Original Research Article

Application on a Factor Derived from Hilbert and Carleson Measure on Hardy Spaces

Abstract.

We follow S. Ye and G. Feng [16], for μ_j to a positive Borel measure on the interval $[0,1)$. The Hankel matrix $\mathcal{H}_{\mu_j} = \left((\mu_j)_{n,k}\right)_{j,n,k\geq 0}$ with entries $(\mu_j)_{n,k} = (\mu_j)_{n+k}$, where $(\mu_j)_n = \int_{[0,1)} t^n d\mu_j(t)$, the operator is formally induced

$$\sum_j \mathcal{DH}_{\mu_j}(f_j)(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_j \left((\mu_j)_{n,k} a_k\right)(n + 1)z^n$$

in the space of each analytical function $f_j(z) = \sum_{k=0}^{\infty} a_k z_n$ in the unit disc $\mathbb{D}$. We classify positive Borel measures on $[0,1)$ as such $\mathcal{DH}_{\mu_j}\left(f_j\right)(z) = \int_{[0,1)} \frac{f_j(t)}{(1-tz)^2} d\mu_j(t)$ for all in Hardy spaces $H^{1+\epsilon}(0 \leq \epsilon < \infty)$, and we describe those for which $\mathcal{DH}_{\mu_j}$ is a bounded* operator from $H^{1+\epsilon}(0 \leq \epsilon < \infty)$ into $H^{1+2\epsilon}(\epsilon \geq 0)$.

Keywords: Hardy spaces; Carleson measure; derivative Hilbert operator.

1. Introduction

For μ_j be a positive Borel measure on the interval $[0,1)$. The Hankel matrix $\mathcal{H}_{\mu_j} = \left((\mu_j)_{n,k}\right)_{n,k\geq 0}$ with entries $(\mu_j)_{n,k} = (\mu_j)_{n+k}$, where $(\mu_j)_n = \int_{[0,1)} t^n d\mu_j(t)$, for analytic functions $f_j(z) = \sum_{k=0}^{\infty} a_k z^n$, the generalized Hilbert operator define as

$$\sum_j \mathcal{H}_{\mu_j}(f_j)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\sum_j \left(\mu_j\right)_{n,k} a_k\right) z^n. \tag{1}$$

When the right side is logical and defines an analytical function in $\mathbb{D}$.

In recent decades, the generalized Hilbert operator $\mathcal{H}_{\mu_j}$ which is induced by the Hankel matrix $\mathcal{H}_{\mu_j}$ has been studied extensively. Galanopoulos and Peláez [7], characterized the Borel measure μ_j for which the Hankel operator $\mathcal{H}_{\mu_j}$ is a bounded operator on H^1. Then Chatzifountas, Girela, and Peláez [1] extended this with Hardy spaces $H^{1+\epsilon}$ with $0 \leq \epsilon < \infty$. In [9], Girela and Merchán studied factorization that operates on certain fixed areas of analytic functions on disk (see [16]).

In 2021, Ye and Zhou [17] firstly used the Hankel matrix defined and the Derivative-Hilbert operator $\mathcal{DH}_{\mu_j}$ as

$$\sum_j \mathcal{DH}_{\mu_j}(f_j)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\sum_j (\mu_j)_{n,k} a_k\right)(n + 1)z^n.$$

Another generalized Hilbert-integral operator related to $\mathcal{DH}_{\mu_j}$ denoted by $\mathcal{I}_{(\mu_j)_{1+\epsilon}}((1+\epsilon) \in \mathbb{N}^+)$is defined by

$$\sum_j \mathcal{I}(\mu_j)_{1+\epsilon}(f_j)(z) = \int_{[0,1)} \sum_j \frac{f_j(t)}{(1-tz)^{1+\epsilon}} d\mu_j(t).$$

whenever the right Hanright-Handes sense and defines an analytic functions in $\mathbb{D}$. We can easily see that the case $\epsilon = 0$ is the integral representation of the generalized Hilbert operator. Ye and Zhou characterized the measure μ_j for which $\mathcal{I}(\mu_j)_2$ and $\mathcal{DH}_{\mu_j}$ are bounded on Bloch space [17] and Bergman spaces [18].

We consider the operators

$$\mathcal{DH}_{\mu_j}, \mathcal{I}_{(\mu_j)_2}: H^{1+\epsilon} \to H^{1+2\epsilon},\ 0 \leq \epsilon < \infty,\ \epsilon \geq 0.$$

$$\mathcal{DH}_{\mu_j}, \mathcal{I}_{(\mu_j)_2}: H^{1+\epsilon} \to B_{1-\epsilon},\ 0 \leq \epsilon < 1.$$

The aim is to study the boundedness of $\mathcal{I}_{(\mu_j)_2}$ and $\mathcal{DH}_{\mu_j}$.

We characterize the positive Borel measures μ_j for which the operator which $\mathcal{I}_{(\mu_j)_2}$ and $\mathcal{DH}_{\mu_j}$ is well defined in the Hardy spaces $H^{1+\epsilon}$. Then we give the necessary and sufficient conditions such that operator $\mathcal{DH}_{\mu_j}$ is bounded from the Hardy space $H^{1+\epsilon}(0 \le \epsilon < \infty)$ into the space $H^{1+2\epsilon}(\epsilon \ge 0)$, or from $H^{1+\epsilon}(0 \le \epsilon < 1)$ into $B_{1-\epsilon}(0 < \epsilon < 1)$ (see [16]).

2. Notation and Preliminaries

For $\mathbb{D}$ denote the open unit disk of the complex plane, and let $H(\mathbb{D})$ denote the set of all analytic functions in $\mathbb{D}$.

If $0 < \epsilon < 1$ and $f_j \in H(\mathbb{D})$, we set

$$\sum_j M_{1+\epsilon}(1-\epsilon, f_j) = \left(\frac{1}{2\pi}\int_0^{2\pi} \sum_j \left|f_j\left((1-\epsilon)e^{i\theta}\right)\right|^{1+\epsilon} d\theta\right)^{\frac{1}{1+\epsilon}}, \ 0 \le \epsilon < \infty.$$

$$\sum_j M_\infty(1-\epsilon, f_j) = \sup_{|z|=1-\epsilon} \sum_j |f_j(z)|.$$

For $0 \le \epsilon \le \infty$, the Hardy space $H^{1+\epsilon}$ consists of those $f_j \in H(\mathbb{D})$ such that

$$\| \sum_j f_j \|_{H^{1+\epsilon}} \stackrel{\text{def}}{=} \sup_{0<\epsilon<1} \sum_j M_{1+\epsilon}(1-\epsilon, f_j) < \infty.$$

We refer to [5] for the notation and results regarded Hardy spaces.

For $0 < \epsilon < 1$, we let $B_{1-\epsilon}$ [15] denote the space consisting of those $f_j \in H(\mathbb{D})$ for which

$$\left\|\sum_j f_j\right\|_{B_{1-\epsilon}} = \int_0^1 \sum_j \epsilon^{-\frac{1-2\epsilon}{1-\epsilon}} M_1(1-\epsilon, f_j) d(1-\epsilon) < \infty.$$

The Banach space $B_{1-\epsilon}$ is the "containing Banach space" of $H^{1-\epsilon}$, that is, $H^{1-\epsilon}$ is a dense subspace of $B_{1-\epsilon}$, and the two spaces have the same continuous linear functionals. (We mention [15] as general references for the $B_{1-\epsilon}$ spaces.)

The space $BMOA$ consists of those functions $f_j \in H^1$ whose boundary values limit the mean oscillation on $\partial\mathbb{D}$ as defined by John and Niirenberg. There are many characterizations of $BMOA$ functions. We mention the following (see [16]).

For $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ be a Möbius transformations. If f_j is an analytic function in $\mathbb{D}$, then $f_j \in BMOA$ if and only if

$$\left\|\sum_j f_j\right\|_{BMOA} \stackrel{\text{def}}{=} \sum_j |f_j(0)| + \sum_j \| f_j \|_* < \infty.$$

where

$$\left\|\sum_j f_j\right\|_* \stackrel{\text{def}}{=} \sup_{a\in\mathbb{D}} \sum_j \|f_j \circ \varphi_a - f_j(a)\|_{H^2}.$$

The seminorm $\| \ \|_*$ is conformally invariant. If

$$\lim_{|a|\to 1} \sum_j \|f_j \circ \varphi_a - f_j(a)\|_{H^2} = 0,$$

then we say that f_j belongs to the space VMOA (function analytic for vanishing mean oscillation). We refer to 8 for the theory of $BMOA$ functions.

We recall that a functions $f_j \in H(\mathbb{D})$ is said to be a Bloch function if

$$\left\|\sum_j f_j\right\|_{\mathcal{B}} \stackrel{\text{def}}{=} \sum_j |f_j(0)| + \sup_{z\in\mathbb{D}} \sum_j (1-|z|^2)\left|f_j'(z)\right| < \infty.$$

The space of all Bloch functions is denoted by $\mathcal{B}$. The classic reference for Bloch's functions theory is [10, 14]. The relationship between these spaces which we gave above is well known,

$$H^{\infty} \subsetneqq BMOA \subsetneqq \mathcal{B}, \qquad BMOA \subsetneqq \bigcap_{0\le\epsilon<\infty} H^{1+\epsilon}.$$

Let us recall the knowledge of the Carleson measure, which is a very useful tool in the study of Banach spaces of analytic functions. For $0 \le \epsilon < \infty$, a positive Borel measure μ_j on $\mathbb{D}$ will be called an $(1+\epsilon)$-Carleson measure if there exists a positive constant C such that

$$\mu_j(S(I)) \le C|I|^{1+\epsilon}.$$

The Carleson square $S(I)$ is defined as

$$S(I) = \left\{z = (1-\epsilon)e^{i\theta} : e^{i\theta} \in I; 1 - \frac{|I|}{2\pi} \le 1 - \epsilon \le 1\right\}.$$

where I is an interval of $\partial\mathbb{D}$, $|I|$ denotes the length of I. If μ_j satisfies $\lim_{|I|\to 0} \frac{\mu_j(S(I))}{II|^{1+\epsilon}} = 0$, we call μ_j is a vanishing $(1+\epsilon)$-Carleson measure.

