

Original Research Article

FIXED POINT THEOREMS IN
COMPLETE CONE RANDOM METRIC
SPACES OVER BANACH ALGEBRA

Abstract

We define fixed point results in Random cone metric space(RCMS) over Banach algebra. Also we give related corollaries and illustrate the examples.

Keywords: Banach algebra, Random cone metric space, Random fixed point, Cyclic (α, β) - admissible mapping, C-class function.

2010 AMS classification: Primary 47H10; Secondary 47H09, 47H10.

1 INTRODUCTION

Huang and Zhang [1] introduced cone metric spaces which are generalizations of metric spaces, and they extended Banach's contraction principle to such spaces, whereafter many authors (for example, [?, 2-4] and references therein) studied fixed point theorems and generalized the concept of metric spaces by replacing the set of real numbers with an ordered Banach space, hence they have defined the cone metric spaces. They also described the convergence of sequences and introduced the notion of completeness in cone metric spaces. They have proved some fixed point theorems of contractive mappings on complete cone metric space with the assumption of normality of a cone.

In 2008, the assumption of normality in cone normal spaces is removed by Rezapour and Hambarani [4], which is an important event in developing

fixed point theory in cone metric spaces. Liu and Xu [5] introduced the notion of cone metric spaces over Banach algebras, which is a modification of the concept of cone metric spaces over real Banach spaces [1], and proved the existence of fixed points for mappings defined on such spaces, and they gave an example that fixed point results in metric spaces and in cone metric spaces are not equivalent.

I Beg [14, 15] defined approximation of random fixed point in Normed spaces and also use semi contractivity to find random fixed point results. Itoh [16] proved random fixed point theorems in Banach spaces with the application of random differential equations. Smriti mehta et al [13] established fixed point results in Cone random metric space and also parashar [18] proved fixed point theorems in the setting of cone random metric spaces. Khan et al [20] implimented Cone C -class function in RCMS and Rashwan et al [19] determined Common fixed point results in RCMS using the integral type of Akram contraction.

Seong-Hoon Cho [17] introduced the notions of C -class functions and cyclic (α, β) - admissible mappings in Banach algebras by using complete cone metric spaces.

In this paper, we use reference article [17] for random cone metric spaces(RCMS) instead of cone metric spaces

2 PRELIMINARIES

Definition 1 *A Banach space \mathfrak{A} is called real Banach algebra (with unit) if there exist a multiplication $\mathfrak{A} \times \mathfrak{A} \rightarrow \mathfrak{A}$ such that*

$\forall x, y, z \in \mathfrak{A}$ and $\alpha \in \mathbb{R}$

(i) $(xy)z = x(yz);$

(ii) $x(y + z) = xy + yz$ and $(x + y)z = xz + yz;$

(iii) $\alpha(xy) = (\alpha x)y = x(\alpha y);$

(iv) *there exist $e \in \mathfrak{A}$ such that $xe = ex = x$, where e is an identity element;*

(v) $\|e\| = 1;$

(vi) $\|xy\| \leq \|x\| \|y\|.$

There is $x \in \mathfrak{A}$ is called invertible element if there exist x^{-1} such that $xx^{-1} = x^{-1}x = e$

Definition 2 Let M be a non-empty set and the mapping $d_B : \Omega \times M \rightarrow \mathcal{P}$, where \mathcal{P} is a cone, $\omega \in \Omega$ be a selector, satisfy the following conditions:

1. $d_B(x(\omega), y(\omega)) \geq 0$ and $d_B(x(\omega), y(\omega)) = 0$ if and only if $x(\omega) = y(\omega)$ for all $x(\omega), y(\omega) \in \Omega \times M$,
2. $d_B(x(\omega), y(\omega)) = d_B(y(\omega), x(\omega))$ for all $x, y \in M, \omega \in \Omega$ and $x(\omega), y(\omega) \in \Omega \times M$,
3. $d_B(x(\omega), y(\omega)) \leq d_B(x(\omega), z(\omega)) + d_B(z(\omega), y(\omega))$ for all $x, y \in M$ and $\omega \in \Omega$ be a selector,
4. for any $x, y \in X, \omega \in \Omega, d_B(x(\omega), y(\omega))$ is non-increasing and left continuous.

Then d_B is called random cone metric on M and (M, d_B) is called a random cone metric space (RCMS).

Proposition 1 [6] Let \mathfrak{A} be a Banach algebra and let $x \in \mathfrak{A}$. If the spectral radius $\rho(x)$ of x is less than 1, i.e.,

$$\rho(x) = \lim_{n \rightarrow \infty} \|x^n\|^{1/n} = \inf_{n \geq 1} \|x^n\|^{1/n} < 1 \quad (1)$$

then $e - x \in \text{Inv}(\mathfrak{A})$, where $\text{Inv}(\mathfrak{A})$ is the set of all invertible elements of \mathfrak{A} and

$$(e - x)^{-1} = \sum_{n=0}^{\infty} x^n \quad (2)$$

Remark 1 Let \mathfrak{A} be a Banach algebra. Then the following are satisfied

- (i) for $x \in A, \rho(x) \leq \|x\|$,
- (ii) In proposition 1, $\rho(x) < 1$ is replaced by $\|x\| < 1$.

