{2n+1}, \varrho_{2n+2})\big) \le \varphi\big(H(\mathfrak{I}\varrho_{2n}, \mathfrak{L}\varrho_{2n+1})\big) + (\mathfrak{h}_2 + \mathfrak{h}_5 + \omega\mathfrak{h}_4)^{2n+1}$$

$$\le \left(\psi\left(\varphi\left(\begin{array}{c} \mathfrak{h}_1 \frac{D_b(\varrho_{2n}, \mathfrak{I}\varrho_{2n})}{1 + D_b(\varrho_{2n}, \mathfrak{I}\varrho_{2n})} + \mathfrak{h}_2 \frac{D_b(\varrho_{2n+1}, \mathfrak{L}\varrho_{2n+1})}{1 + D_b(\varrho_{2n+1}, \mathfrak{L}\varrho_{2n+1})} + \mathfrak{h}_3 \frac{D_b(\varrho_{2n}, \mathfrak{L}\varrho_{2n+1})}{1 + D_b(\varrho_{2n}, \mathfrak{L}\varrho_{2n+1})} \\ +\mathfrak{h}_4 \frac{D_b(\varrho_{2n+1}, \mathfrak{I}\varrho_{2n})}{1 + D_b(\varrho_{2n+1}, \mathfrak{I}\varrho_{2n})} + \mathfrak{h}_5 D_b(\varrho_{2n}, \varrho_{2n+1}) \end{array} \right)\right) \atop +(\mathfrak{h}_2 + \mathfrak{h}_5 + \omega\mathfrak{h}_4)^{2n+1} \right)$$

$$= \left(\psi\left(\varphi\left(\begin{array}{c} \mathfrak{h}_1 \frac{D_b(\varrho_{2n}, \varrho_{2n+1})}{1 + D_b(\varrho_{2n}, \varrho_{2n+1})} + \mathfrak{h}_2 \frac{D_b(\varrho_{2n+1}, \varrho_{2n+2})}{1 + D_b(\varrho_{2n+1}, \varrho_{2n+2})} + \mathfrak{h}_3 \frac{D_b(\varrho_{2n}, \varrho_{2n+2})}{1 + D_b(\varrho_{2n}, \varrho_{2n+2})} \\ +\mathfrak{h}_4 \frac{D_b(\varrho_{2n+1}, \varrho_{2n+1})}{1 + D_b(\varrho_{2n+1}, \varrho_{2n+1})} + \mathfrak{h}_5 D_b(\varrho_{2n}, \varrho_{2n+1}) \end{array} \right)\right) \atop +(\mathfrak{h}_2 + \mathfrak{h}_5 + \omega\mathfrak{h}_4)^{2n+1} \right)$$

$$\le \left(\psi\left(\varphi\left(\begin{array}{c} \mathfrak{h}_1 D_b(\varrho_{2n}, \varrho_{2n+1}) + \mathfrak{h}_2 D_b(\varrho_{2n+1}, \varrho_{2n+2}) \\ +\mathfrak{h}_3 \omega[D_b(\varrho_{2n}, \varrho_{2n+1}) + D_b(\varrho_{2n+1}, \varrho_{2n+2})] \\ +\mathfrak{h}_5 D_b(\varrho_{2n}, \varrho_{2n+1}) \end{array} \right)\right) \atop +(\mathfrak{h}_2 + \mathfrak{h}_5 + \omega\mathfrak{h}_4)^{2n+1} \right)$$

$$= \left(\psi\left(\varphi\left(\begin{array}{c} (\mathfrak{h}_1 + \mathfrak{h}_5 + \omega\mathfrak{h}_3) D_b(\varrho_{2n}, \varrho_{2n+1}) \\ +(\mathfrak{h}_2 + \omega\mathfrak{h}_3) D_b(\varrho_{2n+1}, \varrho_{2n+2}) \end{array} \right)\right) \atop +(\mathfrak{h}_2 + \mathfrak{h}_5 + \omega\mathfrak{h}_4)^{2n+1} \right)$$

which implies that

$$(e - \mathfrak{h}_2 - \omega\mathfrak{h}_3)\varphi\big(D_b(\varrho_{2n+1}, \varrho_{2n+2})\big) \le \left(\begin{array}{c} (\mathfrak{h}_1 + \mathfrak{h}_5 + \omega\mathfrak{h}_3)\psi\Big(\varphi\big(D_b(\varrho_{2n}, \varrho_{2n+1})\big)\Big) \\ +(\mathfrak{h}_2 + \mathfrak{h}_5 + \omega\mathfrak{h}_4)^{2n+1} \end{array} \right) \quad (1.10)$$

