ACTION OF THE SYMMETRIC GROUP S_7 ON UNORDERED PAIRS

ABSTRACT

In this paper some properties of Symmetric group $G = S_7$ on $X^{(2)}$ are investigated. It is shown that G acts transitively, primitively but not doubly transitively on $X^{(2)}$. The orbits of $G_{\{1,2\}}$ acting on $X^{(2)}$ and the orbits of G acting on $X^{(2)} \times X^{(2)}$ are found to be 3. Suborbital graphs corresponding to the action of G on $X^{(2)} \times X^{(2)}$ are constructed. Some theoretic properties of these graphs are discussed.

Keywords: *Transitive, Orbits, Suborbits, Suborbital graphs*

1.0 Introduction

Let $G = S_7$ acting naturally on $X = \{1,2,3,4,5,6,7\}$. Then G acts on $X^{(2)}$, the set of 21 unordered pairs from the set X by the rule

$$g\{x,y\} = \{gx, gy\},\ \forall\ g \in G, \{x,y\} \in X^{(2)}.$$

1.1 Definition

Let X be a set. The group G acts on the left on X if for each $g \in G$ and each $x \in X$ there corresponds a unique element $gx \in X$ such that -:

i. $(g_1 g_2)x = g_1(g_2 x), \forall g_1, g_2 \in G$ and $x \in X$

ii. For any $x \in X$ $1x = x$, where 1 is the identity in G.

1.2 Definition

If the action of a group G on a set X has only one orbit, then we say that G acts transitively on X. In other words, G acts transitively on X if for every pair of points $x, y \in X$, there exist $g \in G$ such that $gx = y$.

1.3 Definition

A group G is said to act doubly transitively on a set X if and only if given $a,b,c,d \in X$ with $a \neq b$ and $c \neq d$, then there exists $g \in G$ such that $ga = c$ and $gb=d$.

1.4 Definition

Let *G* act transitively on a set *X*. Then a subset *B* of *X* is a block if $gB = B$ or $gB \cap B = \phi$ for $g \in G$. Clearly the set *X* and the singleton subsets of *X* form blocks; these blocks are called trivial blocks. If these are the only blocks, then we say that *G* acts primitively on *X*. Otherwise *G* acts imprimitively.

1.5 Theorem[Cauchy-Frobenius Lemma]

Let *G* be a finite group acting on a set *X*. Then the number orbits of *G* is $\frac{I}{|G|}\sum_{g\in G}|Fix(g)|$, where $|Fix(g)|$ denotes the number of points in *X* fixed by *g*.

[Harary, 1955]

1.6 Theorem Two permutations in S_n are conjugate if and only if $g \in G$ has cycle type $(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n)$, then the number of permutations in S_n conjugate to *g* is $\frac{n!}{\prod_{i=1}^{n} \alpha_i ! i^{\alpha_i}}$.

[Krishnamurthy, 1985]

1.7Theorem

G_x has an orbit different from {*x*} and paired with itself if and only if *G* has even order.

We notice that *G* acts on $X \times X$ by $g(x, y) = (gx, gy),\ g \in G, x, y \in X$. If $O \subseteq X \times X$ is a *G*-Orbit, then for a fixed $x \in X$, $\Delta = \{y \in X | (x.y) \in O\}$ is a G_x-Orbit. Conversely, if $\Delta \subset X$ is a G_x-Orbit, then $O = \{(gx, gy) | g \in G,\ y \in \Delta\}$ is a *G*-orbit on $X \times X$. We say Δ corresponds to O.

The *G*-orbits on $X \times X$ are called suborbitals. Let $O_i \subseteq X \times X, i = 0,1,..., r-1$ be a suborbital. Then we form a graph Γ_i, by taking *X* as the set of vertices of Γ_i and by including a directed edge from *x*to $y (x, y \in X)$ if and only if $(x, y) \in O_i$. Thus each suborbital O_i determines a suborbital graph Γ_i. Now $O_i^* = \{(x, y) | (y, x) \in O_i\}$ is a *G*-Orbit. Let Γ_i^* be the suborbital graph corresponding to the suborbital O_i^*. Let the suborbits Δ_i $(i = 0,1,..., r-1)$ correspond to the suborbital O_i . Then Γ_i is undirected if Δ_i is self-paired and Γ_i is directed if Δ_i is not self-paired. [Wielandt, 1964]

Theorem 1.8 [Kamuti, 2004]

The counting polynomial for digraphs with *p* points is $d_p(x) = Z\left(S_p^{[2]}, 1+x\right)$.

