

Inference on the Parameters of Zubair-Exponential Distribution with Application to Survival Times of Guinea Pigs

Abstract

In this paper, we derived a sub-model of Zubair-G family of distribution named Zubair-Exponential distribution with two parameters. Simulation of the Estimates of the parameters based on some classical methods are obtained. The likelihood equations and the maximum likelihood estimator as well as asymptotic confidence interval are derived. Bayes estimates with the estimates of the associated greatest posterior density credible interval are derived using squared error Loss (SEL), Linear-Exponential (LINEX) and Generalized Entropy Loss (GEL) functions. Using the Metropolis-Hasting algorithm and the method of Markov Chain Monte Carlo (MCMC), estimates of Bayes are summarized. To determine the performance of the estimates, a Monte Carlo simulation study is carried out and maximum likelihood estimates, their standard errors and measures of fitness using real data on survival times of Guinea pigs are obtained. The proposed distribution has a better fit based on Akaike Information criterion (AIC) and the Bayesian Information criterion (BIC).

Keywords— Zubair-Exponential distribution, Bayesian estimation, Maximum likelihood estimation, Markov Chain Monte Carlo, Squared Error Loss function, Linear-Exponential loss function, Generalized Entropy loss function

1 Introduction

To achieve a better fit, economy of parameter, flexibility and tractability of model, various authors have explored avenues of providing compact distributions that model lifetime data. Among them are Zubair [1], Onyekwere and Obulezi [2], Onyekwere et al [3]. Some authors have further studied various classical methods of estimation. Dey [4] studied the maximum likelihood estimation procedure together with Maximum product of spacing, Cramer von mises, Least squares, weighted least squares, Anderson Darling, Right-Tailed Anderson Darling and Percentile estimation methods. Recently, there are more advances in Bayesian estimation procedures. However, in Bayesian estimation, there are not many approaches as there are for classical estimation. The loss functions used in Bayesian estimation are generally grouped into symmetric and asymmetric functions. The Squared Error Loss function (SEL) is popular symmetric loss function because it attaches equal loss weight to over-and under-estimation. Next, is the Linear Exponential (LINEX) loss function which is asymmetric was proposed by Varian [5] and the Generalized Entropy Loss (GEL) function.

Nikolay et al [6] extended the work of Zubair [1] with manipulations on the cumulative lifetime function to derive theorems that lends Zubair-G family of distributions to wider applications. Pavlov [7] used cumulative distribution function of the Lomax distribution as a baseline to have a new submodel of the Zubair-G family of distributions. Nasiru and Abubakari [8] proposed the Marshall-Olkin Zubair-G family of distributions as a modified extension. Rahneva et al [9] worked on Investigations on the Zubair-family with baseline Ghosh-Bourguignon's extended Burr XII cumulative sigmoid.

This work is motivated by the Zubair-G family of distributions proposed by Zubair [1]. The baseline here being the well-known Exponential distribution which compares favourable with the sub-model Zubair-Weibull distribution, an extension studied by Zubair [1].

The rest of the paper is organized as follows; the probability density and the associated functions of Zubair-Exponential distribution is presented in section 2, in section 3, we derive the mathematical properties. Classical estimation of the parameters is carried out in section 4 while the Bayesian estimation is studied in section 5. In the end, section 6 is dedicated to simulation study and real life data analysis with conclusion following it.