Also,

$$\varphi\big(D_b(\varrho_{2n+2}, \varrho_{2n+1})\big) \le \varphi\big(\,H(\mathfrak{L}\varrho_{2n+1}, \mathfrak{I}\varrho_{2n})\big) + (\mathfrak{h}_2 + \mathfrak{h}_5 + \omega\mathfrak{h}_4)^{2n+1}$$

$$\le \left(\psi\left(\varphi\left(\begin{array}{c} \mathfrak{h}_1 \frac{D_b(\varrho_{2n+1}, \mathfrak{L}\varrho_{2n+1})}{1 + D_b(\varrho_{2n+1}, \mathfrak{L}\varrho_{2n+1})} + \mathfrak{h}_2 \frac{D_b(\varrho_{2n}, \mathfrak{I}\varrho_{2n})}{1 + D_b(\varrho_{2n}, \mathfrak{I}\varrho_{2n})} + \mathfrak{h}_3 \frac{D_b(\varrho_{2n+1}, \mathfrak{I}\varrho_{2n})}{1 + D_b(\varrho_{2n+1}, \mathfrak{I}\varrho_{2n})} \\ +\mathfrak{h}_4 \frac{D_b(\varrho_{2n}, \mathfrak{L}\varrho_{2n+1})}{1 + D_b(\varrho_{2n}, \mathfrak{L}\varrho_{2n+1})} + \mathfrak{h}_5 D_b(\varrho_{2n+1}, \varrho_{2n}) \end{array} \right)\right) \atop +(\mathfrak{h}_2 + \mathfrak{h}_5 + \omega\mathfrak{h}_4)^{2n+1} \right)$$

$$= \left(\psi\left(\varphi\left(\begin{matrix}\mathfrak{h}_1 \frac{D_b(\varrho_{2n+1},\varrho_{2n+2})}{1+D_b(\varrho_{2n+1},\varrho_{2n+2})} + \mathfrak{h}_2 \frac{D_b(\varrho_{2n},\varrho_{2n+1})}{1+D_b(\varrho_{2n},\varrho_{2n+1})} + \mathfrak{h}_3 \frac{D_b(\varrho_{2n+1},\varrho_{2n+1})}{1+D_b(\varrho_{2n+1},\varrho_{2n+1})} \\ +\mathfrak{h}_4 \frac{D_b(\varrho_{2n},\varrho_{2n+2})}{1+D_b(\varrho_{2n},\varrho_{2n+2})} + \mathfrak{h}_5 D_b(\varrho_{2n+1},\varrho_{2n})\end{matrix}\right)\right) \\ +(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)^{2n+1}\right)$$

$$\leq \left(\psi\left(\varphi\left(\begin{matrix}\mathfrak{h}_1 D_b(\varrho_{2n+1},\varrho_{2n+2}) + \mathfrak{h}_2 D_b(\varrho_{2n},\varrho_{2n+1}) \\ +\mathfrak{h}_4\omega[D_b(\varrho_{2n},\varrho_{2n+1}) + D_b(\varrho_{2n+1},\varrho_{2n+2})] \\ +\mathfrak{h}_5 D_b(\varrho_{2n+1},\varrho_{2n})\end{matrix}\right)\right) \\ +(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)^{2n+1}\right)$$

$$= \left(\psi\left(\varphi\left(\begin{matrix}(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)D_b(\varrho_{2n},\varrho_{2n+1}) \\ +(\mathfrak{h}_1+\omega\mathfrak{h}_4)D_b(\varrho_{2n+1},\varrho_{2n+2})\end{matrix}\right)\right) \\ +(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)^{2n+1}\right)$$

$$(e-\mathfrak{h}_1+\omega\mathfrak{h}_4)\varphi\big(D_b(\varrho_{2n+2},\varrho_{2n+1})\big) \leq \left(\begin{matrix}(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)\psi\left(\varphi\big(D_b(\varrho_{2n},\varrho_{2n+1})\big)\right) \\ +(\mathfrak{h}_2+\mathfrak{h}_5+\omega\mathfrak{h}_4)^{2n+1}\end{matrix}\right) \quad (1.11)$$

Add up (1.10) and (1.11) yields that

$$\begin{pmatrix}2e-\mathfrak{h}_1-\mathfrak{h}_2 \\ -\omega\mathfrak{h}_3-\omega\mathfrak{h}_4\end{pmatrix}\varphi\big(D_b(\varrho_{2n+1},\varrho_{2n+2})\big) \leq \left(\begin{matrix}\begin{pmatrix}2\mathfrak{h}_5+\mathfrak{h}_1+\mathfrak{h}_2 \\ +\omega\mathfrak{h}_3+\omega\mathfrak{h}_4\end{pmatrix}\psi\left(\varphi\big(D_b(\varrho_{2n},\varrho_{2n+1})\big)\right) \\ +(2\mathfrak{h}_5+\mathfrak{h}_1+\mathfrak{h}_2+\omega\mathfrak{h}_3+\omega\mathfrak{h}_4)^{2n+1}\end{matrix}\right) \quad (1.12)$$

Denote $\mathfrak{h}_1+\mathfrak{h}_2+\omega\,\mathfrak{h}_3+\omega\mathfrak{h}_4=\mathfrak{h}$, then (1.12) yields that

$$(2e-\mathfrak{h})\varphi\big(D_b(\varrho_{2n+1},\varrho_{2n+2})\big) \leq \left(\begin{matrix}(2\mathfrak{h}_5+\mathfrak{h})\psi\left(\varphi\big(D_b(\varrho_{2n},\varrho_{2n+1})\big)\right) \\ +(2\mathfrak{h}_5+\mathfrak{h})^{2n+1}\end{matrix}\right) \quad (1.13)$$

