ACTION OF THE SYMMETRIC GROUP S_7 ON UNORDERED PAIRS

ABSTRACT

In this paper we investigate some properties of Symmetric group $G = S_7$ on $X^{(2)}$. We show that G acts transitively, primitively but not doubly transitively on $X^{(2)}$. We find the orbits of $G_{\{1,2\}}$ acting on $X^{(2)}$ and the orbits of G acting on $X^{(2)} \times X^{(2)}$ to be 3. We also give a construction of suborbital graphs corresponding to the action of G on $X^{(2)} \times X^{(2)}$. Some theoretic properties of these graphs are discussed.

***Keywords**: Transitive, Orbits, Suborbits, Suborbital graphs*

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1.0 Introduction

Let $G = S_7$ acting naturally on $X = \{1,2,3,4,5,6,7\}$. Then G acts on $X^{(2)}$, the set of 21 unordered pairs from the set X by the rule

$$g\{x,y\} = \{gx, gy\}, \forall\ g \in G, \{x,y\} \in X^{(2)}.$$

1.1 Definition

Let X be a set. The group G acts on the left on X if for each $g \in G$ and each $x \in X$ there corresponds a unique element $gx \in X$ such that -:

i. $(g_1 g_2)x = g_1(g_2 x), \forall g_1, g_2 \in G$ and $x \in X$

ii. For any $x \in X$ $1x = x$, where 1 is the identity in G.

1.2 Definition

If the action of a group G on a set X has only one orbit, then we say that G acts transitively on X. In other words, G acts transitively on X if for every pair of points $x, y \in X$, there exist $g \in G$ such that $gx = y$.

1.3 Definition

A group G is said to act doubly transitively on a set X if and only if given $a, b, c, d \in X$ with $a \neq b$ and $c \neq d$, then there exists $g \in G$ such that $ga = c$ and $gb=d$.

1.4 Definition

Let G act transitively on a set X. Then a subset B of X is a block if $gB = B$ or $gB \cap B = \phi$ for $g \in G$. Clearly the set X and the singleton subsets of X form blocks; these blocks are called trivial blocks. If these are the only blocks, then we say that G acts primitively on X. Otherwise G acts imprimitively.

1.5 Theorem [Cauchy-Frobenius Lemma]

Let G be a finite group acting on a set X. Then the number orbits of G is $\frac{I}{|G|}\sum_{g \in G}|Fix(g)|$,

where $|Fix(g)|$ denotes the number of points in X fixed by g.

[Harary, [1]]

1.6 Theorem Two permutations in S_n are conjugate if and only if $g \in G$ has cycle type $(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n)$, then the number of permutations in S_n conjugate to g is $\frac{n!}{\prod_{i=1}^{n} \alpha_i ! i^{\alpha_i}}$.

[Krishnamurthy, [2]]

1.7 Theorem

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G_x has an orbit different from $\{x\}$ and paired with itself if and only if G has even order.

We notice that G acts on $X \times X$ by $g(x,y) = (gx, gy), g \in G, x, y \in X$. If $O \subseteq X \times X$ is a G-Orbit, then for a fixed $x \in X$, $\Delta = \{y \in X \mid (x.y) \in O\}$ is a G_x-Orbit. Conversely, if $\Delta \subset X$ is a G_x-Orbit, then $O = \{(gx, gy) \mid g \in G, y \in \Delta\}$ is a G-orbit on $X \times X$. We say Δ corresponds to O.

The G-orbits on $X \times X$ are called suborbitals. Let $O_i \subseteq X \times X, i = 0,1,...,r-1$ be a suborbital. Then we form a graph Γ_i, by taking X as the set of vertices of Γ_i and by including a directed edge from x to y $(x, y \in X)$ if and only if $(x, y) \in O_i$. Thus each suborbital O_i determines a suborbital graph Γ_i. Now $O_i^* = \{(x,y) \mid (y,x) \in O_i\}$ is a G-Orbit. Let Γ_i^* be the suborbital graph corresponding to the suborbital O_i^*. Let the suborbits Δ_i $(i = 0,1,...,r-1)$ correspond to the suborbital O_i. Then Γ_i is undirected if Δ_i is self-paired and Γ_i is directed if Δ_i is not self-paired. [Wielandt, [3]]

2.0 RESULTS AND DISCUSSION

2.1 Some properties of the action of G on $X^{(2)}$

Lemma 2.1.1

G acts transitively on $X^{(2)}$.