2 Zubair-Exponential Distribution

Let $X \sim \text{Zubair} - G(\alpha, \xi)$ due to Zubair [1] then using Exponential distribution as a baseline distribution the probability density function (pdf), cumulative distribution function (cdf), survival function (sf) and hazard rate function (hrf), reversed hazard rate function (rhrf) and Odd function (chrf) of Zubair-Exponential distribution are respectively

$$f(x; \alpha, \theta) = \frac{2\alpha\theta e^{-\theta x}(1 - e^{-\theta x})e^{\alpha(1 - e^{-\theta x})^2}}{e^\alpha - 1}; \quad x, \quad \alpha, \quad \theta > 0 \quad (1)$$

$$F(x; \alpha, \theta) = \frac{e^{\alpha(1 - e^{-\theta x})^2} - 1}{e^\alpha - 1} \quad (2)$$

$$S(x; \alpha, \theta) = \frac{e^\alpha - e^{\alpha(1 - e^{-\theta x})^2}}{e^\alpha - 1} \quad (3)$$

$$h(x; \alpha, \theta) = \frac{2\alpha\theta e^{-\theta x}(1 - e^{-\theta x})e^{\alpha(1 - e^{-\theta x})^2}}{e^\alpha - e^{\alpha(1 - e^{-\theta x})^2}} \quad (4)$$

$$rh(x; \alpha, \theta) = \frac{2\alpha\theta e^{-\theta x}(1 - e^{-\theta x})e^{\alpha(1 - e^{-\theta x})^2}}{e^{\alpha(1 - e^{-\theta x})^2} - 1} \quad (5)$$

and

$$O(x; \alpha, \theta) = \frac{e^{\alpha(1 - e^{-\theta x})^2} - 1}{e^\alpha - e^{\alpha(1 - e^{-\theta x})^2}} \quad (6)$$