For μ_j be a positive Borel measure on $\mathbb{D}$. For $0 \le \epsilon < \infty$ we say that μ_j is $(1+\epsilon)$-logarithmic $(1+\epsilon)$-Carleson measure, if there exists a positive constant C such that

$$\frac{\mu_j(S(I))\left(\log \frac{2\pi}{|I}\right)^{1+\epsilon}}{|I|^{1+\epsilon}} \le C \; I \subset \partial\mathbb{D}.$$

If $\mu_j(S(I))\left(\log \frac{2\pi}{II}\right)^{1+\epsilon} = o(|I|^{1+\epsilon})$, as $|I| \to 0$, we say that μ_j is vanishing $(1+\epsilon)$-logarithmic $(1+\epsilon)$-Carleson measure 13, 18].

Suppose μ_j is a $(1+\epsilon)$-Carleson measure on $\mathbb{D}$, we show that the identity mapping i is well defined from $H^{1+\epsilon}$ into $L^{1-\epsilon}(\mathbb{D}, \mu_j)$. Let $\mathcal{N}(\mu_j)$ be the norm of i. For $0 < \epsilon < 1$, let

$$d(\mu_j)_{1-\epsilon}(z) = \chi_{1-\epsilon<|z|<1}(t)d\mu_j(t).$$

Then μ_j is a vanishing $(1+\epsilon)$-Carleson measure if and only if

$$\mathcal{N}\left((\mu_j)_{1-\epsilon}\right) \to 0 \text{ as } 1-\epsilon \to 1^-. \tag{2}$$

A positive Borel measure on $[0,1)$ also can be seen as a Borel measure on $\mathbb{D}$ by identifying it with the measure μ_j defined by

$$\tilde{\mu}_j(E) = \mu_j(E \bigcap [0,1)).$$

for any Borel subset E of $\mathbb{D}$. Then a positive Borel measure μ_j on $[0,1)$ can be seen as a $(1+\epsilon)$-Carleson measure on $\mathbb{D}$, if

$$\mu_j([t,1)) \le C(1-t)^{1+\epsilon}, \qquad 0 \le t < 1.$$

Also, we have similar statements for vanishing $(1+\epsilon)$-Carleson measures, $(1+\epsilon)$-logarithmic $(1+\epsilon)$-Carleson and vanishing $(1+\epsilon)$-logarithmic $(1+\epsilon)$-Carleson measures.

As usual, during this paper, C refers to a positive constant that depends only on the displayed parameters but is not necessarily the same from case to case, for any given $\epsilon > 0, \frac{1+\epsilon}{\epsilon}$ will denote the conjugate index of $1+\epsilon$, that is, $\epsilon = 0$ (see [16]).

3. Terms such as $\mathcal{D}_{\mu_j}$ are Well Defined in Solid Spaces

We obtain the sufficient condition such this$\mathcal{DH}_{\mu_j}$ are well-defined in $H^{1+\epsilon}(0 \le \epsilon < \infty)$ and obtain that $\mathcal{DH}_{\mu_j}(f_j) = \mathcal{I}_{(\mu_j)_2}(f_j)$, for all $f_j \in H^{1+\epsilon}$, with the certain condition (see [16]).

We show recall two results about the coefficients of functions in Hardy spaces.

Lemma 3.1. [16], [5, p.98] If

$$f_j(z) = \sum_{n=0}^{\infty} a_n z^n \in H^{1-\epsilon},\; 0 \le \epsilon < 1,$$

then

$$a_n = o\left(n^{\frac{2-\epsilon}{1-\epsilon}}\right),$$

and

$$|a_n| \le Cn^{\frac{2-\epsilon}{1-\epsilon}} \|f_j\|_{H^{1-\epsilon}}$$

Lemma 3.2. [16], [5, p.95] If

$$f_j(z) = \sum_{n=0}^{\infty} a_n z^n \in H^{2-\epsilon},\ 0 \le \epsilon < 2,$$

then $\sum n^{-\epsilon}|a_n|^{2-\epsilon} < \infty$ and

$$\left\{\sum_{n=0}^{\infty} (n+1)^{-\epsilon}|a_n|^{2-\epsilon}\right\}^{\frac{1}{2-\epsilon}} \le C \sum_j \| f_j \|_{H^{2-\epsilon}}^{2-\epsilon}.$$

Theorem 3.3. Suppose $0 \le \epsilon < \infty$ and let μ_j be a positive Borel measure on $[0,1)$. Then the power series in (1) defines a well-defined analytic function in $\mathbb{D}$ for every $f_j \in H^{1+\epsilon}$ in any of the two following cases (see [16]).

(a) The measure μ_j is a $\frac{1}{1-\epsilon}$-Carleson measure, if $0 \le \epsilon < 1$.

(b) The measure μ_j is a 1-Carleson measure, if $0 < \epsilon < \infty$.

Furthermore, in such cases, we have that

$$\sum_j \mathcal{DH}_{\mu_j}(f_j)(z) = \int_{[0,1)} \sum_j \frac{f_j(t)}{(1-tz)^2} d\mu_j(t) = \sum_j \mathcal{I}(\mu_j)_2(f_j)(z). \tag{3}$$

Proof First recall a well-known result of Hastings [11]: For $0 < \epsilon < \infty, \mu_j$ is a $\frac{1+2\epsilon}{1+\epsilon}$-Carleson measure if and only if there exists a positive constant C such that

$$\left\{\int_{[0,1)} \sum_j |f_j(t)|^{1+2\epsilon} d\mu_j(t)\right\}^{\frac{1}{1+2\epsilon}} \le C \sum_j \| f_j \|_{H^{1+\epsilon}}, \text{ for all } f_j \in H^{1+\epsilon}. \tag{4}$$

(a) Suppose that $0 \le \epsilon < 1$ and μ_j is a $\frac{1}{1-\epsilon}$-Carleson measure. Then (4) gives

$$\int_{[0,1)} \sum_j |f_j(t)| d\mu_j(t) \le C \sum_j ||f_j \|_{H^{1-\epsilon}}, \text{ for all } f_j \in H^{1-\epsilon}.$$

Fix $f_j(z) = \sum_{k=0}^{\infty} a_k z^k \in H^{1-\epsilon}$ and z with $|z| < 1-\epsilon, 0 < \epsilon < 1$. It follows that

$$\begin{aligned}\int_{[0,1)} \sum_j \frac{|f_j(t)|}{|1-tz|^2} d\mu_j(t) &\le \frac{1}{\epsilon^2} \int_{[0,1)} \sum_j |f_j(t)| d\mu_j(t) \\ &\le C \frac{1}{\epsilon^2} \sum_j \| f_j \|_{H^{1-\epsilon}}.\end{aligned}$$

This implies that the integral $\int_{[0,1)} \frac{f_j(t)}{(1-tz)^2} d\mu_j(t)$ uniformly converges and that

$$\begin{aligned}\sum_j \mathcal{I}(\mu_j)_2(f_j)(z) &= \int_{[0,1)} \sum_j \frac{f_j(t)}{(1-tz)^2} d\mu_j(t) \\ &= \sum_{n=0}^{\infty} (n+1) \left(\int_{[0,1)} \sum_j t^n f_j(t) d\mu_j(t) \right) z^n.\end{aligned} \tag{5}$$

Take $f_j(z) = \sum_{k=0}^{\infty} a_k z^k \in H^{1-\epsilon}$. Since μ_j is $\frac{1}{1-\epsilon}$-Carleson measure, by [1] Proposition 1 and Lemma 3.1, we have that there exists $C > 0$ such that

$$\sum_j |(\mu_j)_{n,k}| = \sum_j |(\mu_j)_{n+k}| \le \frac{C}{(k+1)^{\frac{1}{1-\epsilon}}}$$

$$|a_k| \le C(k+1)^{\frac{\epsilon}{1-\epsilon}} \text{ for all } n, k.$$

Then it follows that, for every n,

$$(n+1)\sum_{k=0}^{\infty}\sum_{j} |(\mu_j)_{n,k}||a_k| \leq C(n+1)\sum_{k=0}^{\infty} \frac{|a_k|}{(k+1)^{\frac{1}{1-\epsilon}}} = C(n+1)\sum_{k=0}^{\infty} \frac{|a_k|^{1-\epsilon}|a_k|^{\epsilon}}{(k+1)^{\frac{1}{1-\epsilon}}}$$

$$\leq C(n+1)\sum_{k=0}^{\infty} \frac{|a_k|^{1-\epsilon}(k+1)^{\frac{\epsilon^2}{1-\epsilon}}}{(k+1)^{\frac{1}{1-\epsilon}}}$$

$$= C(n+1)\sum_{k=0}^{\infty} (k+1)^{-(1+\epsilon)}|a_k|^{1-\epsilon}$$

and then by Lemma 3.2, we deduce that

$$(n+1)\sum_{k=0}^{\infty}\sum_{j} |(\mu_j)_{n,k}a_k| \leq C(n+1)\sum_{j} \| f_j \|_{H^{1-\epsilon}}^{1-\epsilon}.$$

This implies that $\mathcal{DH}_{\mu_j}$ is a well defined for all $z \in \mathbb{D}$ and that

$$\begin{aligned}\sum_{j} \mathcal{DH}_{\mu_j}(f_j)(z) &= \sum_{n=0}^{\infty} (n+1)\left(\sum_{k=0}^{\infty}\sum_{j} (\mu_j)_{n,k}a_k\right)z^n \\ &= \sum_{n=0}^{\infty} (n+1)\int_{[0,1)}\sum_{j} t^n f_j(t)d\mu_j(t)z^n \\ &= \int_{[0,1)}\sum_{j} \frac{f_j(t)}{(1-tz)^2}d\mu_j(t).\end{aligned} \tag{6}$$

This give that $\sum_j \mathcal{DH}_{\mu_j}(f_j) = \sum_j \mathcal{I}_{(\mu_j)_2}(f_j)$.