Therefore,

$$\varphi(D_b(\varrho_n,\varrho_{n+1})) \leq \varphi\big(H(\Im\varrho_{n-1},\mathfrak{L}\varrho_n)\big) + (2\mathfrak{h}_5+\mathfrak{h}_1+\mathfrak{h}_2+\omega\mathfrak{h}_3+\omega\mathfrak{h}_4)^n$$

$$\begin{pmatrix}2e-\mathfrak{h}_1-\mathfrak{h}_2 \\ -\omega\mathfrak{h}_3-\omega\mathfrak{h}_4\end{pmatrix}\varphi(D_b(\varrho_n,\varrho_{n+1})) \leq \left(\begin{matrix}\begin{pmatrix}2\mathfrak{h}_5+\mathfrak{h}_1+\mathfrak{h}_2 \\ +\omega\mathfrak{h}_3+\omega\mathfrak{h}_4\end{pmatrix}\psi\left(\varphi\big(D_b(\varrho_{n-1},\varrho_n)\big)\right) \\ +\begin{pmatrix}2\mathfrak{h}_5+\mathfrak{h}_1+\mathfrak{h}_2 \\ +\omega\mathfrak{h}_3+\omega\mathfrak{h}_4\end{pmatrix}^n\end{matrix}\right) \quad (1.14)$$

Denote $\mathfrak{h}_1+\mathfrak{h}_2+\omega\,\mathfrak{h}_3+\omega\mathfrak{h}_4=\mathfrak{h}$, then (1.14) yields that

$$(2e-\mathfrak{h})\varphi\big(D_b(\varrho_n,\varrho_{n+1})\big) \leq (2\mathfrak{h}_5+\mathfrak{h})\psi\left(\varphi\big(D_b(\varrho_{n-1},\varrho_n)\big)\right) + (2\mathfrak{h}_5+\mathfrak{h})^n \quad (1.15)$$

Note that

$$2\rho(\mathfrak{h}) \leq (\omega+1)\rho(\mathfrak{h}) \leq 2\omega\rho(\mathfrak{h}_5) + (\omega+1)\rho(\mathfrak{h}) < 2$$

$\rho(\mathfrak{h}) < 1 < 2$, then by lemma 2.16 it follows that $(2e-\mathfrak{h})$ is invertible. Furthermore,

$$(2e-\mathfrak{h})^{-1} = \sum_{i=0}^{\infty}\frac{\mathfrak{h}^i}{2^{i+1}}$$

By multiplying in both sides of (2.15) by $(2e-\mathfrak{h})^{-1}$, we get

$$\varphi\big(D_b(\varrho_n,\varrho_{n+1})\big) \leq \left(\begin{matrix}(2e-\mathfrak{h})^{-1}(2\mathfrak{h}_5+\mathfrak{h})\psi\left(\varphi\big(D_b(\varrho_{n-1},\varrho_n)\big)\right) \\ +(2e-\mathfrak{h})^{-1}(2\mathfrak{h}_5+\mathfrak{h})^n\end{matrix}\right) \quad (1.16)$$

Let $\gamma = (2e-\mathfrak{h})^{-1}(2\mathfrak{h}_5+\mathfrak{h})$, then by (1.16) we get

$$\begin{aligned}\varphi\big(D_b(\varrho_n,\varrho_{n+1})\big) &\leq \psi(\gamma\varphi(D_b(\varrho_{n-1},\varrho_n))) + \gamma^n \\ &\leq \gamma\psi\left(\gamma\varphi\big(D_b(\varrho_{n-2},\varrho_{n-1})\big)\right) + 2\gamma^n\end{aligned}$$

$$= \psi\left(\gamma^2\varphi\big(D_b(\varrho_{n-2},\varrho_{n-1})\big)\right) + 2\gamma^n$$
$$\vdots$$
$$\le \gamma^n\psi\left(\varphi\big(D_b(\varrho_0,\varrho_1)\big)\right) + n\gamma^n.$$

Since $\mathfrak{h}_5$ commutes with $\mathfrak{h}$, it follows that

$$\begin{aligned}(2e-\mathfrak{h})^{-1}(2\mathfrak{h}_5+\mathfrak{h}) &= \left(\textstyle\sum_{i=0}^{\infty}\frac{\mathfrak{h}^i}{2^{i+1}}\right)(2\mathfrak{h}_5+\mathfrak{h})\\ &= 2\left(\textstyle\sum_{i=0}^{\infty}\frac{\mathfrak{h}^i}{2^{i+1}}\right)\mathfrak{h}_5 + \left(\textstyle\sum_{i=0}^{\infty}\frac{\mathfrak{h}^{i+1}}{2^{i+1}}\right)\\ &= 2\mathfrak{h}\left(\textstyle\sum_{i=0}^{\infty}\frac{\mathfrak{h}^i}{2^{i+1}}\right) + \mathfrak{h}\left(\textstyle\sum_{i=0}^{\infty}\frac{\mathfrak{h}^i}{2^{i+1}}\right)\\ &= (2\mathfrak{h}_5+\mathfrak{h})\left(\textstyle\sum_{i=0}^{\infty}\frac{\mathfrak{h}^i}{2^{i+1}}\right) = (2\mathfrak{h}_5+\mathfrak{h})(2e-\mathfrak{h})^{-1}\end{aligned}$$

then, $(2e-\mathfrak{h})^{-1}$ commutes with $(2\mathfrak{h}_5+\mathfrak{h})$. Note by Lemma 2.5 and Lemma 3.1 that