Definition 3 Let \mathfrak{A} be a Banach algebra and \mathcal{P} be a subset of \mathfrak{A} is called cone if the followings are satisfied:

1. \mathcal{P} is non-empty and closed subset of \mathfrak{A} and $\{0, e\} \in \mathcal{P}$;
2. $ax + by \in \mathcal{P}$, whenever $x, y \in \mathcal{P}$ and $a, b \in \mathbb{R}(a, b \geq 0)$;
3. $\mathcal{P}^2 = \mathcal{P}\mathcal{P}$;
4. $\mathcal{P} \cap (-\mathcal{P}) = 0$.

Given a cone $\mathcal{P} \subset \mathfrak{A}$, we define a partial ordering \leq with respect to \mathcal{P} by $x \leq y$ if and only if $y - x \in \mathcal{P}$.

We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}(\mathcal{P})$, where $\text{int}(\mathcal{P})$ denotes the interior of \mathcal{P} . A cone is \mathcal{P} called normal if there exist a number $K \geq 1$ such that for all $x, y \in \mathfrak{A}$, $\|x\| \leq K\|y\|$ whenever $0 \leq x \leq y$.

A cone is \mathcal{P} called regular if every increasing sequence which is bounded from above is converges. if v_n is a sequence such that form some $v \in \mathfrak{A}$

$$v_1 \leq v_2 \leq \dots \leq v. \tag{3}$$

Then there exist $v \in \mathfrak{A}$ such that

$$\lim_{n \rightarrow \infty} \|v_n - v\| = 0 \tag{4}$$

equivalently, a cone \mathcal{P} is regular if and only if every decreasing sequence which is bounded from below is convergent.

Also every regular cone is normal.

Definition 4 Let (M, d_B) be a RCMS and Φ be the class of all continuous function $\phi : \text{int}(\mathcal{P}) \cup \{0\} \rightarrow \text{int}(\mathcal{P}) \cup \{0\}$ satisfy the following:

$$(\phi_1) \quad \phi^{-1}(0) = 0;$$

$$(\phi_2) \quad \phi(t) \ll t \text{ for all } t \in \text{int}(\mathcal{P}).$$

Remark 2 Let \mathfrak{A} be a Banach algebra and $\mathcal{P} \subset \mathfrak{A}$ be a cone. If $x \in \mathcal{P}$ and $\rho(x) < 1$, then $e - x \in \text{Inv}(\mathfrak{A})$

Example 1 Let $K > 1$ be given. Consider the real vector space with

$$E = \{ax + b : a, b \in R; x \in [1 - \frac{1}{k}, 1]\}$$

with supremum norm and the cone

$$P = \{ax + b : a \geq 0, b \leq 0\}$$

in E . The cone P is regular and so normal.

Definition 5 Let (M, d_B) be a random cone metric space(RCMS) and $\{x_n(\omega)\} \subset \Omega \times M$ be a sequence and $x(\omega) \in \Omega \times M$. The it satisfies the followings:

1. $\lim_{n \rightarrow \infty} x_n(\omega) = x(\omega)$ if and only if for any $r \in \text{int}(\mathcal{P})$, there exists N such that for all $n > N$, $r - d_B(x_n(\omega), x(\omega)) \in \text{int}(\mathcal{P})$ -
2. $\{x_n(\omega)\}$ is a Cauchy sequence if and only if for any $r \in \text{int}(\mathcal{P})$ there exist N such that for all $n, m > N$, $r - d_B(x_n(\omega), x_m(\omega)) \in \text{int}(\mathcal{P})$.

Note that if $\lim_{n \rightarrow \infty} d_B(x_n(\omega), x(\omega)) = 0$, then $\lim_{n \rightarrow \infty} x_n(\omega) = x(\omega)$. The converse is true if \mathcal{P} is a normal cone. Also, if \mathcal{P} is a normal cone, then $\{x_n(\omega)\}$ is a Cauchy sequence in $\Omega \times M$ if and only if $\lim_{n, m \rightarrow \infty} d_B(x_n(\omega), x_m(\omega)) = 0$.

Definition 6 Let Ψ be the family of all continuous function $\psi: \mathcal{P} \rightarrow \mathcal{P}$ such that

(ψ_1) ψ is strictly increasing, i.e, $x < y \iff \psi(x) < \psi(y)$;

(ψ_2) $\psi^{-1}(\{0\}) = 0$ Note that, if $\psi(x) \leq \psi(y)$, then $x \leq y$.

Definition 7 Let \mathcal{C} be a family of all continuous functions $f: \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$ such that

(1) $f(s, t) \leq s \quad \forall s, t \in \mathcal{P}$

(2) $f(s, t) \leq t \implies$ either $s = 0$ or $t = 0 \quad \forall s, t \in \mathcal{P}$

Then we say that $f \in \mathcal{C}$ is C -class function.