2.0 RESULTS AND DISCUSSION

2.1 Some properties of the action of G on $X^{(2)}$

Lemma 2.1.1

G acts transitively on $X^{(2)}$.

Proof

By Definition 1.2, we only need to use the Theorem 1.5 to show that the action of *G* on $X^{(2)}$ has one orbit.

Let $g \in G$ have cycle type $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, then the number of permutations in G having the same cycle type as *g* is given by Theorem 1.6 and

$$|Fix(g)| = \binom{\alpha_1}{2} + \alpha_2 .$$

We now have the following table

Table .1: Permutation in *G* and the number of fixed points

Permutation *g* in *G*	Number of permutations	$\lvert Fix(g)\rvert$
I	1	21
(ab)	21	11
(abc)	70	6
(abcd)	210	3
(abcde)	504	1
(abcdef)	840	0
(abcdefg)	720	0
(ab)(cd)(ef)	105	3
(ab)(cd)(efg)	210	2
(ab)(cdefg)	504	1

(ab)(cd)	105	5
(ab)(cdef)	630	1
(abc)(def)	280	0
(ab)(cde)	420	2
(abc)(defg)	420	0

Now applying Theorem 1.5we get:

number of orbits of G acting on $X^{(2)} = \frac{1}{|G|}\sum_{g\in G}|Fix(g)|$

$$= \frac{1}{7!}\big[1\times 21 + 11\times 21 + 6\times 70 + 3\times 210 + 1\times 504 + 0\times 840 + 0\times 720 + 3\times 105 + 2\times 210 + 1\times 504 + 5\times 105 + 1\times 630 + 0\times 280 + 2\times 420 + 0\times 420\big]$$

$$= \frac{1}{7!}\big[21 + 231 + 420 + 630 + 504 + 315 + 420 + 504 + 525 + 630 + 840\big]$$

$$= \frac{1}{5040}\big[5040\big]$$

$$= 1$$

Therefore G acts transitively on $X^{(2)}$.

Lemma 2.1.2

G does not act doubly transitively on $X^{(2)}$.

Proof

By Definition 1.3, G does not act doubly transitively on $X^{(2)}$ because for example there is no element of G which takes {1,2} to {1,2} and {1,3} to {4,5}.

Lemma 2.1.3

G acts primitively on $X^{(2)}$.

Proof

By definition 1.4, if G acts imprimitively on $X^{(2)}$ then its blocks of imprimitivity have length 3 or 7. Now if B is a block containing {1,2}, then for $g \in G_{\{1,2\}}$, $gB = B$ or $gB \cap B = \phi$. But

$\{1,2\} \in B \cap gB$, so $gB = B$. Therefore B is invariant under $G_{\{1,2\}}$, so B is a union of some orbits of $G_{\{1,2\}}$. Since the orbits of $G_{\{1,2\}}$ have lengths 1, 10, 10 (as we shall see in Section 2.2), this gives a contradiction. Hence G acts primitively on $X^{(2)}$.

2.2 Orbits of G acting on $X^{(2)} \times X^{(2)}$ and the corresponding suborbital graphs

Lemma 2.2.1

The number of orbits of $G_{\{1,2\}}$ acting on $X^{(2)}$ is 3.