$$\begin{aligned}\rho(\gamma) &= \rho\big((2\mathfrak{h}_5+\mathfrak{h})(2e-\mathfrak{h})^{-1}\big)\\ &\le \rho((2e-\mathfrak{h})^{-1})\,\rho\big((2\mathfrak{h}_5+\mathfrak{h})\big)\\ &\le \frac{1}{2-\rho(\mathfrak{h})}[2\rho(\mathfrak{h}_5)+\rho(\mathfrak{h})]\\ &= \frac{1}{2-\rho(\mathfrak{h}_1+\mathfrak{h}_2+\omega\,\mathfrak{h}_3+\omega\mathfrak{h}_4)}[2\rho(\mathfrak{h}_5)+\rho(\mathfrak{h}_1+\mathfrak{h}_2+\omega\,\mathfrak{h}_3+\omega\mathfrak{h}_4)]\\ &< \frac{1}{\omega},\quad [since\ 2\omega\rho(\mathfrak{h}_5)+(\omega+1)\big(\rho(\mathfrak{h}_1)+\rho(\mathfrak{h}_2)+\omega\,\rho(\mathfrak{h}_3)+\omega\,\rho(\mathfrak{h}_4)\big)<2\,]\end{aligned}$$

which establishes that $(e-\omega\gamma)$ is invertible and $\|\gamma^m\| \to 0(m\to\infty)$. Hence, for any $m\ge 1$, $\mathfrak{p}\ge 1$ and $\gamma\in\mathfrak{P}$ with $\rho(\gamma)<\frac{1}{\omega}$, we have that

$$\begin{aligned}\phi\left(D_b\big(\varrho_m,\varrho_{m+\mathfrak{p}}\big)\right) &\le \psi\big(\phi\omega[D_b(\varrho_m,\varrho_{m+1})+D_b(\varrho_{m+1},\varrho_{m+\mathfrak{p}})]\big)\\ &\le \omega\psi\big(\phi D_b(\varrho_m,\varrho_{m+1})\big)+\omega^2\psi\left(\phi\begin{bmatrix}D_b(\varrho_{m+1},\varrho_{m+2})\\ +D_b\big(\varrho_{m+2},\varrho_{m+\mathfrak{p}}\big)\end{bmatrix}\right)\\ &\le \begin{pmatrix}\omega\psi\big(\phi D_b(\varrho_m,\varrho_{m+1})\big)+\omega^2\psi\big(\phi D_b(\varrho_{m+1},\varrho_{m+2})\big)\\ +\omega^3\psi\big(\phi D_b(\varrho_{m+2},\varrho_{m+3})\big)+\cdots\\ +\omega^{\mathfrak{p}-1}\psi\left(\phi D_b\big(\varrho_{m+\mathfrak{p}-2},\varrho_{m+\mathfrak{p}-1}\big)\right)\\ +\omega^{\mathfrak{p}-1}\psi\left(\phi D_b\big(\varrho_{m+\mathfrak{p}-1},\varrho_{m+\mathfrak{p}}\big)\right)\end{pmatrix}\\ &\le \begin{pmatrix}\omega\gamma^m\psi\big(\phi D_b(\varrho_0,\varrho_1)\big)+\omega^2\gamma^{m+1}\psi\big(\phi D_b(\varrho_0,\varrho_1)\big)\\ +\omega^3\gamma^{m+2}\psi\big(\phi D_b(\varrho_0,\varrho_1)\big)+\cdots\\ +\omega^{\mathfrak{p}-1}\gamma^{m+\mathfrak{p}-2}\psi\big(\phi D_b(\varrho_0,\varrho_1)\big)\\ +\omega^{\mathfrak{p}}\gamma^{m+\mathfrak{p}-1}\psi\big(\phi D_b(\varrho_0,\varrho_1)\big)\end{pmatrix}\\ &= \omega\gamma^m[e+\omega\gamma+\omega^2\gamma^2+\cdots+(\omega\gamma)^{\mathfrak{p}-1}]\psi\big(\phi D_b(\varrho_0,\varrho_1)\big)\\ &\le \omega\gamma^m(e-\omega\gamma)^{-1}\psi\big(\phi D_b(\varrho_0,\varrho_1)\big).\end{aligned}$$

In view, $\|\omega\gamma^m\psi\big(\phi D_b(\varrho_0,\varrho_1)\big)\| \le \|\omega\gamma^m\|\|\psi\big(\phi D_b(\varrho_0,\varrho_1)\big)\| \to 0\ (m\to\infty)$, by lemma 2.4, we have $\{\omega\gamma^m\psi\big(\phi D_b(\varrho_0,\varrho_1)\big)\}$ is a $\mathfrak{c}$-sequence. Next by using Lemma 3.2 and lemma 2.15, we conclude that $\{\varrho_n\}$ is a Cauchy sequence. Since (X,D_b) is complete, there exists $\mathfrak{s}\in X$ such that $\varrho_n\to\mathfrak{s}$. We shall prove that $\mathfrak{s}$ is a common fixed point of $\mathfrak{I}$ and $\mathfrak{L}$.