Note that $f(0, 0) = 0$.

Definition 8 (Measurable Function)

Let (Ω, Σ) be a measurable space with Σ - a sigma algebra of subsets of Ω and M be a non-empty subset of a metric space $X = (X, d)$. Let 2^M be the family of non-empty subsets of M and $C(M)$ the family of all non-empty closed subsets of M . A mapping $G: \Omega \rightarrow 2^M$ is called measurable if for each open subset U of M , $G^{-1}(U) \in \Sigma$, where $G^{-1}(U) = \{\omega \in \Omega : G(\omega) \cap U \neq \emptyset\}$

Definition 9 (Measurable Selector)

A mapping $\xi: \Omega \rightarrow M$ is called a measurable selector of a measurable mapping $G: \Omega \rightarrow 2^M$ if ξ is measurable and $\xi(\omega) \in G(\omega)$ for each $\omega \in \Omega$

Definition 10 (Random Operator)

The mapping $T: \Omega \times M \rightarrow X$ is said to be a random operator if and only if for each fixed $x \in M$, the mapping $T(\cdot, x): \Omega \rightarrow X$ is measurable.

Definition 11 (Continuous Random Operator)

A random operator $T : \Omega \times M \rightarrow X$ is said to be continuous random operator if for each fixed $x \in M$, and $\omega \in \Omega$, the mapping $T(\omega, \cdot) : X \rightarrow X$ is continuous.

Definition 12 (Random fixed point)

A measurable mapping $\xi : \Omega \rightarrow M$ is a random fixed point of a random operator $T : \Omega \times M \rightarrow X$ if and only if $T(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$

Example 2 Let $M = R$ and $P = \{x \in M : x \geq 0\}$, also $\Omega = [0, 1]$ and Σ be the sigma algebra of Lebegsue's measurable subset of $[0, 1]$. Let $X = [0, \infty)$ and define a mapping $d_B : (\Omega \times X) \times (\Omega \times X) \rightarrow P$ by $d_B(x(\omega), y(\omega)) = |x(\omega) - y(\omega)|$. Then (X, d_B) is a random cone metric space.

Lemma 1 [8] Let (M, d_B) be a random cone metric space with regular cone \mathcal{P} such that $d_B(x(\omega), y(\omega)) \in \text{int}(\mathcal{P})$ for $x \neq y$. Let $\phi \in \Phi$. If the sequence $\{x_n(\omega)\}$ is not Cauchy sequence, then there exist $r \in \text{int}(\mathcal{P})$ and two sub sequence $\{x_{m(k)}(\omega)\}$ and $\{x_{n(k)}(\omega)\}$ of $\{x_n(\omega)\}$ such that $\{m(k)\}$ is the smallest index for which $m(k) > n(k) > k$,

$$\phi(r) - d_B(x_{m(k)}(\omega), x_{n(k)}(\omega)) \notin \text{int}(\mathcal{P}) \tag{5}$$

and

$$\phi(r) - d_B(x_{m(k)-1}(\omega), x_{n(k)}(\omega)) \in \text{int}(\mathcal{P}) \tag{6}$$

Moreover if $\lim_{n \rightarrow \infty} d_B(x_n(\omega), x_{n+1}(\omega)) = 0$, then we have

$$(1) \lim_{n \rightarrow \infty} d_B(x_{m(k)}(\omega), x_{n(k)}(\omega)) = \phi(r)$$

$$(2) \lim_{n \rightarrow \infty} d_B(x_{m(k)+1}(\omega), x_{n(k)+1}(\omega)) = \phi(r).$$

Let (M, d_B) be a RCMS . A mapping $T : \Omega \times M \rightarrow M$ is called continuous $x(\omega) \in M$ if for any $V \in \tau$ containing $Tx(\omega)$, there exist $U \in \tau$ containing $x(\omega)$ such that $TU \subset V$, where τ is the topology on M induced by the random cone metric d_B . that is

$$\begin{aligned} \tau &= \{U \subset M : \forall x(\omega) \in U, \exists B \in \beta, x(\omega) \in B \subset U\}, \\ \beta &= \{B(x(\omega), r) : x(\omega) \in M, \forall r \in \text{int}(\mathcal{P})\}, \\ B(x(\omega), r) &= \{y \in M : d_B(x(\omega), y(\omega)) \ll r\} \end{aligned}$$

Remark 3 Note, T is continuous if and only if it is sequentially continuous, i.e., $\lim_{n \rightarrow \infty} Tx_n(\omega) = Tx(\omega)$ for any sequence $\{x_n(\omega)\} \subset M$ with $\lim_{n \rightarrow \infty} x_n(\omega) = x(\omega)$, where (M, d_B) first countable topological space then converse holds.