Proof

We need to apply the Theorem 1.4 to get the number of orbits of $G_{\{1,2\}}$ on $X^{(2)}$. A permutation in $G_{\{1,2\}}$ is either of the form $(1)(2)(\ldots)(\ldots)\ldots(\ldots)$ or $(12)(\ldots)\ldots(\ldots)$. Thus $g \in G_{\{1,2\}}$, if either it fixes or transposes 1 and 2. If g has cycle type $(\alpha_1, \alpha_2, \ldots \alpha_n)$ and g is of the form $(1)(2)(\ldots)(\ldots)\ldots(\ldots)$, then the number of permutations in $G_{\{1,2\}}$ with the same cycle type as g is

$$\frac{(n-2)!}{1^{(\alpha_1-2)}(\alpha_1-2)!\prod_{i=2}^{n}\alpha_i i^{\alpha_i}},$$

and if is of the form (12) (..) …(…) then the number of permutations in $G_{\{1,2\}}$ with the same type as g is

$$\frac{(n-2)!}{1^{\alpha_1}\alpha_1!(\alpha_2-2)!2^{(\alpha_2-2)}\prod_{i=3}^{n}\alpha_i i^{\alpha_i}}$$

Now $|Fix(g)| = \binom{\alpha_1}{2} + \alpha_2$

in each case

We now have the following table

Table 2: Permutation in $G_{\{1,2\}}$ and the number of fixed points

Permutation in $G_{\{1,2\}}$	**Number of permutations**	$\lvert Fix(g) \rvert$

I	1	21
(1)(2)(ab)(c)(d)(e)	10	11
(12)(a)(b)(c)(d)(e)	1	11
(12)(ab)(cd)(e)	15	3
(1)(2)(abc)(d)(e)	20	6
(1)(2)(abcd)(e)	30	3
(1)(2)(abcde)	24	1
(1)(2)(ab)(cde)	20	2
(1)(2)(ab)(d)(e)	15	5
(12)(ab)(c)(d)(e)	10	5
(12)(abc)(de)	20	2
(12)(abcd)(e)	30	1
(12)(abcde)	24	1
(12)(abc)(d)(e)	20	2
Total	**240**	

Now applying Theorem 1.5 we get;

number of orbits of $G_{\{1,2\}}$ on $X^{(2)}$

$$= \frac{1}{\left|G_{\{1,2\}}\right|} \sum_{g \in G_{\{1,2\}}} \left|Fix(g)\right|$$

$$= \frac{1}{240}[1\times 21 + 10\times 11 + 1\times 11 + 15\times 3 + 20\times 6 + 30\times 3 + 24\times 1$$

$$+20\times 2 + 15\times 5 + 10\times 5 + 20\times 2 + 30\times 1 + 24\times 1 + 20\times 2]$$

$$= \frac{1}{240}\times 720$$
$$= 3$$

The three orbits of $G_{\{1,2\}}$ acting on $X^{(2)}$ found in Lemma 2.2.1 are;

$Orb_{G_{\{1,2\}}}\{1,2\} = \{\{1,2\}\} = \Delta_0$, the trivial orbit.

$Orb_{G_{\{1,2\}}}\{1,3\} = \{\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{1,7\},\{2,3\},\{2,4\},\{2,5\},\{2,6\},\{2,7\}\} = \Delta_1$

which is the set of all unordered pairs containing exactly one of 1 or 2.

$Orb_{G_{\{1,2\}}}\{3,4\}=\{\{3,4\},\{3,5\},\{3,6\},\{3,7\},\{4,5\},\{4,6\},\{4,7\},\{5,6\},\{5,7\},\{6,7\}\}=\Delta_2$, the set of all unordered pairs containing neither 1 nor 2.

Thus the rank of *G* on $X^{(2)}$ is 3 and the subdegrees are 1, 10, 10.

We now concentrate on the non-trivial suborbits Δ_1 and Δ_2, since the suborbital graphs corresponding to Δ_0 is the null graph and therefore not very interesting.

Since $|G|=7!$ is even, then by Theorem 1.7 , Δ_1 and Δ_2 are self-paired and therefore their corresponding suborbital graphs Γ_1 and Γ_2 are undirected.

By the theory developed in Theorem 1.7, the suborbital O_1 corresponding to the suborbit Δ_1 is $O_1=\{(g\{1,2\},g\{1,3\})|g\in G\}$. Thus the suborbital graph Γ_1 corresponding to the suborbital O_1 has two 2-element subsets *S* and *T* from $X=\{1,2,...,7\}$ adjacent if and only if $|S\cap T|=1$.