$$\varphi(D_b(\mathfrak{s},\mathfrak{I}\mathfrak{s})) \le \psi\big(\varphi(\,\omega\,[D_b(\mathfrak{s},\varrho_{2n+1})+D_b(\varrho_{2n+1},\mathfrak{I}\mathfrak{s})])\big)$$

$$
\begin{aligned}
&\leq \psi\big(\varphi(\omega\ [D_b(\mathfrak{s},\varrho_{2n+1}) + \mathcal{H}(\varrho_{2n+1},\mathfrak{J}\mathfrak{s})])\big)\\
\varphi(D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})) &\leq \psi\big(\varphi(\omega\ [D_b(\mathfrak{s},\varrho_{2n+1}) + D_b(\varrho_{2n+1},\mathfrak{L}\mathfrak{s})])\big) \qquad (1.17)\\
&\leq \psi\big(\varphi(\omega\ [D_b(\mathfrak{s},\varrho_{2n+1}) + \mathcal{H}(\varrho_{2n},\mathfrak{L}\mathfrak{s})])\big)
\end{aligned}
$$

Where,

$$
\varphi\big(\mathcal{H}(\varrho_{2n},\mathfrak{L}\mathfrak{s})\big) \leq \psi\left(\varphi\left(\begin{matrix}\mathfrak{h}_1\frac{D_b(\varrho_{2n},\mathfrak{J}\varrho_{2n})}{1+D_b(\varrho_{2n},\mathfrak{J}\varrho_{2n})} + \mathfrak{h}_2\frac{D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})}{1+D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})}\\ +\mathfrak{h}_3\frac{D_b(\varrho_{2n},\mathfrak{L}\mathfrak{s})}{1+D_b(\varrho_{2n},\mathfrak{L}\mathfrak{s})} + \mathfrak{h}_4\frac{D_b(\mathfrak{s},\mathfrak{J}\varrho_{2n})}{1+D_b(\mathfrak{s},\mathfrak{J}\varrho_{2n})} + \mathfrak{h}_5 D_b(\varrho_{2n},\mathfrak{s})\end{matrix}\right)\right) (1.18)
$$

Using (1.18) in (1.17) and letting as $n \to \infty$, we obtain,

$$
\begin{aligned}
\varphi(D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})) &\leq \psi(\varphi(\omega\ D_b(\mathfrak{s},\mathfrak{s}))\\
&\leq \psi\left(\varphi\left(\omega\left[\begin{matrix}\mathfrak{h}_1\frac{D_b(\mathfrak{s},\mathfrak{s})}{1+D_b(\mathfrak{s},\mathfrak{s})} + \mathfrak{h}_2\frac{D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})}{1+D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})} + \mathfrak{h}_3\frac{D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})}{1+D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})}\\ +\mathfrak{h}_4\frac{D_b(\mathfrak{s},\mathfrak{s})}{1+D_b(\mathfrak{s},\mathfrak{s})} + \mathfrak{h}_5 D_b(\mathfrak{s},\mathfrak{s})\end{matrix}\right]\right)\right)\\
&= \psi\left(\varphi\left(\omega\left[\begin{matrix}\mathfrak{h}_1 D_b(\mathfrak{s},\mathfrak{s}) + \mathfrak{h}_2 D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s}) + \mathfrak{h}_3 D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})\\ +\mathfrak{h}_4 D_b(\mathfrak{s},\mathfrak{s}) + \mathfrak{h}_5 D_b(\mathfrak{s},\mathfrak{s})\end{matrix}\right]\right)\right)\\
&= \psi\big(\varphi(\omega\ [\mathfrak{h}_2 D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s}) + \mathfrak{h}_3 D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})])\big)\\
&\leq \psi\Big(\varphi\big(\omega(\mathfrak{h}_2+\mathfrak{h}_3) D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})\big)\Big)
\end{aligned}
$$

Which implies that

$$
\psi\big(\phi D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})\big) \leq 0
$$

Then by Lemma 2.2, we deduce that $\psi\big(\phi D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s})\big) = 0$, that is $\mathfrak{L}(\mathfrak{s}) = \mathfrak{s}$. Similarly, $\mathfrak{J}(\mathfrak{s}) = \mathfrak{s}$. Hence $\mathfrak{J}\mathfrak{s} = \mathfrak{L}\mathfrak{s} = \mathfrak{r}$. In the following we shall prove $\mathfrak{L}$ and $\mathfrak{J}$ have a unique point of coincidence. Such that $\mathfrak{s} \neq \mathfrak{s}^*$ then from (1.1) we have

$$
\begin{aligned}
\varphi\big(D_b(\mathfrak{s},\mathfrak{s}^*)\big) &\leq\ \varphi(\mathcal{H}(\mathfrak{J}\mathfrak{s},\mathfrak{L}\mathfrak{s}^*))\\
&\leq \psi\left(\varphi\left(\begin{matrix}\mathfrak{h}_1\dfrac{D_b(\mathfrak{s},\mathfrak{J}\mathfrak{s})}{1+D_b(\mathfrak{s},\mathfrak{J}\mathfrak{s})} + \mathfrak{h}_2\dfrac{D_b(\mathfrak{s}^*,\mathfrak{L}\mathfrak{s}^*)}{1+D_b(\mathfrak{s}^*,\mathfrak{L}\mathfrak{s}^*)} + \mathfrak{h}_3\dfrac{D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s}^*)}{1+D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s}^*)}\\ +\mathfrak{h}_4\dfrac{D_b(\mathfrak{s}^*,\mathfrak{J}\mathfrak{s})}{1+D_b(\mathfrak{s}^*,\mathfrak{J}\mathfrak{s})} + \mathfrak{h}_5 D_b(\mathfrak{s},\mathfrak{s}^*)\end{matrix}\right)\right)\\
&\leq \psi\left(\varphi\left(\begin{matrix}\mathfrak{h}_1 D_b(\mathfrak{s},\mathfrak{J}\mathfrak{s}) + \mathfrak{h}_2 D_b(\mathfrak{s}^*,\mathfrak{L}\mathfrak{s}^*) + \mathfrak{h}_3 D_b(\mathfrak{s},\mathfrak{L}\mathfrak{s}^*)\\ +\mathfrak{h}_4 D_b(\mathfrak{s}^*,\mathfrak{J}\mathfrak{s}) + \mathfrak{h}_5 D_b(\mathfrak{s},\mathfrak{s}^*)\end{matrix}\right)\right)\\
&\leq \psi\Big(\varphi\big(\mathfrak{h}_3 D_b(\mathfrak{s},\mathfrak{s}^*) + \mathfrak{h}_4 D_b(\mathfrak{s},\mathfrak{s}^*) + \mathfrak{h}_5 D_b(\mathfrak{s},\mathfrak{s}^*)\big)\Big)\\
&= (\mathfrak{h}_3+\mathfrak{h}_4+\mathfrak{h}_5)\psi\big(\phi D_b(\mathfrak{s},\mathfrak{s}^*)\big)
\end{aligned}
$$