A point is called cluster point of $\mathfrak{A} \subset M$ if for any $r \in \mathfrak{A}$ with $0 \ll r$,

$$(B(x(\omega), r) - [x(\omega)]) \cap \mathfrak{A} \neq \emptyset. \quad (7)$$

Definition 13 Let $\alpha, \beta : \Omega \times M \rightarrow \mathcal{P}$ be functions, where M is a set. We say that $T : \Omega \times M \rightarrow M$ is a cyclic (α, β) -admissible mapping if

- (1) $\alpha(x(\omega)) - e \in \mathcal{P}$, where $x \in M \implies \beta(Tx(\omega)) - e \in \mathcal{P}$
- (2) $\beta(x(\omega)) - e \in \mathcal{P}$, where $x(\omega) \in M \implies \alpha(Tx(\omega)) - e \in \mathcal{P}$

3 Main Result

Theorem 1 Let (M, d_B) be complete RCMS and let $T : \Omega \times M \rightarrow M$ be mapping such that

$$\psi(d_B(Tx(\omega), Ty(\omega))) \leq f(\psi(d_B(x(\omega), y(\omega))), \phi(d_B(x(\omega), y(\omega)))) \quad (8)$$

for all $x(\omega), y(\omega) \in \Omega \times M$ with $\alpha(x(\omega))\beta(y(\omega)) - e \in \mathcal{P}$, where $f \in \mathcal{C}$, $\psi \in \Psi$ and $\phi \in \Phi$.

Also, the following are satisfied

- (i) T is cyclic (α, β) -admissible;
- (ii) There exists $x_0(\omega) \in \Omega \times M$ such that $\alpha(x_0(\omega)) - e \in \mathcal{P}$ and $\beta(x_0(\omega)) - e \in \mathcal{P}$;
- (iii) Either T is continuous or if $\{x_n(\omega)\} \subset \Omega \times M$ is a sequence with $\beta(x_n(\omega)) - e \in \mathcal{P} \quad \forall n = 1, 2, 3, \dots$ and $x(\omega)$ is a cluster point of $\{x_n(\omega)\}$, then

$$\beta(x(\omega)) - e \in \mathcal{P}.$$

Then T has a fixed point. Moreover if

$$\alpha(x(\omega)) - e \in \mathcal{P} \quad \text{and} \quad \beta(x(\omega)) - e \in \mathcal{P} \quad (9)$$

for all fixed points $x(\omega), y(\omega)$ of T , then T has a unique fixed point.

Proof 1 Define a sequence $\{x_n(\omega)\}$ in $\Omega \times M$ by $x_n(\omega) = Tx_{n-1}(\omega) = T^n x_0(\omega)$ for all $n = 1, 2, 3, \dots$. If there exist N such that $x_N(\omega) = x_{N+1}(\omega)$, then $x_N(\omega) = Tx_N(\omega)$, i.e., $x_N(\omega)$ is a fixed point of T .

Hence we assume that $x_n(\omega) \neq x_{n+1}(\omega) \forall n = 1, 2, 3, \dots$.

For any $x_0(\omega) \in \Omega \times M$, then by (α, β) -admissible mapping we get $\alpha(x_0(\omega)) - e \in \mathcal{P}$, it implies that $\beta(Tx_0(\omega)) - e = \beta(x_1(\omega)) - e \in \mathcal{P}$. Again if $x_1(\omega) \in \Omega \times M$, then by (α, β) -admissible mapping we get $\alpha(x_2(\omega)) - e = \alpha(Tx_1(\omega)) - e \in \mathcal{P}$, it implies that $\beta(Tx_2(\omega)) - e = \beta(x_3(\omega)) - e \in \mathcal{P}$, inductively we have

$$\alpha(x_{2n}(\omega)) - e \in \mathcal{P} \quad \text{and} \quad \beta(x_{2n+1}(\omega)) - e \in \mathcal{P} \quad \forall n = 0, 1, 2, \dots \quad (10)$$

Similarly $\beta(x_0(\omega)) - e \in \mathcal{P}, \alpha(x_1(\omega)) - e \in \mathcal{P}$. On continuing this way we have,

$$\beta(x_{2n}(\omega)) - e \in \mathcal{P} \quad \text{and} \quad \alpha(x_{2n+1}(\omega)) - e \in \mathcal{P} \quad \forall n = 0, 1, 2, \dots \quad (11)$$

Hence,

$$\alpha(x_n(\omega)) - e \in \mathcal{P} \quad \text{and} \quad \beta(x_n(\omega)) - e \in \mathcal{P}. \quad (12)$$

Since, $\alpha(x_{n+1}(\omega)) - e \in \mathcal{P}$ and $\beta(x_n(\omega)) - e \in \mathcal{P}$

$$\alpha(x_{n+1}(\omega))\beta(x_n(\omega)) - \alpha(x_{n+1}(\omega)) - \beta(x_n(\omega)) + e = (\alpha(x_{n+1}(\omega)) - e)(\beta(x_n(\omega)) - e) \quad (13)$$