Proof

By Definition 1.2, we only need to use the Theorem 1.5 to show that the action of G on $X^{(2)}$ has one orbit.

Let $g \in G$ have cycle type $(\alpha_1, \alpha_2, ..., \alpha_n)$, then the number of permutations in G having the same cycle type as g is given by Theorem 1.6 and

UNDER PEER REVIEW

$$\left|Fix(g)\right| = \binom{\alpha_1}{2} + \alpha_2 .$$

We now have the following table

Table .1: Permutation in *G* and the number of fixed points

Permutation *g* in *G*	**Number of permutations**	$\left\|\boldsymbol{Fix}(\boldsymbol{g})\right\|$
I	1	21
(ab)	21	11
(abc)	70	6
(abcd)	210	3
(abcde)	504	1
(abcdef)	840	0
(abcdefg)	720	0
(ab)(cd)(ef)	105	3
(ab)(cd)(efg)	210	2
(ab)(cdefg)	504	1
(ab)(cd)	105	5
(ab)(cdef)	630	1
(abc)(def)	280	0
(ab)(cde)	420	2
(abc)(defg)	420	0

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Now applying Theorem 1.5 we get:

number of orbits of G acting on $X^{(2)} = \frac{1}{|G|}\sum_{g\in G}\left|Fix(g)\right|$

$$= \frac{1}{7!}\left[1\times 21 + 11\times 21 + 6\times 70 + 3\times 210 + 1\times 504 + 0\times 840 + 0\times 720 + 3\times 105 + 2\times 210 + 1\times 504 \right.$$
$$\left. + 5\times 105 + 1\times 630 + 0\times 280 + 2\times 420 + 0\times 420\right]$$

$$= \frac{1}{7!}\left[21 + 231 + 420 + 630 + 504 + 315 + 420 + 504 + 525 + 630 + 840\right]$$

$$= \frac{1}{5040}\left[5040\right]$$
$$= 1$$

Therefore G acts transitively on $X^{(2)}$.

Lemma 2.1.2

G does not act doubly transitively on $X^{(2)}$.

Proof

By Definition 1.3, G does not act doubly transitively on $X^{(2)}$ because for example there is no element of G which takes {1,2} to {1,2} and {1,3} to {4,5}.

Lemma 2.1.3

G acts primitively on $X^{(2)}$.

Proof

By definition 1.4, if G acts imprimitively on $X^{(2)}$ then its blocks of imprimitivity have length 3 or 7. Now if B is a block containing {1,2}, then for $g \in G_{\{1,2\}}$, $gB = B$ or $gB \cap B = \phi$. But $\{1,2\} \in B \cap gB$, so $gB = B$. Therefore B is invariant under $G_{\{1,2\}}$, so B is a union of some orbits of $G_{\{1,2\}}$. Since the orbits of $G_{\{1,2\}}$ have lengths 1, 10, 10 (as we shall see in Section 2.2), this gives a contradiction. Hence G acts primitively on $X^{(2)}$.

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2.2 Orbits of G acting on $X^{(2)} \times X^{(2)}$ and the corresponding suborbital graphs

Lemma 2.2.1

The number of orbits of $G_{\{1,2\}}$ acting on $X^{(2)}$ is 3.

Proof

We need to apply the Theorem 1.4 to get the number of orbits of $G_{\{1,2\}}$ on $X^{(2)}$. A permutation in $G_{\{1,2\}}$ is either of the form $(1)(2)(\ldots)(\ldots)\ldots(\ldots)$ or $(12)(\ldots)\ldots(\ldots)$. Thus $g \in G_{\{1,2\}}$, if either it fixes or transposes 1 and 2. If g has cycle type $(\alpha_1, \alpha_2, \ldots \alpha_n)$ and g is of the form $(1)(2)(\ldots)(\ldots)\ldots(\ldots)$, then the number of permutations in $G_{\{1,2\}}$ with the same cycle type as g is

$$\frac{(n-2)!}{1^{(\alpha_1-2)}(\alpha_1-2)!\prod_{i=2}^{n}\alpha_i i^{\alpha_i}},$$

and if is of the form (12) (..) …(…) then the number of permutations in $G_{\{1,2\}}$ with the same type as g is

$$\frac{(n-2)!}{1^{\alpha_1}\alpha_1!(\alpha_2-2)!2^{(\alpha_2-2)}\prod_{i=3}^{n}\alpha_i i^{\alpha_i}}$$