On the other hand the suborbital O_2 corresponding to the suborbit Δ_2 is $O_2=\{(g\{1,2\},g\{3,4\})|g\in G\}$ and suborbital graph Γ_2 corresponding to O_2 has two 2-element subsets *S* and *T* from *X=(12…7)* adjacent if and only if $|S\cap T|=0$

The two graphs are also complementary. A little calculation shows that the two graphs are regular of degree 10.

Since vertices for example {2,7}, {2,5} and {2,3} are connected in Γ_1 and say {4,7}{3,6} and {2,5} are connected in Γ_2, then these graphs are of girth 3.

2.3 Cycle index formula for $S_7^{(2)}$

Here the cycle index formula for $S_n^{(2)}$ is derived. First the technique used to get the disjoint cycle structures of permutations in S_n when it acts on $X^{[2]}$ is sketched.

2.3.1 Derivation of cycle index formula for $S_n^{[2]}$

Derivation of the cycle index polynomial for a pair group $S_n^{(2)}$, the group induced when the symmetric group S_n acts on unordered pair from the set $X = \{1, 2, ..., n\}$, appears in Harary, [1955]. This polynomial has been used extensively in enumerating various types of graphs. Let (G,X) be a finite permutation group and we denote by $X^{(2)}$ the set of 2-element subsets of X. If g is a permutation in (G,X) we may want to know the disjoint cycle structure of the permutation g' induced by g on $X^{(2)}$.

Let $mon(g) = t_1^{\alpha_1} t_2^{\alpha_2} ... t_n^{\alpha_n}$, our aim is to find $mon(g')$ from which the disjoint cycle structure of g' can straight forwardly be obtained. To do this there are two separate contributions from g to the corresponding term of $mon(g')$ which we need to consider.

Case I From pairs of points, both lying in a common cycle of g.

Case II From pairs of points, one in each of two different cycles of g.

It is convenient to divide the first contributions into:

Case I (a) Those pairs from the odd cycles and

Case I (b) Those from the even cycles.

Case I(a)

Let $\theta = (123...2m+1)$ be an odd cycle in g, then the permutation θ' in $\left(G, X^{(2)}\right)$ induced by θ is as follows.

$$(123\cdots 2m+1)\to\begin{cases}\{1,2\}\to\{2,3\}\to\{3,4\}\to\cdots\cdots\cdots\cdots\to\{2m+1,1\}\\ \{1,3\}\to\{2,4\}\to\{3,5\}\to\cdots\cdots\cdots\to\{2m,1\}\to\{2m+1,2\}\\ \{1,4\}\to\{2,5\}\to\{3,6\}\to\cdots\to\{2m-1,1\}\to\{2m,2\}\to\{2m+1,3\}\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \{1,m+1\}\to\{2,m+2\}\to\cdots\to\{m+2,1\}\to\cdots\cdots\to\{2m+1,m\}\end{cases}$$

Hence $t_{2m+1}\to s_{2m+1}^{m}$.

So if we have α_{2m+1} cycles of length $2m+1$ in g, the pairs of points lying in common cycles contribute:

$$t_{2m+1}^{\alpha_{2m+1}}\to s_{2m+1}^{m\alpha_{2m+1}} \qquad (1)$$

for odd cycles.

Case I(b) If $\theta=(123...2m)$, then we get θ' as follows

$$(123...2m)\to\begin{cases}\{1,2\}\to\{2,3\}\to\{3,4\}\to\cdots\cdots\cdots\cdots\to\{2m,1\}\\ \{1,3\}\to\{2,4\}\to\{3,5\}\to\cdots\cdots\to\{2m-1,1\}\to\{2m,2\}\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \{1,m-1\}\to\{2,m\}\to\cdots\to\{m+3,1\}\to\cdots\to\{2m,m-2\}\\ \{1,m\}\to\{2,m+1\}\to\cdots\to\{m+2,1\}\to\cdots\to\{2m,m-1\}\\ \{1,m+1\}\to\{2,m+2\}\to\cdots\cdots\cdots\cdots\to\{m,2m\}\end{cases}$$

Hence $t_{2m}\to s_m s_{2m}^{m-1}$

So if α_{2m} is the number of cycles of length $2m$ in g, the pairs of points lying in common cycles contribute:

$$t_{2m}^{\alpha_{2m}}\to\left(s_m^1 s_{2m}^{m-1}\right)^{\alpha_{2m}} \qquad (2)$$