Also,

$$\alpha(x_{n+1}(\omega)) + \beta(x_n(\omega)) - 2e = (\alpha(x_{n+1}(\omega)) - e)(\beta(x_n(\omega)) - e) \in \mathcal{P}. \quad (14)$$

Hence,

$$\alpha(x_{n+1}(\omega))\beta(x_n(\omega)) - e \in \mathcal{P} \quad \forall n = 1, 2, 3, \dots \quad (15)$$

And so it follows from (8), we have

$$\begin{aligned} \psi(d_B(x_n(\omega), x_{n+1}(\omega))) &= \psi(d_B(Tx_{n-1}(\omega), Tx_n(\omega))) \\ &\leq f(\psi(d_B(x_{n-1}(\omega), x_n(\omega))), \phi(d_B(x_{n-1}(\omega), x_n(\omega)))) \\ &\leq \psi(d_B(x_{n-1}(\omega), x_n(\omega))) \quad \forall n = 1, 2, 3, \dots \end{aligned} \quad (16)$$

since, ψ is strictly increasing,

$$d_B(x_n(\omega), x_{n+1}(\omega)) \leq d_B(x_{n-1}(\omega), x_n(\omega)) \quad \forall n = 1, 2, 3, \dots \quad (17)$$

Hence $\{d_B(x_n(\omega), x_{n+1}(\omega))\}$ is a decreasing sequence. Since \mathcal{P} is regular, there exists $s \in \mathcal{P}$ such that

$$\lim_{n \rightarrow \infty} d_B(x_{n-1}(\omega), x_n(\omega)) = s \quad (18)$$

By lemma 3.1 of [8], we have

$$s \in \text{int}(\mathcal{P}) \cup \{0\} \quad (19)$$

Assume that $s \in \text{int}(\mathcal{P})$. By taking limit $n \rightarrow \infty$ in (16) and using continuity of f, ψ and ϕ , we obtain

$$\psi(s) \leq f(\psi(s), \phi(s)) \leq \psi(s) \quad (20)$$

which implies

$$\psi(s) - f(\psi(s), \phi(s)) \in \mathcal{P} \cap (-\mathcal{P}) \quad (21)$$

$$f(\psi(s), \phi(s)) = \psi(s). \quad (22)$$

Either $\psi(s) = 0$ or $\phi(s) = 0$. Hence $s = 0$, which is a contradiction to our assumption

$$s \notin \text{int}(\mathcal{P}). \quad (23)$$

Thus we have

$$\lim_{n \rightarrow \infty} d_B(x_n(\omega), x_{n-1}(\omega)) = 0 \quad (24)$$

We show that $\{x_n(\omega)\}$ is Cauchy sequence. On the contrary, assume $\{x_n(\omega)\}$ is not Cauchy sequence.

By lemma (1), there exists $r \in \text{int}(\mathcal{P})$ and two subsequences $\{x_{m(k)}(\omega)\}$ and $\{x_{n(k)}(\omega)\}$ of $\{x_n(\omega)\}$ such that $m(k)$ is the smallest index for which, for all $k \in \mathbb{N}$, $m(k) > n(k) > k$ from (12) we have, $\alpha(x_{m(k)})\beta(x_{n(k)}) - e \in \mathcal{P}$ and from (8) with $x(\omega) = x_{m(k)}(\omega)$ and $y(\omega) = x_{n(k)}(\omega)$, we have

$$\psi(d_B(Tx_{m(k)}(\omega), Tx_{n(k)}(\omega))) \leq f(\psi(d_B(x_{m(k)}(\omega), x_{n(k)}(\omega))), \phi(d_B(x_{m(k)}(\omega), x_{n(k)}(\omega)))) \quad (25)$$

By letting $k \rightarrow \infty$ and by using Lemma (1) and using continuity of ψ, ϕ and f , we have

$$\psi(\phi(r)) \leq f(\psi(\phi(r)), \phi(\phi(r))) \leq \psi(\phi(r)) \quad (26)$$

which implies

$$\psi(\phi(r)) - f(\psi(\phi(r)), \phi(\phi(r))) \in \mathcal{P} \cap (-\mathcal{P}) \quad (27)$$

Hence,

$$f(\psi(\phi(r)), \phi(\phi(r))) = \psi(\phi(r)) \quad (28)$$

which implies either $\psi(\phi(r)) = 0$ or $\phi(\phi(r)) = 0$. Thus $\phi(r) = 0$ and $r = 0$, which is a contradiction

Therefore $\{x(\omega)\}$ is a Cauchy sequence, it follows the completeness of M there exists

$$x(\omega) = \lim_{n \rightarrow \infty} x_n(\omega) \in \Omega \times M. \quad (29)$$

If it T is continuous, then $\lim_{n \rightarrow \infty} x_n(\omega) = Tx(\omega)$ and so $x(\omega) = Tx(\omega)$. Assume that (iii) holds, then we have

$$\beta x - e \in \mathcal{P} \quad (30)$$

Since T is (α, β) -admissible, $\alpha(x_n(\omega)) - e \in \mathcal{P} \quad \forall n = 1, 2, 3, \dots$