(b) When $0 < \epsilon < \infty$, since μ_j is a 1-Carleson measure, (4) holds, then the argument used in the proof of (a) gives that, for every $f_j \in H^{1+\epsilon}, I_{(\mu_j)_2}$ is well defined analytic function in $\mathbb{D}$ and we have (5).

And since μ_j is 1-Carleson measure by Theorem 3 in [1], we know

$$(n+1)\sum_{k=0}^{\infty}\sum_{j} (\mu_j)_{n,k}a_k = (n+1)\int_{[0,1)}\sum_{j} t^n f_j(t)d\mu_j(t),$$

which implies that $\mathcal{DH}_{\mu_j}$ is a well defined for all $z \in \mathbb{D}$, and $\mathcal{DH}_{\mu_j}(f_j) = \mathcal{I}_{(\mu_j)_2}(f_j)$ (see [16]).

4. Bounededness of $\mathcal{DH}_{\mu_j}$ Acting on Hardy Spaces

We mainly characterize those measures μ_j for which $\mathcal{DH}_{\mu_j}$ is a bounded (resp., compact) operator from $H^{1+\epsilon}$ into $H^{1+2\epsilon}$ for some $1+\epsilon$ and $1+2\epsilon$.

Theorem 4.1. [16] Suppose $0 \leq \epsilon < 1$ and let μ_j be a positive Borel measure on $[0,1)$ which satisfies the condition in Theorem 3.3.

(a) If $\epsilon > 0$, then $\mathcal{DH}_{\mu_j}$ is a bounded operator from $H^{1-\epsilon}$ into $H^{1+\epsilon}$ if and only if μ_j is a $\left(\frac{\epsilon+(1-\epsilon)(1+2\epsilon)}{\epsilon(1-\epsilon)}\right)$ Carleson measure.

(b) If $\epsilon = 0$, then $\mathcal{DH}_{\mu_j}$ is a bounded operator from $H^{1-\epsilon}$ into H^1 if and only if μ_j is a $\left(\frac{2-\epsilon}{1-\epsilon}\right)$ Carleson measure.

(c) If $0 < \epsilon < 1$, then $\mathcal{DH}_{\mu_j}$ is a bounded operator from $H^{1-\epsilon}$ into $B_{1-\epsilon}$ if and only if μ_j is a $\left(\frac{2-\epsilon}{1-\epsilon}\right)$-Carleson measure.

Proof. Suppose $0 \leq \epsilon < 1$. Since μ_j satisfies the condition in Theorem 3.3, as in the proof of Theorem 3.3, we obtain that

$$\int_{[0,1)}\sum_{j} |f_j(z)|d\mu_j(t) < \infty, \quad \text{for all } f_j \in H^{1-\epsilon}$$

Hence, it follows that

$$\int_0^{2\pi}\int_{[0,1)}\sum_j \left|\frac{f_j(t)g_j(e^{i\theta})}{(1-(1-\epsilon)e^{i\theta}t)^2}\right| d\mu_j(t)d\theta$$
$$\leq \frac{1}{\epsilon^2}\int_{[0,1)}\sum_j |f_j(t)|d\mu_j(t)\int_0^{2\pi}|g_j(e^{i\theta})|d\theta \tag{7}$$
$$\leq C\sum_j \frac{\| g_j \|_{H^1}}{\epsilon^2} < \infty, \quad 0 < \epsilon \leq 1, \quad f_j \in H^{1-\epsilon}, \quad g_j \in H^1.$$

Using Theorem 3.3, (7) and Fubini's theorem, and Cauchy's integral representation of H^1, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\sum_j \overline{\mathcal{DH}_{\mu_j}(f_j)((1-\epsilon)e^{\iota\theta})}g_j(e^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left(\int_{[0,1)}\sum_j \frac{\overline{f_j(t)}d\mu_j(t)}{(1-(1-\epsilon)e^{-i\theta}t)^2}\right)g_j(e^{i\theta})d\theta$$
$$= \frac{1}{2\pi}\int_{[0,1)}\sum_j \overline{f_j(t)}\int_{|e^{i\theta}|=1}\frac{g_j(e^{i\theta})e^{i\theta}}{(e^{i\theta}-(1-\epsilon)t)^2}de^{i\theta}d\mu_j(t)$$
$$= \frac{1}{2\pi}\int_{[0,1)}\sum_j \overline{f_j(t)}(g_j((1-\epsilon)t)(1-\epsilon)t)'d\mu_j(t)$$
$$= \frac{1}{2\pi}\int_{[0,1)}\sum_j \overline{f_j(t)}\left(g_j((1-\epsilon)t)+(1-\epsilon)tg_j'((1-\epsilon)t)\right)d\mu_j(t), \tag{8}$$
$$0 < \epsilon \leq 1, \quad f_j \in H^{1-\epsilon}, \quad g_j \in H^1.$$

(a) First consider $\epsilon > 0$. Using (8) and the duality theorem [5] for $H^{1+\epsilon}$ which says that $(H^{1+\epsilon})^* \cong H^{\frac{1+\epsilon}{\epsilon}}$ and $\left(H^{\frac{1+\epsilon}{\epsilon}}\right)^* \cong H^{1+\epsilon}(\epsilon > 0)$, under the Cauchy pairing

$$\sum_j < f_j, g_j > = \frac{1}{2\pi}\int_0^{2\pi}\sum_j \overline{f_j(e^{\iota\theta})}g_j(e^{i\theta})d\theta, \quad f_j \in H^{1+\epsilon}, g_j \in H^{\frac{1+\epsilon}{\epsilon}}. \tag{9}$$

We obtain that $\mathcal{DH}_{\mu_j}$ is a bounded operator from $H^{1+\epsilon}$ into $H^{1+\epsilon}$ if and only if there exists a positive constant C such that

$$\left|\int_{[0,1)}\sum_j \overline{f_j(t)}(g_j(t)+tg_j'(t))d\mu_j(t)\right| \leq C\sum_j \| f_j \|_{H^{1+\epsilon}}\| g_j \|_{H^{\frac{1+\epsilon}{\epsilon}}}, \quad f_j \in H^{1+\epsilon}, g_j \in H^{\frac{1+\epsilon}{\epsilon}}. \tag{10}$$

Assume that $\mathcal{DH}_{\mu_j}$ is a bounded operator from $H^{1+\epsilon}$ into $H^{1+\epsilon}$. Take the families of text functions

$$(f_j)_a(z) = \left(\frac{1-a^2}{(1-az)^2}\right)^{\frac{1}{1+\epsilon}}, \ (g_j)_a(z) = \left(\frac{1-a^2}{(1-az)^2}\right)^{\frac{\epsilon}{1+\epsilon}}, \ 0 < a < 1.$$

A calculation shows that $\{(f_j)_a\} \subset H^{1+\epsilon}, \{(g_j)_a\} \subset H^{\frac{1+\epsilon}{\epsilon}}$ and

$$\sup_{a\in[0,1)} \| f_j \|_{H^{1+\epsilon}} < \infty \text{ and } \sup_{a\in[0,1)} \| g_j \|_{H^{\frac{1+\epsilon}{\epsilon}}} < \infty,$$

It follows that

$$
\begin{aligned}
\infty \quad & > C \sup_{a\in[0,1)} \sum_j \| f_j \|_{H^{1+\epsilon}} \sup_{a\in[0,1)} \sum_j \| g_j \|_{H^{\frac{1+\epsilon}{\epsilon}}} \\
& \geq \left| \int_{[0,1)} \sum_j \overline{(f_j)_a(t)} \left((g_j)_a(t) + t (g_j)'_a(t) \right) d\mu_j(t) \right| \\
& \geq \int_{[a,1)} \sum_j \left(\frac{1-a^2}{(1-at)^2} \right)^{\frac{1}{1+\epsilon}} \left(\left(\frac{1-a^2}{(1-at)^2} \right)^{\frac{\epsilon}{1+\epsilon}} + \frac{2\epsilon t^2}{1+\epsilon} \left(\frac{1-a^2}{(1-at)^{\frac{1+3\epsilon}{\epsilon}}} \right)^{\frac{\epsilon}{1+\epsilon}} \right) d\mu_j(t) \qquad (11) \\
& \geq \frac{1}{(1-a^2)^{1+\epsilon}} \sum_j \mu_j([a,1))
\end{aligned}
$$

This is equivalent to saying that μ_j is a $(1+\epsilon)$-Carleson measure. On the other hand, suppose μ_j is a $(1+\epsilon)$-Carleson measure, it is well known that any functions $g_j \in H^{\frac{1+\epsilon}{\epsilon}}$ [5] has the property

$$
\left| \sum_j g_j(z) \right| \leq C \frac{\sum_j \| g_j \|_{H^{\frac{1+\epsilon}{\epsilon}}}}{(1-|z|)^{\frac{\epsilon}{1+\epsilon}}}. \qquad (12)
$$

By the Cauchy formula, we can obtain that

$$
\left| \sum_j g_j'(z) \right| \leq C \sum_j \frac{\| g_j \|_{H^{\frac{1+\epsilon}{\epsilon}}}}{(1-|z|)^{\frac{1+2\epsilon}{1+\epsilon}}}. \qquad (13)
$$

Let $dv(t) = \frac{1}{(1-t)^{\frac{1+2\epsilon}{1+\epsilon}}} d\mu_j(t)$. Using Lemma 3.2 of [12], we obtain that ν is an $\frac{1}{1+\epsilon}$-Carleson.