Case II

Consider two distinct cycles of length *a* and *b* in (*G*,*X*). If *x* belongs to an *a*-cycle θ_a of *g* and *y* belongs to a *b*-cycle θ_b of *g*, then the least positive integer β, for which $g^{\beta}x = x$ and also $g^{\beta}y = y$ is $l(a,b)$, the least common multiple of *a* and *b*. So the element {*x*,*y*} belongs to an $l(a,b)$ cycle of g'. The number of such $l(a,b)$-cycles contributed by *g* on $\theta_a \times \theta_b$ to g' is $\frac{ab}{l(a,b)} = d(a,b)$, the *gcd* of *a* and *b*.

In particular when $a = b = m$, the contribution by *g* on $\theta_a \times \theta_b$ to g' is *m* cycles of length *m*.

Thus when $a \neq b$ we have

$$t_a^{\alpha_a} t_b^{\alpha_b} \to s_{l(a,b)}^{\alpha_a \alpha_b d(a,b)} \tag{3}$$

and when *a*=*b*=*m*

$$t_m^{\alpha_l} \to s_m^{m\binom{\alpha_m}{2}} \tag{4}$$

Now we simply need to multiply the right-sides of (1) – (4) over all applicable cases. Collecting the like terms and simplifying gives $mon(g')$ and hence the disjoint cycle structure of g'.

And thus the cycle index of $S_n^{(2)}$ is

$$Z\left(S_n^{(2)}\right) = \frac{1}{n!}\sum_{\alpha}\frac{n!}{\prod_{m=1}^{n}\alpha_m!m^{\alpha_m}}\prod_{m=1}^{\left[\frac{n}{2}\right]}\left(S_m S_{2m}^{m-1}\right)^{\alpha_{2m}}$$

$$\prod_{m=0}^{\left[\frac{(n-1)}{2}\right]} S_{2m+1}^{m\alpha_{2m+1}} \prod_{m=1}^{\left[\frac{n}{2}\right]} S_m^{m\binom{\alpha_m}{2}} \prod_{1\le a<b<n-1} S_{l(a,b)}^{d(a,b)\alpha_a\alpha_b}$$

2.3.2 Cycle index of the pair group $S_7^{[2]}$

Theresults in Section 2.3.1 are used to derive the cycle index formula for $S_n^{(2)}$.

We now use the results in the previous section to derive the cycle index formula for $S_7^{(2)}$.

Table 3 Permutation S_7 and monomial contributions to $S_7^{(2)}$

Permutations S_7	**Monomial**	**Number of permutations**	**Monomial contributions to $S_7^{(2)}$**
I	t_1^7	1	s_1^{21}
(ab)	$t_1^5t_2$	21	$s_1^{11}s_2^5$
(abc)	$t_1^4t_3$	70	$s_1^6s_3^5$
(abcd)	$t_1^3t_4$	210	$s_1^3s_2s_4^4$
(abcde)	$t_1^2t_5$	504	$s_1^1s_5^4$
(abcdef)	t_1t_6	840	$s_3^1s_6^3$
(abcdefg)	t_7	720	s_7^3
(ab)(cd)(ef)	$t_1t_2^3$	105	$s_1^3s_2^9$
(ab)(cd)(efg)	$t_1^2t_3$	210	$s_1^2s_2^2s_3^1s_6^2$
(ab)(cdefg)	t_2t_5	504	$s_1^1s_5^2s_{10}^1$
(ab)(cd)	$t_1^3t_2^2$	105	$s_1^5s_2^8$
(ab)(cdef)	$t_1t_2t_4$	630	$s_1s_2^2s_4^4$
(abc)(def)	$t_1t_3^2$	280	s_3^7
(ab)(cde)	$t_1^2t_2t_3$	420	$s_1^2s_2^2s_3^3s_6$
(abc)(defg)	t_3t_4	420	$s_2s_3s_4s_{12}$