Using (12) we have

$$\alpha(x_n(\omega))\beta(x(\omega)) - e \in \mathcal{P} \quad \forall n = 1, 2, 3, \dots \quad (31)$$

From (8) we have

$$\psi(d_B(Tx_{n+1}(\omega), Tx(\omega))) \leq f(\psi(d_B(x_n(\omega), Tx(\omega))), \phi(d_B(x_n(\omega), x(\omega)))) \quad (32)$$

Letting $n \rightarrow \infty$ in (32) and using continuity of f, ψ and ϕ , we have

$$\begin{aligned} \psi(d_B(x(\omega), Tx(\omega))) &\leq f(\psi(0), \phi(0)) = 0, \\ \psi(d_B(x(\omega), Tx(\omega))) &\in -\mathcal{P} \end{aligned} \quad (33)$$

which implies

$$\psi(d_B(x(\omega), Tx(\omega))) \in \mathcal{P} \cap -\mathcal{P} \quad (34)$$

Thus

$$\begin{aligned} \psi(d_B(x(\omega), Tx(\omega))) &= 0 \\ d_B(x(\omega), Tx(\omega)) &= 0 \end{aligned} \quad (35)$$

Hence

$$x(\omega) = Tx(\omega)$$

For the uniqueness of fixed point, suppose we have another fixed point $x'(\omega)$ then from (9) $\alpha(x(\omega)) - e \in \mathcal{P}$ and $\beta(x'(\omega)) - e \in \mathcal{P}$. Hence $\alpha(x(\omega))\beta(x'(\omega)) - e \in \mathcal{P}$.

From (8), we have

$$\begin{aligned} \psi(d_B(x(\omega), x'(\omega))) &= \psi(d_B(Tx, Tx')) \\ &\leq f(\psi(d_B(x(\omega), x'(\omega))), \phi(d_B(x(\omega), x'(\omega)))) \\ &\leq \psi(d_B(x(\omega), x'(\omega))) \end{aligned}$$

This implies that

$$\psi(d_B(x(\omega), x'(\omega))) - f(\psi(d_B(x(\omega), x'(\omega))), \phi(d_B(x(\omega), x'(\omega)))) \in \mathcal{P} \cap -\mathcal{P}. \quad (36)$$

thus

$$f(\psi(d_B(x(\omega), x'(\omega))), \phi(d_B(x(\omega), x'(\omega)))) = \psi(d_B(x(\omega), x'(\omega))) \quad (37)$$

which implies

either $\psi(d_B(x(\omega), x'(\omega))) = 0$ or $\phi(d_B(x(\omega), x'(\omega))) = 0$.

So $d(x(\omega), x'(\omega)) = 0$

Hence

$x(\omega) = x'(\omega)$

Example 3 Let $\mathfrak{A} = \mathbb{R}^2$, $\mathcal{P} = \{(x_1(\omega), x_2(\omega)) \in \mathfrak{A} : x_1(\omega), x_2(\omega) \geq 0\}$ and $\|x(\omega)\| = |x_1(\omega)| + |x_2(\omega)| \quad \forall x = (x_1(\omega), x_2(\omega)) \in \mathfrak{A}$.

Define the multiplication xy of $x = (x_1(\omega), x_2(\omega))$ and $y = (y_1(\omega), y_2(\omega))$,

$$xy = (x_1(\omega)y_1(\omega) - x_2(\omega)y_2(\omega) - x_2(\omega)y_1(\omega)).$$

Then \mathfrak{A} is a Banach algebra with unit $e = (0, 1)$, \mathcal{P} is regular cone and $\text{int}(\mathcal{P}) = \{(x_1(\omega), x_2(\omega)) \in \mathfrak{A} : x_1(\omega), x_2(\omega) > 0\}$

Let $M = \mathbb{R}^2$ and $d_B : \Omega \times M \rightarrow \mathcal{P}$ be a mapping defined by

$$d(x, y) = \max\{|x_1(\omega) - y_2(\omega)|, |x_2(\omega) - y_1(\omega)|\}.$$

Then (M, d_B) is a complete RCMS and $d(x(\omega), y(\omega)) \in \text{int}(\mathcal{P}) \quad \forall x(\omega), y(\omega) \in \Omega \times M$ with $x(\omega) \neq y(\omega)$. Define a mapping $T : \Omega \times M \rightarrow \mathcal{P}$ by

$$Tx = \begin{cases} \frac{1}{10}x, & x = (x_1(\omega), x_2(\omega)) \in \mathcal{P} \quad \text{and} \quad \|x\| < 1 \\ e^{(x_1(\omega), x_2(\omega))}, & \text{otherwise.} \end{cases}$$

Let $\phi(t) = \frac{1}{3}t$ and $\psi(t) = t \ \forall \ t = (t_1, t_2) \in \mathcal{P}$.