Set $(\mathfrak{h}_3+\mathfrak{h}_4+\mathfrak{h}_5) = \zeta$, then it follows that

$$
\varphi\big(D_b(\mathfrak{s},\mathfrak{s}^*)\big) \leq \zeta\psi\Big(\varphi\big(D_b(\mathfrak{s},\mathfrak{s}^*)\big)\Big) \leq \cdots \leq \zeta^n\psi\Big(\varphi\big(D_b(\mathfrak{s},\mathfrak{s}^*)\big)\Big) \qquad (1.19)
$$

Because of

$$
2\rho(\mathfrak{h}_5) + 2\rho(\mathfrak{h}) \leq 2\omega\rho(\mathfrak{h}_5) + (\omega+1)\rho(\mathfrak{h}\ ) < 2,
$$

It follows that $\rho(\mathfrak{h}_5) + \rho(\mathfrak{h}) < 1$. Since $\mathfrak{h}_5$ commutes with $\mathfrak{h}$, then by Lemma 2.5,

$$
\rho(\mathfrak{h}_5 + \mathfrak{h}) \leq \rho(\mathfrak{h}_5) + \rho(\mathfrak{h}) < 1.
$$

Accordingly, by Lemma 2.5, we speculate that $\{(\mathfrak{h}_5+\mathfrak{h})^n\}$ is a $\mathfrak{c}$-sequence. Noticing that $\zeta \le \mathfrak{h}_5+\mathfrak{h}$ leads to $\zeta^n \le (\mathfrak{h}_5+\mathfrak{h})^n$, we claim that $\{\zeta^n\}$ is a $\mathfrak{c}$-sequence. Consequently, in view (1.19), it easy to see $\psi\big(\phi D_b(\mathfrak{s},\mathfrak{s}^*)\big)=0$, that is $\mathfrak{s}=\mathfrak{s}^*$.

Finally, if $(\mathfrak{I},\mathfrak{L})$ is weakly compatible, then Lemma 2.19, we claim that $\mathfrak{L}$ and $\mathfrak{I}$ have a unique common fixed point.

Corollary.3.6

Let (X,D_b) be a cone b-metric space over Banach algebra $\mathfrak{A}$ and $\mathfrak{P}$ be a solid cone in $\mathfrak{A}$ with the coefficient $\omega \ge 1$. $\mathfrak{h}_i \in \mathfrak{P}$ $(i=1,2,\dots 4,)$ be a generalized Lipschitz constant with $2\omega\rho(\mathfrak{h}_4)+(\omega+1)\rho(\mathfrak{h}_1+\omega\,\mathfrak{h}_2+\omega\mathfrak{h}_3)<2$. Suppose that $\mathfrak{h}_4$ commutes with $\mathfrak{h}_1+\omega\,\mathfrak{h}_2+\omega\mathfrak{h}_3$ and the mappings $\mathfrak{I},\mathfrak{L}: X\to \mathfrak{CB}(X)$ be generalized multi-valued $(\phi,\mathfrak{F})$-contraction mapping, satisfies that

$$\phi\big(\mathcal{H}(\mathfrak{I}\varrho,\mathfrak{L}\varsigma)\big)\le \mathfrak{F}(\phi(\mathcal{M}(\varrho,\varsigma))$$

where,

$$\mathcal{M}(\varrho,\varsigma)=\mathfrak{h}_1\frac{D_b(\varrho,\mathfrak{I}\varrho)}{1+D_b(\varrho,\mathfrak{I}\varrho)}+\mathfrak{h}_2\frac{D_b(\varrho,\mathfrak{L}\varsigma)}{1+D_b(\varrho,\mathfrak{L}\,\varsigma)}+\mathfrak{h}_3\frac{D_b(\varsigma,\mathfrak{I}\varrho)}{1+D_b(\varsigma,\mathfrak{I}\varrho)}+\mathfrak{h}_4 D_b(\varrho,\varsigma)$$

Where, $\mathfrak{F}\in\Psi$, $\phi\in\Phi$ such that $\forall\,\varrho,\varsigma\in X$. Moreover, if $\mathfrak{I}$ and $\mathfrak{L}$ are weakly compatible, then $\mathfrak{I}$ and $\mathfrak{L}$ have a unique common fixed point.