This together with (12) and (13) we obtain that

$$
\begin{aligned}
\left| \int_{[0,1)} \sum_j \overline{f_j(t)} \left(g_j(t) + t g_j'(t) \right) d\mu_j(t) \right| & \leq C \sum_j \| g_j \|_{H^{\frac{1+\epsilon}{\epsilon}}} \int_{[0,1)} |f_j(t)| \left(\frac{1}{(1-t)^{\frac{\epsilon}{1+\epsilon}}} + \frac{t}{(1-t)^{\frac{1+2\epsilon}{1+\epsilon}}} \right) d\mu_j(t) \\
& \leq C \sum_j \| g_j \|_{H^{\frac{1+\epsilon}{\epsilon}}} \int_{[0,1)} |f_j(t)| dv(t) \\
& \leq C \sum_j \| g_j \|_{H^{\frac{1+\epsilon}{\epsilon}}} \| f_j \|_{H^{1+\epsilon}}, \; g_j \in H^{\frac{1+\epsilon}{\epsilon}} f_j \in H^{1+\epsilon}. \qquad (14)
\end{aligned}
$$

Hence (10) holds and then $\mathcal{DH}_{\mu_j}$ is a bounded operator from $H^{1+\epsilon}$ into $H^{1+\epsilon}$.

(b) We shall use Fefferman's duality theorem, which says that $(H^1)^* \cong BMOA$ and $(VMOA)^* \cong H^1$, under the Cauchy pairing

$$
\sum_j < f_j, g_j > = \lim_{1-\epsilon\to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \sum_j \overline{f_j((1-\epsilon)e^{\iota\theta})} g_j(e^{i\theta}) d\theta, \qquad f_j \in H^1,
$$
$$
g_j \in BMOA (\text{resp. } VMOA). \qquad (15)
$$

Using the duality theorem and (8) it follows that $\mathcal{DH}_{\mu_j}$ is a bounded operator from $H^{1+\epsilon}$ into H^1 if and only if there exists a positive constant C such that

$$
\left| \int_{[0,1)} \sum_j \overline{f_j(t)} \left(g_j((1-\epsilon)t) + (1-\epsilon) t g_j'((1-\epsilon)t) \right) d\mu_j(t) \right| \leq C \sum_j \| f_j \|_{H^{1+\epsilon}} \| g_j \|_{BMOA}, \qquad (16)
$$
$$
0 < \epsilon \leq 1, \qquad f_j \in H^{1+\epsilon}, \qquad g_j \in VMOA.
$$

Suppose that $\mathcal{DH}_{\mu_j}$ is a bounded operator from $H^{1+\epsilon}$ into H^1. Take the families of text functions

$$
(f_j)_a(z) = \left(\frac{1-a^2}{(1-az)^2} \right)^{\frac{1}{1+\epsilon}}, \; (g_j)_a(z) = \log \frac{e}{1-az}, \; 0 < a < 1.
$$

A calculation shows that $\{(f_j)_a\} \subset H^{1+\epsilon}$, $\{(g_j)_a\} \subset VMOA$ and

$$\sup_{a\in[0,1)}\sum_j \| f_j \|_{H^{1+\epsilon}} < \infty \text{ and } \sup_{a\in[0,1)}\sum_j \| g_j \|_{BMOA} < \infty.$$

We let $1-\epsilon \in [a,1)$, and obtain

$$\begin{aligned}\infty &> C\sup_{a\in[0,1)}\sum_j \| f_j \|_{H^{1+\epsilon}} \sup_{a\in[0,1)} \| g_j \|_{BMOA} \\ &\geq \left|\int_{[0,1)}\sum_j \overline{(f_j)_a(t)}\left((g_j)_a((1-\epsilon)t) + (1-\epsilon)t(g_j)_a'((1-\epsilon)t)\right)d\mu_j(t)\right| \\ &\geq \int_{[a,1)}\sum_j \left(\frac{1-a^2}{(1-at)^2}\right)^{\frac{1}{1+\epsilon}}\left(\log\frac{e}{1-a(1-\epsilon)t} + \frac{a(1-\epsilon)t}{1-a(1-\epsilon)t}\right)d\mu_j(t) \\ &\geq \frac{1}{(1-a^2)^{\frac{2+\epsilon}{1+\epsilon}}}\sum_j \mu_j([a,1)).\end{aligned} \tag{17}$$

This is equivalent to saying that μ_j is a $\left(\frac{2+\epsilon}{1+\epsilon}\right)$-Carleson measure.

On the other hand, suppose μ_j is a $\left(\frac{2+\epsilon}{1+\epsilon}\right)$-Carleson measure. It is well known that any functions $g_j \in \mathcal{B}$ [14] has the property

$$\left|\sum_j g_j(z)\right| \leq C\sum_j \| g_j \|_{\mathcal{B}} \log\frac{e}{1-|z|}, \text{ and } \left|\sum_j g_j'(z)\right| \leq C\sum_j \frac{\| g_j \|_{\mathcal{B}}}{1-|z|} \text{ for all } z\in\mathbb{D}. \tag{18}$$

Let $d\nu(t) = \frac{1}{1-t}d\mu_j(t)$, then ν is an $\frac{1}{1+\epsilon}$-Carleson. Using (16), (18) and $BMOA \subset \mathcal{B}$, Theorem 5.2 in [8], we obtain that

$$\begin{aligned}&\left|\int_{[0,1)}\sum_j \overline{f_j(t)}\left(g_j((1-\epsilon)t) + (1-\epsilon)tg_j'((1-\epsilon)t)\right)d\mu_j(t)\right| \\ &\qquad\leq C\sum_j \| g_j \|_{\mathcal{B}} \int_{[0,1)} |f_j(t)|\left(\log\frac{1}{1-t} + \frac{t}{1-t}\right)d\mu_j(t) \\ &\qquad\leq C\sum_j \| g_j \|_{BMOA} \int_{[0,1)} |f_j(t)|d\nu(t) \\ &\qquad\leq C\sum_j \| g_j \|_{BMOA} \| f_j \|_{H^{1+\epsilon}},\ f_j\in H^{1+\epsilon}, g_j \in VMOA.\end{aligned} \tag{19}$$

Hence (16) holds and then $\mathcal{DH}_{\mu_j}$ is a bounded operator from $H^{1+\epsilon}$ into H^1.

(c) If $0<\epsilon<1$. Now we recall Theorem 10 in [15] that $B_{1-\epsilon}$ they can be identified with the duplication of a particular subspace X of H^∞ under the pairing

$$\sum_j < f_j, g_j > = \lim_{1-\epsilon\to 1^-}\frac{1}{2\pi}\int_0^{2\pi}\sum_j \overline{f_j((1-\epsilon)e^{i\theta})}g_j(e^{i\theta})d\theta,\ f_j\in B_{1-\epsilon}, g_j\in X. \tag{20}$$

This together with (8) and (20), we obtain that $\mathcal{DH}_{\mu_j}$ is a bounded operator from $H^{1-\epsilon}$ into $B_{1-\epsilon}$ if and only if there exists a positive constant C such that

$$\begin{gathered}\left|\int_{[0,1)}\sum_j \overline{f_j(t)}\left(g_j((1-\epsilon)t) + (1-\epsilon)tg_j'((1-\epsilon)t)\right)d\mu_j(t)\right| \leq C\sum_j \| f_j \|_{H^{1-\epsilon}} \| g_j \|_{H^\infty}, \\ 0<\epsilon\leq 1, \quad f_j\in H^{1-\epsilon}, \quad g_j\in H^\infty.\end{gathered} \tag{21}$$

Suppose that $\mathcal{DH}_{\mu_j}$ is a bounded operator from $H^{1-\epsilon}$ into $B_{1-\epsilon}$, take the text function families

$$(f_j)_a(z) = \left(\frac{1-a^2}{(1-az)^2}\right)^{\frac{1}{1-\epsilon}},\ (g_j)_a(z) = \frac{1-a^2}{1-az},\ 0<a<1.$$

A calculation shows that $\{(f_j)_a\} \subset H^{1-\epsilon}, \{(g_j)_a\} \subset H^\infty$ and

$$\sup_{a\in[0,1)}\sum_j \| f_j \|_{H^{1-\epsilon}} < \infty \text{ and } \sup_{a\in[0,1)}\sum_j \| g_j \|_{H^\infty} < \infty.$$