We define two functions $\alpha, \beta : M \rightarrow \mathcal{P}$ by

$$\alpha(x) = \beta(x) = \begin{cases} e, & x = (x_1, x_2) \in \mathcal{P} \text{ and } \|x\| < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, T is (α, β) -admissible. Let $f(s, t) = s - \phi(s) \ \forall \ s, t \in \mathcal{P}$, then we have

$$\begin{aligned} & f(\psi(d_B(x(\omega), y(\omega))), \phi(d_B(x(\omega), y(\omega)))) - \psi(d_B(Tx(\omega), Ty(\omega))) \\ &= f\left(\max\{|x_1(\omega) - y_2(\omega)|, |x_2(\omega) - y_1(\omega)|\}, \frac{1}{3} \max\{|x_1(\omega) - y_2(\omega)|, |x_2(\omega) - y_1(\omega)|\}\right) \\ &\quad - \frac{1}{10} \max\{|x_1(\omega) - y_2(\omega)|, |x_2(\omega) - y_1(\omega)|\} \\ &= \max\{|x_1(\omega) - y_2(\omega)|, |x_2(\omega) - y_1(\omega)|\} - \frac{1}{3} \max\{|x_1(\omega) - y_2(\omega)|, |x_2(\omega) - y_1(\omega)|\} \\ &\quad - \frac{1}{10} \max\{|x_1(\omega) - y_2(\omega)|, |x_2(\omega) - y_1(\omega)|\} \\ &= \max\{|x_1(\omega) - y_2(\omega)|, |x_2(\omega) - y_1(\omega)|\} - \frac{1}{3} \max\{|x_1(\omega) - y_2(\omega)|, |x_2(\omega) - y_1(\omega)|\} \\ &\quad - \frac{1}{10} \max\{|x_1(\omega) - y_2(\omega)|, |x_2(\omega) - y_1(\omega)|\} \\ &= \frac{17}{30} \max\{|x_1(\omega) - y_2(\omega)|, |x_2(\omega) - y_1(\omega)|\} \end{aligned}$$

Hence, $\psi(d_B(Tx(\omega), Ty(\omega))) \leq f(\psi(d_B(x(\omega), y(\omega))), \phi(d_B(x(\omega), y(\omega)))) \ \forall \ x(\omega), y(\omega) \in \Omega \times M$ with $\alpha(x(\omega))\beta(y(\omega)) - e \in \mathcal{P}$. Let $\{x_n(\omega)\} = \left| \frac{1}{n+6} \right| + \left| \frac{3}{n+6} \right| \ \forall n = 1, 2, 3, \dots$

Then $x_n \in \mathcal{P}$ and $\|x_n\| < 1 \ \forall n = 1, 2, 3, \dots$ Hence

$$\beta(x_n) - e = (0, 0) \in \mathcal{P} \ \forall n = 1, 2, 3, \dots$$

Then $\lim_{n \rightarrow \infty} x_n = (0, 0)$ and $(0, 0)$ is a cluster point of $\{x_n\}$,

$$\beta(x_n) - e = (0, 0) \in \mathcal{P}.$$

Hence, all the condition of theorem (1) are satisfied and $(0, 0)$ is the fixed point of T .

Corollary 1 Let M be complete RCMS and let $T : \Omega \times M \rightarrow M$ be a mapping such that

$$\psi(d_B(\alpha(x(\omega))\beta(x(\omega))Tx(\omega), Ty(\omega))) \leq f(\psi(d_B(x(\omega), Tx(\omega))), \phi(d_B(y(\omega), Ty(\omega))))(38)$$

for all $x(\omega), y(\omega) \in \Omega \times M$, where $\psi \in \Psi$ and $\phi \in \Phi$.

Also, suppose that conditions (i), (ii), and (iii) of Theorem (1) hold. Then T has a fixed point.

Moreover, if (9) holds, then T has a unique fixed point.

Corollary 2 Let M be complete RCMS and let $T : \omega \times M \rightarrow M$ be a mapping such that

$$\psi(d_B(\alpha(x(\omega))\beta(x(\omega))Tx(\omega), Ty(\omega))) \leq f(\psi(d_B(x(\omega), Tx(\omega))), \phi(d_B(y(\omega), Tx(\omega))))(39)$$

for all $x(\omega), y(\omega) \in \Omega \times M$, where $\psi \in \Psi$ and $\phi \in \Phi$.

Also, suppose that conditions (i), (ii), and (iii) of Theorem (1) hold. Then T has a fixed point.

Moreover, if (9) holds, then T has a unique fixed point.