Now $|Fix(g)| = \binom{\alpha_1}{2} + \alpha_2$

in each case

We now have the following table

Table 2: Permutation in $G_{\{1,2\}}$ and the number of fixed points

Permutation in $G_{\{1,2\}}$	**Number of permutations**	$\lvert Fix(g)\rvert$

UNDER PEER REVIEW

I	1	21
(1)(2)(ab)(c)(d)(e)	10	11
(12)(a)(b)(c)(d)(e)	1	11
(12)(ab)(cd)(e)	15	3
(1)(2)(abc)(d)(e)	20	6
(1)(2)(abcd)(e)	30	3
(1)(2)(abcde)	24	1
(1)(2)(ab)(cde)	20	2
(1)(2)(ab)(d)(e)	15	5
(12)(ab)(c)(d)(e)	10	5
(12)(abc)(de)	20	2
(12)(abcd)(e)	30	1
(12)(abcde)	24	1
(12)(abc)(d)(e)	20	2
Total	**240**	

Now applying Theorem 1.5 we get;

number of orbits of $G_{\{1,2\}}$ on $X^{(2)}$

$$= \frac{1}{\left|G_{\{1,2\}}\right|} \sum_{g \in G_{\{1,2\}}} \left|Fix(g)\right|$$

$$= \frac{1}{240}\left[1\times 21 + 10\times 11 + 1\times 11 + 15\times 3 + 20\times 6 + 30\times 3 + 24\times 1\right.$$

$$\left. + 20\times 2 + 15\times 5 + 10\times 5 + 20\times 2 + 30\times 1 + 24\times 1 + 20\times 2\right]$$

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$$= \frac{1}{240} \times 720$$
$$= 3$$

The three orbits of $G_{\{1,2\}}$ acting on $X^{(2)}$ found in Lemma 2.2.1 are;

$Orb_{G_{\{1,2\}}}\{1,2\} = \{\{1,2\}\} = \Delta_0$, the trivial orbit.

$Orb_{G_{\{1,2\}}}\{1,3\} = \{\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{1,7\},\{2,3\},\{2,4\},\{2,5\},\{2,6\},\{2,7\}\} = \Delta_1$

which is the set of all unordered pairs containing exactly one of 1 or 2.

$Orb_{G_{\{1,2\}}}\{3,4\} = \{\{3,4\},\{3,5\},\{3,6\},\{3,7\},\{4,5\},\{4,6\},\{4,7\},\{5,6\},\{5,7\},\{6,7\}\} = \Delta_2$, the set of all unordered pairs containing neither 1 nor 2.

Thus the rank of G on $X^{(2)}$ is 3 and the subdegrees are 1, 10, 10.

We now concentrate on the non-trivial suborbits Δ_1 and Δ_2, since the suborbital graphs corresponding to Δ_0 is the null graph and therefore not very interesting.

Since $|G| = 7!$ is even, then by Theorem 1.7 , Δ_1 and Δ_2 are self-paired and therefore their corresponding suborbital graphs Γ_1 and Γ_2 are undirected.

By the theory developed in Theorem 1.7, the suborbital O_1 corresponding to the suborbit Δ_1 is $O_1 = \{(g\{1,2\}, g\{1,3\}) | g \in G\}$. Thus the suborbital graph Γ_1 corresponding to the suborbital O_1 has two 2-element subsets S and T from $X = \{1,2,...,7\}$ adjacent if and only if $|S \cap T| = 1$.

On the other hand the suborbital O_2 corresponding to the suborbit Δ_2 is $O_2 = \{(g\{1,2\}, g\{3,4\}) | g \in G\}$ and suborbital graph Γ_2 corresponding to O_2 has two 2-element subsets S and T from $X=(12\ldots7)$ adjacent if and only if $|S \cap T| = 0$

UNDER PEER REVIEW

The two graphs are also complementary. A little calculation shows that the two graphs are regular of degree 10.

Since vertices for example {2,7}, {2,5} and {2,3} are connected in Γ_1 and say {4,7}{3,6} and {2,5} are connected in Γ_2, then these graphs are of girth 3.

REFERENCES

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