We let $1-\epsilon \in [a,1)$, obtain

$$\begin{aligned}
\infty \quad &> C \sup_{a\in[0,1)}\sum_j \| f_j \|_{H^{1-\epsilon}} \sup_{a\in[0,1)} \| g_j \|_{H^\infty} \\
&\geq \left| \int_{[0,1)} \sum_j \overline{(f_j)_a(t)} \left((g_j)_a((1-\epsilon)t) + (1-\epsilon)t (g_j)'_a((1-\epsilon)t) \right) d\mu_j(t) \right| \\
&\geq \int_{[a,1)} \sum_j \left(\frac{1-a^2}{(1-at)^2} \right)^{\frac{1}{1-\epsilon}} \left(\left(\frac{1-a^2}{(1-a(1-\epsilon)t)} \right) + \left(\frac{\text{art}\,(1-a^2)}{(1-a(1-\epsilon)t)^2} \right) \right) d\mu_j(t) \\
&\geq \frac{1}{(1-a^2)^{\frac{2-\epsilon}{1-\epsilon}}} \sum_j \mu_j([a,1)).
\end{aligned} \tag{22}$$

This is equivalent to saying that μ_j is a $\left(\frac{2-\epsilon}{1-\epsilon}\right)$-Carleson measure.

On the other hand, assume μ_j is a $\left(\frac{2-\epsilon}{1-\epsilon}\right)$-Carleson measure, then $d\nu(t) = \frac{1}{1-t} d\mu_j(t)$ as an $\frac{1}{1-\epsilon}$-Carleson. Then we have that

$$\begin{aligned}
&\left| \int_{[0,1)} \sum_j \overline{f_j(t)} \left(g_j((1-\epsilon)t) + (1-\epsilon)t g_j'((1-\epsilon)t) \right) d\mu_j(t) \right| \\
&\leq C \sum_j \| g_j \|_{H^\infty} \int_{[0,1)} |f_j(t)| \left(1 + \frac{t}{1-t} \right) d\mu_j(t) \\
&\leq C \sum_j \| g_j \|_{H^\infty} \int_{[0,1)} |f_j(t)| d\nu(t) \\
&\leq C \sum_j \| g_j \|_{H^\infty} \| f_j \|_{H^{1-\epsilon}},\ f_j \in H^{1-\epsilon}, g_j \in H^\infty
\end{aligned} \tag{23}$$

Hence (21) holds and then $\mathcal{DH}_{\mu_j}$ is a bounded operator from $H^{1-\epsilon}$ into $B_{1-\epsilon}$.

Next, we will consider $0 < \epsilon < \infty$, giving the sufficient condition and the other necessary condition for the restriction $\mathcal{DH}_{\mu_j}$ from $H^{1+\epsilon}$ into $H^{1+2\epsilon}$ respectively.

We give Lemma 4.2 which is useful when we proof of the Theorem 4.3.

Lemma 4.2. Let μ_j be a positive measure on $[0,1)$ and $\epsilon \geq 0$. If μ_j is a $(2+2\epsilon)$-Carleson measure, then

$$\int_{[0,1)} \sum_j \frac{1}{(1-t)^{1+\epsilon}} d\mu_j(t) < \infty.$$

The result is obvious, we omit the details (see [16]).

Theorem 4.3. Let $0 < \epsilon < \infty$ and μ_j be a positive Borel measure on $[0,1)$ which satisfies the condition in Theorem 3.3,

(a)If μ_j is a $\left(\frac{(1+2\epsilon)+(1+\epsilon)(2+2\epsilon)}{(1+\epsilon)(1+2\epsilon)} + 1 + \epsilon\right)$-Carleson measure for any $\epsilon \geq 0$, then $\mathcal{DH}_{\mu_j}$ is a bounded operator from $H^{1+\epsilon}$ into $H^{1+2\epsilon}$.

(b) If $\mathcal{DH}_{\mu_j}$ is a bounded operator from $H^{1+\epsilon}$ into $H^{1+2\epsilon}$, then μ_j is a $\left(\frac{(1+2\epsilon)+(1+\epsilon)(2+2\epsilon)}{(1+\epsilon)(1+2\epsilon)}\right)$-Carleson measure.

Proof. Suppose μ_j is a $\left(\frac{(1+2\epsilon)+(1+\epsilon)(2+2\epsilon)}{(1+\epsilon)(1+2\epsilon)} + 1 + \epsilon\right)$-Carleson measure. Let $d\nu(t) = \frac{1}{1-t} d\mu_j(t)$, then ν is a $\left(\frac{1+2\epsilon(2+\epsilon)}{(1+\epsilon)(1+2\epsilon)} + 1 + \epsilon\right)$-Carleson. And set $\epsilon = 0$, the conjugate exponent of $1+\epsilon$ is $s' = 1 + \frac{2\epsilon(1+\epsilon)}{1+2\epsilon}$ and $\frac{1}{1+\epsilon} + \frac{2\epsilon}{1+2\epsilon} = 1 = \frac{2\epsilon s'}{1+2\epsilon}$. Then by Theorem 9.4 in [5], $H^{1+\epsilon}$ is continuously embedded in $L^{1+\epsilon}(d\nu)$, that is,

$$\left(\int_{[0,1)} \sum_j |f_j(t)|^{1+\epsilon} d\nu(t)\right)^{\frac{1}{1+\epsilon}} \leq C \sum_j \| f_j \|_{H^{1+\epsilon}},\ f_j \in H^{1+\epsilon}, \tag{24}$$

and, by Lemma 4.2,

$$\left(\int_{[0,1)} \frac{1}{(1-t)^{\frac{2\epsilon s'}{1+2\epsilon}}} d\nu(t)\right)^{\frac{1}{s'}} < \infty, \qquad g_j \in H^{\frac{1+2\epsilon}{2\epsilon}}. \tag{25}$$

Using Hölder's inequality with exponents $1+\epsilon$ and s', (24) and (25), we obtain that

$$\left|\int_{[0,1)} \sum_j \overline{f_j(t)}\left(g_j(t) + t g_j(t)'\right) d\mu_j(t)\right| \leq C \sum_j \| g_j \|_{H^{\frac{1+2\epsilon}{2\epsilon}}} \int_{[0,1)} |f_j(t)| \left(\frac{1}{(1-t)^{\frac{1+4\epsilon}{1+2\epsilon}}}\right) d\mu_j(t)$$

$$= C \sum_j \| g_j \|_{H^{\frac{1+2\epsilon}{2\epsilon}}} \int_{[0,1)} |f_j(t)| \left(\frac{1}{(1-t)^{\frac{1+4\epsilon}{1+2\epsilon}}}\right) d\nu(t)$$

$$\leq C \sum_j \| g_j \|_{H^{\frac{1+2\epsilon}{2\epsilon}}} \left(\int_{[0,1)} |f_j(t)|^{1+\epsilon} d\nu(t)\right)^{\frac{1}{1+\epsilon}} \left(\int_{[0,1)} \frac{1}{(1-t)^{\frac{2\epsilon s'}{1+2\epsilon}}} d\nu(t)\right)^{\frac{1}{s'}}$$

$$\leq C \sum_j \| g_j \|_{H^{\frac{1+2\epsilon}{2\epsilon}}} \| f_j \|_{H^{1+\epsilon}},\ f_j \in H^{1+\epsilon}, g_j \in H^{\frac{1+2\epsilon}{2\epsilon}}. \tag{26}$$

Hence, (10) holds and then it follows that $\mathcal{DH}_{\mu_j}$ is a bounded operator from $H^{1+\epsilon}$ into $H^{1+2\epsilon}$.

Conversely, if $\mathcal{DH}_{\mu_j}$ is a bounded operator from $H^{1+\epsilon}$ into $H^{1+2\epsilon}$, then μ_j is a $\left(\frac{(1+2\epsilon)+(1+\epsilon)(2+2\epsilon)}{(1+\epsilon)(1+2\epsilon)}\right)$ Carleson measure. The evidence is the same as that of Theorem 4.1(a). We omit the details here (see [16]).

We also find $\mathcal{DH}_{\mu_j}$ in $H^{1+\epsilon}(0 \leq \epsilon \leq 1)$ have a better conclusion.

Theorem 4.4. Let $0 \leq \epsilon \leq 1$ and μ_j be a positive Borel measure on $[0,1)$ which satisfies the condition in Theorem 3.3. Then $\mathcal{DH}_{\mu_j}$ is a bounded operator in $H^{1+\epsilon}$ if and only if μ_j is a 2-Carleson measure.

Proof Firstly, if $\epsilon = 0$, by Theorem 4.1 we obtain that $\mathcal{DH}_{\mu_j}$ is a bounded operator in H^1 if and only if μ_j is a 2-Carleson measure.

If $\epsilon = 1$, by Theorem 4.3 we only need to show that if μ_j is a 2 -Carleson measure then $\mathcal{DH}_{\mu_j}$ is a bounded operator in H^2.