Proof. Choose $\mathfrak{h}_1=\mathfrak{h}_3=\mathfrak{h}_4=\mathfrak{h}_5=\mathfrak{h}$ and $\mathfrak{h}_2=0$ in theorem 3.5, the proof is valid.

Corollary.3.7

Let (X,D_b) be a cone b-metric space over Banach algebra $\mathfrak{A}$ and $\mathfrak{P}$ be a solid cone in $\mathfrak{A}$ with the coefficient $\omega\ge 1$. $\mathfrak{h}_i\in\mathfrak{P}$ $(i=1,2,\dots 4,)$ be a generalized nonnegative real constant with $2\omega(\mathfrak{h}_4)+(\omega+1)(\mathfrak{h}_1+\omega\,\mathfrak{h}_2+\omega\mathfrak{h}_3)<2$. Let mappings $\mathfrak{I},\mathfrak{L}: X\to\mathfrak{CB}(X)$ be generalized multi-valued $(\phi,\mathfrak{F})$-contraction mapping, satisfies that

$$\phi\big(\mathcal{H}(\mathfrak{I}\varrho,\mathfrak{L}\varsigma)\big)\le\mathfrak{F}(\phi(\mathfrak{h}_1 D_b(\varrho,\mathfrak{I}\varrho)+\mathfrak{h}_2 D_b(\varrho,\mathfrak{L}\varsigma)+\mathfrak{h}_3 D_b(\varsigma,\mathfrak{I}\varrho)+\mathfrak{h}_4 D_b(\varrho,\varsigma))$$

Where, $\mathfrak{F}\in\Psi$, $\phi\in\Phi$ such that $\forall\,\varrho,\varsigma\in X$. Moreover, if $\mathfrak{I}$ and $\mathfrak{L}$ are weakly compatible, then $\mathfrak{I}$ and $\mathfrak{L}$ have a unique common fixed point.

Proof. Taking $\mathfrak{h}_1,\mathfrak{h}_3,\mathfrak{h}_4,\mathfrak{h}_5\in\mathbb{R}^+$ in theorem 3.5, we obtain the desired result.

Corollary.3.8

Let (X,D_b) be a cone b-metric space over Banach algebra $\mathfrak{A}$ and $\mathfrak{P}$ be a solid cone in $\mathfrak{A}$ with the coefficient $\omega\ge 1$. $\mathfrak{h}_i\in\mathfrak{P}$ be a generalized Lipschitz constant with $\rho(\mathfrak{h})<\frac{1}{\omega^2+\omega}$. Suppose that the mappings $\mathfrak{I},\mathfrak{L}: X\to\mathfrak{CB}(X)$ be generalized multi-valued $(\phi,\mathfrak{F})$-contraction mapping, satisfies that

$$\phi\big(\mathcal{H}(\mathfrak{I}\varrho,\mathfrak{L}\varsigma)\big)\le\mathfrak{F}(\phi(\mathfrak{h}\big(D_b(\varrho,\mathfrak{L}\varsigma)+D_b(\varsigma,\mathfrak{I}\varrho)\big)$$

Where, $\mathfrak{F}\in\Psi$, $\phi\in\Phi$ such that $\forall\,\varrho,\varsigma\in X$. Moreover, if $\mathfrak{I}$ and $\mathfrak{L}$ are weakly compatible, then $\mathfrak{I}$ and $\mathfrak{L}$ have a unique common fixed point.

Proof. Putting $\mathfrak{h}_1=\mathfrak{h}_2=\mathfrak{h}_5=0$ and $\mathfrak{h}_3=\mathfrak{h}_4=\mathfrak{h}$ in theorem 3.5, we complete the proof.

Example 3.9

Let $X = [0,1]$ and $\mathfrak{A}$ be the set of all real valued functions on X which also have continuous derivatives on X with the norm $\| \varrho \| = \| \varrho \|_\infty + \| \varrho' \|_\infty$ and the usual multiplication. Let $\mathfrak{P}\{\varrho \in \mathfrak{A} : \varrho(\mathrm{t}) \geq 0, \mathrm{t} \in X\}$. It is clear that $\mathfrak{P}$ is a non normal cone and $\mathfrak{A}$ is a Banach algebra with a unit $e = 1$. Define a mapping $D_b : X \times X \to \mathfrak{A}$ by

$$D_b(\varrho, \varsigma)(\mathrm{t}) = |\varrho - \varsigma|^2 e^{\mathrm{t}}$$

We make a conclusion that (X, D_b) is a complete cone b-metric space over Banach algebra $\mathfrak{A}$ with the coefficient $\omega = 2$. Now define the mappings $\mathfrak{I}, \mathfrak{L} : X \to X$ by

$$\mathfrak{I}(\varrho) = \frac{\varrho}{8}, \qquad \mathfrak{L}(\varsigma) = \frac{\varsigma}{2}$$

Taking $\mathfrak{h}_1 = \frac{1}{12} + \frac{1}{12}\mathrm{t}, \mathfrak{h}_2 = \frac{1}{16} + \frac{1}{16}\mathrm{t}$ and $\mathfrak{h}_3 = \mathfrak{h}_4 = 0, \mathfrak{h}_5 = \frac{1}{8} + \frac{1}{8}\mathrm{t}$. Show that all conditions of Theorem 3.5 are satisfied. Theorem, 0 is the unique common fixed point of $\mathfrak{I}$ and $\mathfrak{L}$.