4 Applications

Theorem 2 Let (M, d_B) be complete RCMS and let $T : \Omega \times M \rightarrow M$ be (α, β) -admissible mapping such that

$$\psi(d_B(Tx(\omega), Ty(\omega))) \leq \psi(d_B(x(\omega), y(\omega))) - \phi(d_B(x(\omega), y(\omega))) \quad (40)$$

for all $x(\omega), y(\omega) \in \Omega \times M$ with $\alpha(x(\omega))\beta(x(\omega)) - e \in \mathcal{P}$, where $\psi \in \Psi$ and $\phi \in \Phi$ such that

$$\phi(\psi(t)) \leq \phi(t) \quad \forall t > 0 \quad (41)$$

$x_0(\omega) \in \Omega \times M$ such that $\alpha(x_0(\omega)) - e \in \mathcal{P}$ and $\beta(x_0(\omega)) - e \in \mathcal{P}$ Suppose that $\alpha(x_0(\omega)) - e \in \mathcal{P}$ and $\beta(x_0(\omega)) - e \in \mathcal{P}$, where $x_0(\omega) \in \Omega \times M$ such that Assume that either

(1) T is continuous or

(2) if $\{x_n(\omega)\} \subset \Omega \times M$ is a sequence such that $\lim_{n \rightarrow \infty} d_B(x_n(\omega), x(\omega)) = 0$ and $\beta(x_n(\omega)) - e \in \mathcal{P} \quad \forall n = 1, 2, 3, \dots$, then $\beta(x(\omega)) - e \in \mathcal{P}$.

Then T has a fixed point in M . If $\alpha(x(\omega))\beta(y(\omega)) - e \in \mathcal{P}$ for all fixed points $x(\omega), y(\omega)$ of T , then T has unique fixed point.

Proof 2 Let $f(s, t) = s - \phi(s)$, $\forall s, t \in \mathcal{P}$ where $k < \frac{3}{5}$. Then f is the C -class function.

For any $x(\omega), y(\omega) \in \Omega \times M$ with $\alpha(x(\omega))\beta(x(\omega)) - e \in \mathcal{P}$

$$\begin{aligned} 0 &\leq \psi(d_B(x(\omega), y(\omega))) - \psi(d_B(x(\omega), y(\omega))) - \psi(d_B(Tx(\omega), Ty(\omega))) \\ &\leq \psi(d_B(x(\omega), y(\omega))) - \phi(\psi(d_B(x(\omega), y(\omega)))) - \psi(d_B(Tx(\omega), Ty(\omega))) \\ &= f(\psi(d_B(x(\omega), y(\omega))), \phi(d_B(x(\omega), y(\omega)))) - \psi(d_B(Tx(\omega), Ty(\omega))). \end{aligned}$$

This result is same as from (8) and also it satisfies all the conditions of theorem (1).

Corollary 3 Let (M, d_B) be complete RCMS and let $T : \Omega \times M \rightarrow M$ be (α, β) -admissible mapping such that

$$d_B(Tx(\omega), Ty(\omega)) \leq d_B(x(\omega), y(\omega)) - \phi(d_B(x(\omega), y(\omega))) \quad (42)$$

for all $x(\omega), y(\omega) \in \Omega \times M$ with $\alpha(x(\omega))\beta(x(\omega)) - e \in \mathcal{P}$, where $\phi \in \Phi$. Suppose that $\alpha(x_0(\omega)) - e \in \mathcal{P}$ and $\beta(x_0(\omega)) - e \in \mathcal{P}$, where $x_0(\omega) \in \Omega \times M$ such that

Assume that either

(1) T is continuous or

(2) if $\{x_n(\omega)\} \subset \Omega \times M$ is a sequence such that $\lim_{n \rightarrow \infty} d_B(x_n(\omega), x(\omega)) = 0$ and $\beta(x_n(\omega)) - e \in \mathcal{P} \forall n = 1, 2, 3, \dots$, then $\beta(x(\omega)) - e \in \mathcal{P}$.

Then T has a fixed point in M . Further if $\alpha(x(\omega))\beta(x(\omega)) - e \in \mathcal{P}$ for all fixed points $x(\omega), y(\omega)$ of T , then T has unique fixed point.

By taking $\alpha(x(\omega)) = \beta(x(\omega)) = e$ in theorem (2), we have the following results.

Next, take $\alpha(x(\omega)) = e$ and $\beta(x(\omega)) = e \forall x(\omega) \in \Omega \times M$ in theorem (2), we have the following corollaries.

Corollary 4 Let M be complete RCMS and let $T : \Omega \times M \rightarrow M$ be a mapping such that

$$\psi(d_B(Tx(\omega), Ty(\omega))) \leq \psi(d_B(x(\omega), y(\omega))) - \phi(d_B(x(\omega), y(\omega))) \quad (43)$$

for all $x(\omega), y(\omega) \in \Omega \times M$, where $\psi \in \Psi$ and $\phi \in \Phi$ with $\phi(\psi(t)) \leq \phi(t) \forall t > 0$.

Then T has unique fixed point in $\Omega \times M$

Corollary 5 *Let M be complete RCMS and let $T : \Omega \times M \rightarrow M$ be a mapping such that*

$$d_B(Tx(\omega), Ty(\omega)) \leq d_B(x(\omega), y(\omega)) - \phi(d_B(x(\omega), y(\omega))) \quad (44)$$

for all $x(\omega), y(\omega) \in \Omega \times M$, where $\phi \in \Phi$. Then T has unique fixed point in $\Omega \times M$.

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