Since $f_j(z) = \sum_{k=0}^{\infty} a_k z^k \in H^2$, we have $\| f_j \|_{H^2}^2 = \sum_{k=0}^{\infty} |a_k|^2$, and when μ_j is a 2-Carleson measure, we have

$$\left|(\mu_j)_{n,k}\right| = \left|(\mu_j)_{n+k}\right| \leq \frac{C}{(n+k+1)^2}. \tag{27}$$

By using classical Hilbert inequality, (1), and (27), we obtain that

$$\sum_j \left\|\mathcal{DH}_{\mu_j}(f_j)\right\|_{H^2}^2 = \sum_{n=0}^{\infty} (n+1)^2 \left|\sum_{k=0}^{\infty} \sum_j (\mu_j)_{n,k} a_k\right|^2 \le \sum_{n=0}^{\infty} (n+1)^2 \left(\sum_{k=0}^{\infty} \sum_j \left|(\mu_j)_{n,k} \| a_k\right|\right)^2$$
$$\le C \sum_{n=0}^{\infty} (n+1)^2 \left(\sum_{k=0}^{\infty} \frac{|a_k|}{(n+k+1)^2}\right)^2$$
$$\le C \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{|a_k|}{(n+k+1)}\right)^2$$
$$\le C \sum_{k=0}^{\infty} |a_k|^2 = C \sum_j \| f_j \|_{H^2}^2. \quad (28)$$

Thus $\mathcal{DH}_{\mu_j}$ is a bounded operator in H^2. Finally, we shall use complex interpolation to present our results. We know that

$$H^{1+\epsilon} = (H^2, H^1)_\theta, \text{ if } 0 < \epsilon < 1 \text{ and } \theta = 1. \quad (29)$$

Using (29) and [21, Theorem 2.4], it follows that $\mathcal{DH}_{\mu_j}$ is a bounded operator in $H^{1+\epsilon}(0 \le \epsilon \le 1)$.

Conjecture 4.6. We conjecture that if μ_j is a 2-Carleson measure then $\mathcal{DH}_{\mu_j}$ is a bounded operator in $H^{2+\epsilon}$ for all $0 \le \epsilon < \infty$ (see [16]).

5. Compactness of $\mathcal{DH}_{\mu_j}$ Acting on Hardy Spaces

We characterize the compactness of the Derivative-Hilbert $\mathcal{DH}_{\mu_j}$. We begin with the following Lemma 5.1 which is useful to deal with compactness.

Lemma 5.1. [16] For $0 \le \epsilon < \infty$. Suppose that $\mathcal{DH}_{\mu_j}$ is a bounded operator from $H^{1+\epsilon}$ into $H^{1+\epsilon}($ resp., $B_{1+\epsilon})$. Then $\mathcal{DH}_{\mu_j}$ is a compact operator if and only if for any bounded sequence $\{(f_j)_n\}$ in $H^{1+\epsilon}$ which converges uniformly to 0 on every compact subset of $\mathbb{D}$, we have $\mathcal{DH}_{\mu_j}((f_j)_n) \to 0$ in $H^{1+\epsilon}($ resp., $B_{1+\epsilon})$

The proof is similar to that of Proposition 3.1 in [2].

Theorem 5.2. Suppose $0 \le \epsilon < 1$ and let μ_j be a positive Borel measure on $[0,1)$ which satisfies the condition in Theorem 3.3,

(a) If $\epsilon > 0$, then $\mathcal{DH}_{\mu_j}$ is a compact operator from $H^{1-\epsilon}$ into $H^{1+\epsilon}$ if and only if μ_j is a vanishing $\left(\frac{\epsilon+(1-\epsilon)(1+2\epsilon)}{\epsilon(1-\epsilon)}\right)$-Carleson measure.

(b) If $\epsilon = 0$, then $\mathcal{DH}_{\mu_j}$ is a compact operator from $H^{1-\epsilon}$ into H^1 if and only if μ_j is a vanishing $\left(\frac{2-\epsilon}{1-\epsilon}\right)$-Carleson measure.

(c) If $0 < \epsilon < 1$, then $\mathcal{DH}_{\mu_j}$ is a compact operator from $H^{1-\epsilon}$ into $B_{1-\epsilon}$ if and only if μ_j is a vanishing $\left(\frac{2-\epsilon}{1-\epsilon}\right)$-Carleson measure.

Proof (a) First consider $\epsilon > 0$. Suppose that $\mathcal{DH}_{\mu_j}$ is a compact operator from $H^{1-\epsilon}$ into $H^{1+\epsilon}$.

Let $\{a_n\} \subset (0,1)$ be any sequence with $a_n \to 1$. We set

$$(f_j)_{a_n}(z) = \left(\frac{1-a_n^2}{(1-a_n z)^2}\right)^{\frac{1}{1-\epsilon}}, \; z \in \mathbb{D}.$$

Then $(f_j)_{a_n}(z) \in H^{1-\epsilon}, \sup_{n\ge1}\|(f_j)_{a_n}\|_{H^{1-\epsilon}} < \infty$ and $(f_j)_{a_n} \to 0$, uniformly on any compact subset of $\mathbb{D}$. Using Lemma 5.1 and bearing in mind that $\mathcal{DH}_{\mu_j}$ is a compact operator from $H^{1-\epsilon}$ into $H^{1+\epsilon}$, we obtain that $\left\{\mathcal{DH}_{\mu_j}\left((f_j)_{a_n}\right)\right\}$ converges to 0 in $H^{1+\epsilon}$. This and (8) imply that

$$\begin{aligned}&\lim_{n\to\infty}\int_{[0,1)}\sum_j \overline{(f_j)_{a_n}(t)}\left(g_j(t)+tg_j'(t)\right)d\mu_j(t)\\ =\ &\lim_{n\to\infty}\int_0^{2\pi}\sum_j \overline{\mathcal{DH}_{\mu_j}\left((f_j)_{a_n}\right)(e^{\iota\theta})}g_j(e^{i\theta})d\theta=0,\ g_j\in H^{\frac{1+\epsilon}{\epsilon}}.\end{aligned} \tag{30}$$

Now we wet

$$(g_j)_{a_n}(z)=\left(\frac{1-a_n^2}{(1-a_nz)^2}\right)^{\frac{\epsilon}{1+\epsilon}},\ z\in\mathbb{D}.$$

It is obvious find that $g_j\in H^{\frac{1+\epsilon}{\epsilon}}$. For every n, fix $1-\epsilon\in(a_n,1)$. Thus,

$$\begin{aligned}&\int_{[0,1)}\sum_j \overline{(f_j)_{a_n}(t)}\left((g_j)_{a_n}(t)+t(g_j)'_{a_n}(t)\right)d\mu_j(t)\\ &\geq C\int_{[a_n,1)}\sum_j\left(\frac{1-a_n^2}{(1-a_nt)^2}\right)^{\frac{1}{1-\epsilon}}\left(\left(\frac{1-a_n^2}{(1-a_nt)^2}\right)^{\frac{\epsilon}{1+\epsilon}}\right.\\ &\left.+\frac{2\epsilon t^2}{1+\epsilon}\left(\frac{1-a_n^2}{(1-a_nt)^{\frac{1+3\epsilon}{\epsilon}}}\right)^{\frac{\epsilon}{1+\epsilon}}\right)d\mu_j(t)\geq\frac{C}{(1-a_n^2)^{\frac{\epsilon+(1-\epsilon)(1+2\epsilon)}{\epsilon(1-\epsilon)}}}\sum_j \mu_j([a_n,1)).\end{aligned}$$

By (30) and the fact $\{a_n\}\subset(0,1)$ is a sequence with $a_n\to1$, as $n\to\infty$, we obtain that

$$\lim_{a\to1^-}\sum_j\frac{1}{(1-a_n^2)^{\frac{\epsilon+(1-\epsilon)(1+2\epsilon)}{\epsilon(1-\epsilon)}}}\mu_j([a_n,1))=0.$$

Thus μ_j is a vanishing $\left(\frac{\epsilon+(1-\epsilon)(1+2\epsilon)}{\epsilon(1-\epsilon)}\right)$-Carleson measure.

On the other hand, suppose that μ_j is a vanishing $\left(\frac{\epsilon+(1-\epsilon)(1+2\epsilon)}{\epsilon(1-\epsilon)}\right)$-Carleson measure. Let $\{(f_j)_n\}_{j,n=1}^{\infty}$ be a sequence of $H^{1-\epsilon}$ functions with $\sup_{n\geq1}\|(f_j)_n\|_{H^{1-\epsilon}}<\infty$ and such that $\{(f_j)_n\}\to0$, uniformly on any compact subset of $\mathbb{D}$. Then by Lemma 5.1, it is enough to show that $\left\{\mathcal{DH}_{\mu_j}((f_j)_n)\right\}\to 0$ in $H^{1+\epsilon}$

Taking $g_j\in H^{\frac{1+\epsilon}{\epsilon}}$ and $1-\epsilon\in[0,1)$, we have

$$\begin{aligned}&\int_{[0,1)}\sum_j |(f_j)_n(t)||(g_j(t)+tg_j'(t)|d\mu_j(t)\\ &=\int_{[0,1-\epsilon)}\sum_j\left|(f_j)_n(t)\right|\left|g_j(t)+tg_j'(t)\right|d\mu_j(t)+\int_{[1-\epsilon,1)}\sum_j\left|(f_j)_n(t)\right|\left|g_j(t)+tg_j'(t)\right|d\mu_j(t).\end{aligned}$$

Then $\int_{[0,1-\epsilon)}\sum_j \left|(f_j)_n(t)\right|\left|g_j(t)+tg_j'(t)\right|d\mu_j(t)$ tends to 0 as $\{(f_j)_n\}\to0$ uniformly on any compact subset of $\mathbb{D}$.