Example 3.10

Let $X = [0,1]$. Define a function $D_b : X \times X \to \mathfrak{A}$ by $D_b(\varrho, \varsigma) = |\varrho - \varsigma|$. Clearly, (X, D_b) is a complete cone b-metric space over Banach algebra $\mathfrak{A}$ with coefficient $\omega = 2$. Now define $\psi : \mathfrak{P} \to \mathfrak{P}$ by $\psi(\mathrm{t}) = \mathrm{t}$ for all $t > 0$. Then $\psi \in \Psi$. Also define $\phi : \mathfrak{P} \to \mathfrak{P}$ by $\phi(\mathrm{t}) = \mathcal{K}\mathrm{t}$ for all $t > 0$. Then ψ is a continuous comparison function.

Define the mapping $\mathfrak{I}, \mathfrak{L} : X \to \mathfrak{CB}(X)$ by $\mathfrak{I}(\varrho) = \frac{\varrho}{8}$, $\mathfrak{L}(\varsigma) = \frac{\varsigma}{2}$ for all $\varrho, \varsigma \in X$. Then,

$$\phi\big(\mathcal{H}(\mathfrak{I}\varrho, \mathfrak{L}\varsigma)\big) \leq \mathfrak{F}(\phi(\mathcal{M}(\varrho, \varsigma))$$

where,

$$\mathcal{M}(\varrho, \varsigma) = \mathfrak{h}_1 \frac{D_b(\varrho, \mathfrak{I}\varrho)}{1 + D_b(\varrho, \mathfrak{I}\varrho)} + \mathfrak{h}_2 \frac{D_b(\varsigma, \mathfrak{L}\varsigma)}{1 + D_b(\varsigma, \mathfrak{L}\varsigma)} + \mathfrak{h}_3 \frac{D_b(\varrho, \mathfrak{L}\varsigma)}{1 + D_b(\varrho, \mathfrak{L}\varsigma)} + \mathfrak{h}_4 \frac{D_b(\varsigma, \mathfrak{I}\varrho)}{1 + D_b(\varsigma, \mathfrak{I}\varrho)} + \mathfrak{h}_5 D_b(\varrho, \varsigma)$$

$$\begin{aligned}
\phi\big(\mathcal{H}(\mathfrak{I}\varrho, \mathfrak{L}\varsigma)\big) &\leq \psi\left(\varphi\left(\begin{matrix}\mathfrak{h}_1 \frac{|\varrho - \mathfrak{I}\varrho|}{1 + |\varrho - \mathfrak{I}\varrho|} + \mathfrak{h}_2 \frac{|\varsigma - \mathfrak{L}\varsigma|}{1 + |\varsigma - \mathfrak{L}\varsigma|} + \mathfrak{h}_3 \frac{|\varrho - \mathfrak{L}\varsigma|}{1 + |\varrho - \mathfrak{L}\varsigma|} \\ + \mathfrak{h}_4 \frac{|\varsigma - \mathfrak{I}\varrho|}{1 + |\varsigma - \mathfrak{I}\varrho|} + \mathfrak{h}_5 |\varrho - \varsigma|\end{matrix}\right)\right) \\
&\leq \psi\left(\varphi\left(\begin{matrix}\mathfrak{h}_1 \frac{\left|\varrho - \frac{\varrho}{8}\right|}{1 + \left|\varrho - \frac{\varrho}{8}\right|} + \mathfrak{h}_2 \frac{\left|\varsigma - \frac{\varsigma}{2}\right|}{1 + \left|\varsigma - \frac{\varsigma}{2}\right|} + \mathfrak{h}_3 \frac{\left|\varrho - \frac{\varsigma}{2}\right|}{1 + \left|\varrho - \frac{\varsigma}{2}\right|} \\ + \mathfrak{h}_4 \frac{\left|\varsigma - \frac{\varrho}{8}\right|}{1 + \left|\varsigma - \frac{\varrho}{8}\right|} + \mathfrak{h}_5 |\varrho - \varsigma|\end{matrix}\right)\right) \\
&\leq \psi\left(\phi\left(\frac{1}{6} |\varrho - \varsigma|\right)\right), \\
&\leq \psi\left(\frac{1}{6} |\varrho - \varsigma|\right) \\
&\leq \frac{\mu}{6} |\varrho - \varsigma|, \qquad\qquad for \;\; 0 < \mu < 1 \\
&\leq \frac{\mu}{6} M(\varrho, \varsigma) = \frac{\mu}{6} \phi\big(M(\varrho, \varsigma)\big) \\
&\leq \psi(\varphi(M(\varrho, \varsigma)).
\end{aligned}$$

Choose $\mathfrak{h}_1 = \mathfrak{h}_2 = \mathfrak{h}_3 = \mathfrak{h}_4 = 0, \mathfrak{h}_5 = \frac{1}{6}$. Note that $\mathfrak{I}$ and $\mathfrak{L}$ commute at the coincidence point $\varrho = 0$. The pair $(\mathfrak{I}, \mathfrak{L})$ is weakly compatible, it is easy to see that all the conditions of theorem 3.5 holds trivially good and 0 is the unique common fixed point of $\mathfrak{I}$ and $\mathfrak{L}$.

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