Thus from the second and the third columns;

$$Z(s_7) = \frac{1}{7!}\Big[t_1^7 + 21t_1^5t_2 + 70t_1^4t_3 + 210t_1^3t_4 + 504t_1^2t_5 + 840t_1t_6 + 720t_7 + 105t_1t_2^3$$

$$+210t_2^2t_3 + 504t_2t_5 + 105t_1^3t_2^2 + 630t_1t_2t_4 + 280t_1t_3^2 + 420t_2t_3t_1^2 + 420t_3t_4\Big]$$

From the third and fourth columns;

$$Z\left(s_7^{(2)}\right)=\frac{1}{7!}\Big[s_1^{21}+21s_1^{11}s_2^5+70s_1^6s_3^5+210s_1^3s_2s_4^4+504s_1^1s_5^4+840s_3^1s_6^3+720s_7^3+105s_1^3s_2^9$$

$$+210s_1^2s_2^2s_3^1s_6^2+504s_1^1s_5^2s_{10}^1+105s_1^5s_2^8+630s_1s_2^2s_4^4+280\,s_3^7+420s_1^2s_2^2s_3^3s_6+420s_2s_3s_4s_{12}\Big]$$

2.3.3 Counting series for unlabelled graph with 7 vertices

Now by Theorem 1.8 and the formula for $Z\left(S_7^{(2)}\right)$ given in the previous section, the counting series for unlabelled graph with 7 vertices is given by

$$Z\left(S_7^{(2)},1+x\right)=\frac{1}{7!}\Big[(1+x)^{21}+21(1+x)^{11}\left(1+x^2\right)^5+70(1+x)^6\left(1+x^3\right)^5$$

$$+210(1+x)^3\left(1+x^2\right)\left(1+x^4\right)^4+504(1+x)\left(1+x^5\right)^4+840\left(1+x^3\right)\left(1+x^6\right)^3$$

$$+720\left(1+x^7\right)^3+105\left(1+x^3\right)\left(1+x^2\right)^9+210\left(1+x^2\right)^2(1+x)^2\left(1+x^3\right)\left(1+x^6\right)^2$$

$$+504(1+x)\left(1+x^5\right)^2\left(1+x^{10}\right)+105(1+x)^5\left(1+x^2\right)^8+630(1+x)\left(1+x^2\right)^2\left(1+x^4\right)^4$$

$$+280\left(1+x^3\right)^7+420(1+x)\left(1+x^2\right)^2\left(1+x^3\right)^3\left(1+x^6\right)$$

$$+420\left(1+x^2\right)\left(1+x^3\right)\left(1+x^4\right)\left(1+x^{12}\right)\Big]$$

$$=\frac{1}{5040}\Big[5040+5040x+10080x^2+25200x^3+50400x^4+105840x^5+206640x^6+327600x^7$$

$$+488880x^8+660240x^9+745920x^{10}+745920x^{11}\pm 660240x^{12}+488880x^{13}+327600x^{14}$$

$$+206640x^{15}+105840x^{16}+50400x^{17}+25200x^{18}+10080x^{19}+5040x^{20}+5040x^{21}\Big]$$

$$=1+x+2x^2+5x^3+10^4+21x^5+41x^6+65x^7+97x^8+131x^9+148x^{10}$$

$$+148x^{11}+131x^{12}+97x^{13}+65x^{14}+41x^{15}+21x^{16}$$

$$+10x^{17}+5x^{18}+2x^{19}+x^{20}+x^{21}$$

3.0 CONCLUSIONS

In this paper some properties of the action of S_7 acting on unordered pairs were investigated; it was shown that S_7 act transitively, primitively but not doubly transitively on unordered pairs. The rank of S_7 when it acts on unordered pair was found to be 3, same as that obtained by Higmann (1964). Again the cycle index of the pair group $S_7^{(2)}$ were obtained. The counting series for unlabelled graph with seven vertices were computed; it was proved that there are 1044 non – isomorphic graphs; same as those obtained by Harary, (1969).

Having investigated some properties of symmetric group S_7 acting on unordered pairs and constructing suborbital graphs corresponding to the action of S_7 on $X^{[2]} \times X^{[2]}$. This work can be extended by investigating some properties of General Linear groups acting on its cosets

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