And by the conclusion in the proof of the boundedness in Theorem 4.1 (a), let $d\nu(t)=\frac{1}{(1-t)^{\frac{1+2\epsilon}{1+\epsilon}}}d\mu_j(t)$. We know that ν is a vanishing $\frac{1}{1-\epsilon}$-Carleson. Then it implies that

$$\begin{aligned}\int_{[1-\epsilon,1)}\sum_j |(f_j)_n(t)||g_j(t)+tg_j'(t)|d\mu_j(t)&\leq C\sum_j\|g_j\|_{H^{\frac{1+\epsilon}{\epsilon}}}\int_{[0,1)}\left|(f_j)_n(t)\right|d\nu_{1-\epsilon}(t)\\ &\leq C\mathcal{N}(\nu_{1-\epsilon})\sum_j\|g_j\|_{H^{\frac{1+\epsilon}{\epsilon}}}\left\|(f_j)_n\right\|_{H^{1-\epsilon}}\leq C\mathcal{N}(\nu_{1-\epsilon})\sum_j\|g_j\|_{H^{\frac{1+\epsilon}{\epsilon}}}\end{aligned}\tag{31}$$

It also tends to 0 by (2). Thus

$$\lim_{n\to\infty}\left|\int_0^{2\pi}\sum_j \overline{\mathcal{DH}_{\mu_j}((f_j)_n)(e^{\iota\theta})}\,g_j(e^{i\theta})d\theta\right| = \lim_{n\to\infty}\int_{[0,1)}\sum_j \left|(f_j)_n(t)\right|\left|g_j(t)+tg_j'(t)\right|d\mu_j(t)$$

$$= 0, \text{ for all } g_j \in H^{\frac{1+\epsilon}{\epsilon}}.$$

It means $\mathcal{DH}_{\mu_j}((f_j)_n) \to 0$ in $H^{1+\epsilon}$, by Lemma 5.1 we obtain $\mathcal{DH}_{\mu_j}$ is a compact operator from $H^{1-\epsilon}$ into $H^{1+\epsilon}$. (b) Let $\epsilon = 0$. Suppose that $\mathcal{DH}_{\mu_j}$ is a compact operator from $H^{1-\epsilon}$ into H^1. Let $\{a_n\} \subset (0,1)$ be any sequence with $a_n \to 1$ and $(f_j)_{a_n}$ defines like in (a). Lemma 5.1 implies that $\left\{\mathcal{DH}_{\mu_j}((f_j)_{a_n})\right\}$ converges to 0 in H^1. Then we have

$$\begin{aligned}&\lim_{n\to\infty}\int_{[0,1)}\sum_j \overline{(f_j)_{a_n}(t)}\left(g_j((1-\epsilon)t)+(1-\epsilon)tg_j'((1-\epsilon)t)\right)d\mu_j(t)\\ =\ &\lim_{n\to\infty}\int_0^{2\pi}\sum_j \overline{\mathcal{DH}_{\mu_j}((f_j)_{a_n})((1-\epsilon)e^{\iota\theta})}\,g_j(e^{i\theta})d\theta = 0,\ g_j \in VMOA.\end{aligned} \tag{32}$$

Set

$$(g_j)_{a_n}(z) = \log\frac{e}{1-a_n z}.$$

It is well known that $g_j \in VMOA$. For $1-\epsilon \in (a_n, 1)$, we deduce that

$$\begin{aligned}&\int_{[0,1)}\sum_j \overline{(f_j)_{a_n}(t)}\left(g_j((1-\epsilon)t)+(1-\epsilon)tg_j'((1-\epsilon)t)\right)d\mu_j(t)\\ &\geq C\int_{[a_n,1)}\sum_j \left(\frac{1-a_n^2}{(1-a_n t)^2}\right)^{\frac{1}{1-\epsilon}}\left(\log\frac{e}{1-a_n(1-\epsilon)t}+\frac{a_n(1-\epsilon)t}{1-a_n(1-\epsilon)t}\right)d\mu_j(t)\\ &\geq \frac{C}{(1-a_n)^{\frac{2-\epsilon}{1-\epsilon}}}\sum_j \mu_j([a_n,1)).\end{aligned}$$

Letting $a_n \to 1^-$ as $n \to \infty$, we have

$$\lim_{a\to 1^-}\sum_j \frac{1}{(1-a_n^2)^{\frac{2-\epsilon}{1-\epsilon}}}\mu_j([a_n,1)) = 0.$$

We can obtain that μ_j is a vanishing $\left(\frac{2-\epsilon}{1-\epsilon}\right)$-Carleson measure.

On the other hand, suppose that μ_j is a vanishing $\left(\frac{2-\epsilon}{1-\epsilon}\right)$-Carleson measure. Let $d\nu(t) = (1-t)^{-1}d\mu_j(t)$, we know that ν is a vanishing $\frac{1}{1-\epsilon}$-Carleson. Let $\{(f_j)_n\}_{j,n=1}^{\infty}$ be a sequence of $H^{1-\epsilon}$ functions with $\sup_{n\geq 1}\|(f_j)_n\|_{H^{1-\epsilon}} < \infty$ and such that $\{(f_j)_n\} \to 0$, uniformly on any compact subset of $\mathbb{D}$. Then by Lemma 5.1, it is enough to show that $\left\{\mathcal{DH}_{\mu_j}((f_j)_n)\right\} \to 0$ in H^1. For every $g_j \in VMOA$, $0 < \epsilon < 1$, using (4) and (18), we deduce that

$$\begin{aligned}&\int_{[0,1)}\sum_j |(f_j)_n(t)||g_j(t)+tg_j'(t)|d\mu_j(t)\\ &= \int_{[0,1-\epsilon)}\sum_j \left|(f_j)_n(t)\right|\left|g_j(t)+tg_j'(t)\right|d\mu_j(t) + \int_{[1-\epsilon,1)}\sum_j \left|(f_j)_n(t)\right|\left|g_j(t)+tg_j'(t)\right|d\mu_j(t).\end{aligned}$$

Then $\int_{[0,1-\epsilon)}|f_j(t)|\,|\,(g_j(t)+tg_j'(t)\,|\,d\mu_j(t)$ tends to 0 as $\{(f_j)_n\} \to 0$ uniformly on any compact subset of $\mathbb{D}$. For second term, arguing as in the proof of the boundedness in Theorem 4.1 (b), we obtain that

$$\int_{[1-\epsilon,1)}\sum_j \left|(f_j)_n(t)\right|\left|g_j(t)+tg_j'(t)\right|d\mu_j(t)$$

$$
\begin{aligned}
&\le C\sum_j \| g_j \|_{BMOA} \int_{[0,1)} |(f_j)_n(t)| \left(\log\frac{1}{1-t}+\frac{t}{1-t}\right) d\mu_j(t)\\
&\le C\sum_j \| g_j \|_{BMOA} \int_{[0,1)} |(f_j)_n(t)| d\nu_{1-\epsilon}(t)\\
&\le C\mathcal{N}(\nu_{1-\epsilon})\sum_j \| g_j \|_{BMOA} \|(f_j)_n\|_{H^{1-\epsilon}}\\
&\le C\mathcal{N}(\nu_{1-\epsilon})\sum_j \| g_j \|_{BMOA}, g_j \in VMOA
\end{aligned} \tag{33}
$$

it also tends to 0 by (2). Thus

$$
\lim_{n\to\infty}\left|\int_0^{2\pi}\sum_j \overline{\mathcal{DH}_{\mu_j}((f_j)_n)(e^{\iota\theta})} g_j(e^{i\theta}) d\theta\right|
$$

$$
= \lim_{n\to\infty}\int_{[0,1)}\sum_j |(f_j)_n(t)||(g_j(t)+tg_j'(t))| d\mu_j(t) = 0, \text{ for all } g_j \in VMOA.
$$

It means $\mathcal{DH}_{\mu_j}((f_j)_n) \to 0$ in H^1, by Lemma 5.1 we obtain $\mathcal{DH}_{\mu_j}$ is a compact operator from $H^{1-\epsilon}$ into H^1.

(c) The proof is the same as that of Theorem 4.1(c) and Theorem 5.2(1). We omit the details here.

Finally, we consider the situation of $\epsilon > 0$, characterize those measures μ_j for which $\mathcal{DH}_{\mu_j}$ is a compact operator from $H^{1+\epsilon}$ into $H^{1+\epsilon}$, and give sufficient and necessary conditions respectively (see [16]).

Theorem 5.3. Let $0 < \epsilon < \infty$ and μ_j be a positive Borel measure on $[0,1)$ which satisfies the condition in Theorem 3.3.

(a) If μ_j is a vanishing $\left(\frac{(1+2\epsilon)+(1+\epsilon)(2+2\epsilon)}{(1+\epsilon)(1+2\epsilon)} + 1 + \epsilon\right)$-Carleson measure for any $\epsilon > 0$, then $\mathcal{DH}_{\mu_j}$ is a compact operator from $H^{1+\epsilon}$ into $H^{1+2\epsilon}$.

(b) If $\mathcal{DH}_{\mu_j}$ is a compact operator from $H^{1+\epsilon}$ into $H^{1+2\epsilon}$, then μ_j is a vanishing $\left(\frac{(1+2\epsilon)+(1+\epsilon)(2+2\epsilon)}{(1+\epsilon)(1+2\epsilon)}\right)$ Carleson measure.

Proof (a) The proof is the same as that of Theorem 5.2(a).We omit the details here.

(b) The proof is similar to that of Theorem 4.3(b) and Theorem 5.2(a). We omit the details here.

Similarly, $\mathcal{DH}_{\mu_j}$ in $H^{1+\epsilon}(0 \le \epsilon \le 1)$ also have a better conclusion (see [16]).

Theorem 5.4. Let $0 \le \epsilon \le 1$ and μ_j be a positive Borel measure on $[0,1)$ which satisfies the condition in Theorem 3.3. Then $\mathcal{DH}_{\mu_j}$ is a compact operator in $H^{1+\epsilon}$ if and only if μ_j is a vanishing 2-Carleson measure.

Proof Firstly, let $\epsilon = 0$, we know that $\mathcal{DH}_{\mu_j}$ is a compact operator in H^1 if and only if μ_j is a vanishing 2-Carleson measure by Theorem 5.2.

Next, let $\epsilon = 1$, by Theorem 5.3, We only need to show if μ_j is a vanishing 2-Carleson measure then $\mathcal{DH}_{\mu_j}$ is a compact operator in H^2.

Assume that μ_j is a vanishing 2-Carleson measure and let $\{(f_j)_{j_0}\}$ be a sequence of functions in H^2 with $\|(f_j)_{j_0}\|_{H^2} \le 1$, for all j_0, and such that $(f_j)_{j_0} \to 0$, uniformly on compact subsets of $\mathbb{D}$. Since μ_j is a vanishing 2-Carleson measure then $(\mu_j)_{n+k} = o\left(\frac{1}{(n+k+1)^2}\right)$, as $n \to \infty$. Say

$$
(\mu_j)_{n,k} = (\mu_j)_{n+k} = \frac{\varepsilon_n}{(n+k+1)^2},\ n = 0,1,2,\ldots
$$

Then $\{\varepsilon_n\} \to 0$. Say that, for every j_0,

$$
(f_j)_{j_0}(z) = \sum_{k=0}^{\infty} a_k^{(j_0)} z^k,\ z \in \mathbb{D}.
$$

By using the classical Hilbert inequality, we have

$$\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{a_k^{(j_0)}}{n+k+1}\right|^2 \leq \pi^2\sum_{k=0}^{\infty}\left|a_k^{(j_0)}\right|^2 \leq \pi^2. \tag{34}$$

Take $\varepsilon > 0$ and next take a natural number N such that

$$n \geq N \ \Rightarrow \ \varepsilon_n^2 < \frac{\varepsilon}{2\pi^2}.$$

We have

$$\begin{aligned}
\sum_j \left\|\mathcal{DH}_{\mu_j}\big((f_j)_{j_0}\big)\right\|_{H^2}^2 &= \sum_{n=0}^{\infty}(n+1)^2\left|\sum_{k=0}^{\infty}\sum_j (\mu_j)_{n,k}a_k^{(j_0)}\right|^2 \\
&= \sum_{n=0}^{N}(n+1)^2\left|\sum_{k=0}^{\infty}\sum_j (\mu_j)_{n,k}a_k^{(j_0)}\right|^2 + \sum_{n=N+1}^{\infty}(n+1)^2\left|\sum_{k=0}^{\infty}\sum_j (\mu_j)_{n,k}a_k^{(j_0)}\right|^2 \\
&\leq \sum_{n=0}^{N}(n+1)^2\left|\sum_{k=0}^{\infty}\sum_j (\mu_j)_{n,k}a_k^{(j_0)}\right|^2 + \sum_{n=0}^{\infty}(n+1)^2\left|\sum_{k=0}^{\infty}\frac{\varepsilon_n a_k^{(j_0)}}{(n+k+1)^2}\right|^2 \\
&\leq \sum_{n=0}^{N}(n+1)^2\left|\sum_{k=0}^{\infty}\sum_j (\mu_j)_{n,k}a_k^{(j_0)}\right|^2 + \frac{\varepsilon}{2\pi^2}\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{a_k^{(j_0)}}{n+k+1}\right|^2 \\
&\leq \sum_{n=0}^{N}(n+1)^2\left|\sum_{k=0}^{\infty}\sum_j \left(\mu_j\right)_{n,k}a_k^{(j_0)}\right|^2 + \frac{\varepsilon}{2}.
\end{aligned} \tag{35}$$

Now, since $(f_j)_{j_0} \to 0$, uniformly on compact subsets of $\mathbb{D}$, it follows that

$$\sum_{n=0}^{N}(n+1)^2\left|\sum_{k=0}^{\infty}\sum_j (\mu_j)_{n,k}a_k^{(j_0)}\right|^2 \to 0, \ \text{as } j_0 \to \infty$$

Then it follows that there exist $(j_0)_0 \in N$ such that $\left\|\mathcal{DH}_{\mu_j}\big((f_j)_{j_0}\big)\right\|_{H^2}^2 < \varepsilon$ for all $j_0 \geq (j_0)_0$. So, we have shown that $\left\|\sum_j \mathcal{DH}_{\mu_j}\left((f_j)_{j_0}\right)\right\|_{H^2}^2 \to 0$. The compactness of $\mathcal{DH}_{\mu_j}$ on H^2 follows (see [16]).

Since we have show that when $\epsilon = 0$, the compactness of $\mathcal{DH}_{\mu_j}$ on H^1. To deal with the cases $0 < \epsilon < 1$, we use again complex interpolation. Let $0 < \epsilon < 1$ and μ_j be a vanishing 2-Carleson measure. Recall that

$$H^{1+\epsilon} = (H^2, H^1)_\theta, \ \text{if } 0 < \epsilon < 1 \text{ and } \theta = \frac{2}{1+\epsilon} - 1.$$

We have also that if $0 < \epsilon < \infty$ then

$$H^2 = (H^{2+\epsilon}, H^1)_{1+\epsilon}.$$

for a certain $1-\epsilon \in (0{,}1)$, namely, $\epsilon = 0$. Since H^2 is reflexive, and $\mathcal{DH}_{\mu_j}$ is compact from H^2 into itself and from H^1 into itself, Theorem 10 of [3] gives that $\mathcal{DH}_{\mu_j}$ is a compact operator in $H^{1+\epsilon}(0 \leq \epsilon \leq 1)$.

References

[1] C. Chatzifountas, D. Girela, J. A. Peláez, A generalized Hilbert matrix acting on Hardy spaces, ´ J Math Anal Appl, 2014, 413(1): 154-168.

[2] C. C. Cowen Jr, B. I. MacCluer, Composition Operators on Spaces of Analytic Functions. CRC Press, 1995.

[3] M. Cwikel, N. J. Kalton, Interpolation of compact operators by the methods of Calderón. and Gustavsson-Peetre, Proceedings of the Edinburgh Mathematical Society, 1995, 38(2): 261-276.

[4] E. Diamantopoulos, A. G. Siskakis, Composition operators and the Hilbert matrix, Studia Math, 2000, 140(2): 191-198.

[5] P. L. Duren, Theory of H^p Spaces, New York: Academic Press, 1970.

[6] P. Galanopoulos, D. Girela, J. A. Peláez, A. G. Siskakis, Generalized Hilbert operators, Ann Acad Sci Fenn Math, 2014, 39: 231-258.
[7] P. Galanopoulos, J. A. Peláez, A Hankel matrix acting on Hardy and Bergman spaces, Studia Math ,2010, 200(3): 201-220.
[8] D. Girela, Analytic functions of bounded mean oscillation, Complex function spaces (Mekrijarvi, 1999), 2001, 4: 61-170.
[9] D. Girela, N. Merchán, A generalized Hilbert operator acting on conformally invariant spaces, Banach J Math Anal ,2018, 12(2): 374-398.
[10] D. Gnuschke-Hauschild, C. Pommerenke, On Bloch functions and gap series, J Reine Angew Math, 1986, 367: 172-186.
[11] W. W. Hastings, A Carleson measure theorem for Bergman spaces, Proc Amer Math Soc, 1975, 52(1): 237-241.
[12] S. Li, J. Zhou, Essential norm of generalized Hilbert matrix from Bloch type spaces to BMOA and Bloch space, AIMS Math, 2021, 6: 3305-3318.
[13] B. MacCluer, R. Zhao, Vanishing logarithmic Carleson measures, Illinois J Math , 2002, 46(2):507-518.
[14] C. Pommerenke, J. Clunie, J. Anderson, On Bloch functions and normal functions, J Reine Angew Math, 1974, 270: 12-37.
[15] B. W. Romberg, P. L. Duren, A. L. Shields, Linear functionals on Hp spaces with $0 < p < 1$, Reine Angew Math, 1969, 238: 32-60.
[16] S. Ye and G. Feng, A Derivative-Hilbert operator acting on Hardy spaces, arXiv:2206.12024v1 [math.CV] 24 Jun 2022.
[17] S. Ye, Z. Zhou. A derivative-Hilbert operator acting on the Bloch space, Complex Anal Oper Theory, 2021, 15(5): 88.
[18] S. Ye, Z. Zhou, A derivative-Hilbert operator acting on Bergman spaces, J Math Anal Appl, 2022, 506(1): 125553.
[19] R. Zhao. On logarithmic Carleson measures, Acta Sci Math, 2003, 69: 605-618.
[20] K. Zhu, Bloch Type Spaces of Analytic Functions, Rocky Mountain J Math, 1993, 23(3): 1143-1177.
[21] Zhu K. Operator theory in function spaces, American Mathematical Soc, 2007.