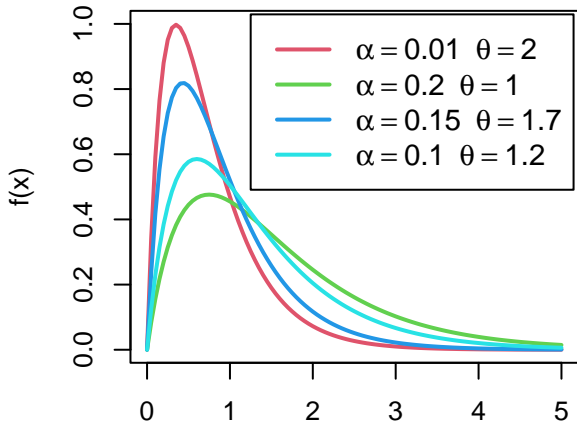


fig 1a: pdf of Zubair-Exponential distribution

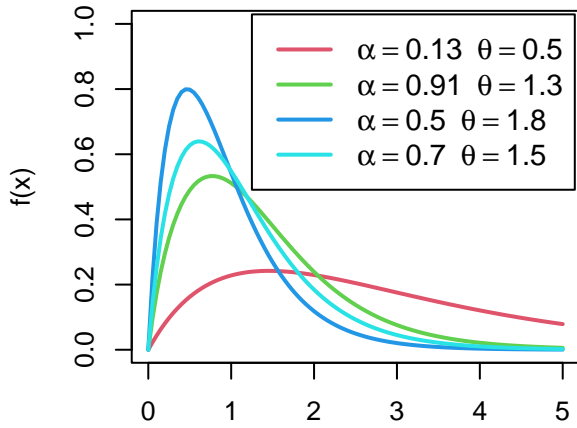


fig 1b: pdf of Zubair-Exponential distribution

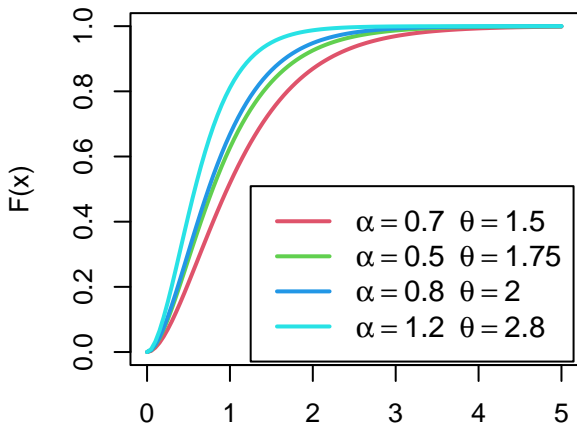


fig 1c: cdf of Zubair-Exponential distribution

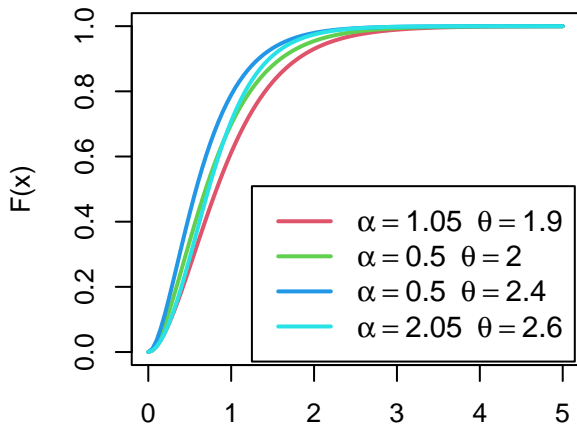


fig 1d: cdf of Zubair-Exponential distribution

Figure 1: pdf and cdf of Zubair-Exponential distribution

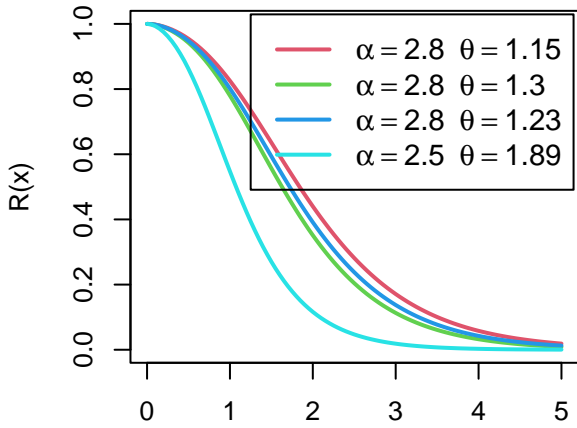


fig 2a: Reliability funct. of Zubair-Exponential dist

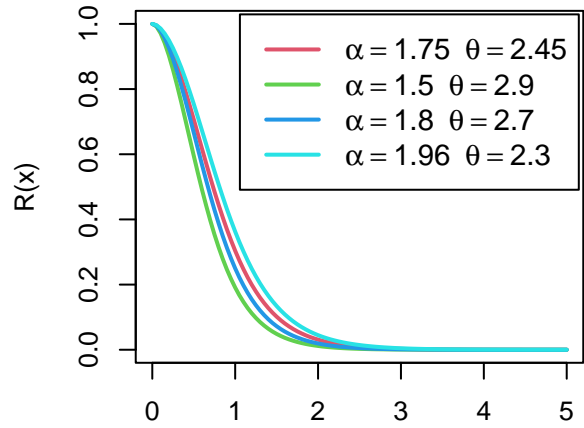


fig 2b: Reliability funct. of Zubair-Exponential dist

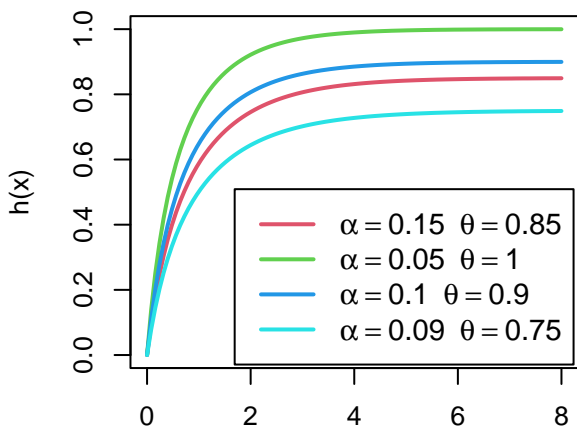


fig 2c: hazard function of Zubair-Exponential dist

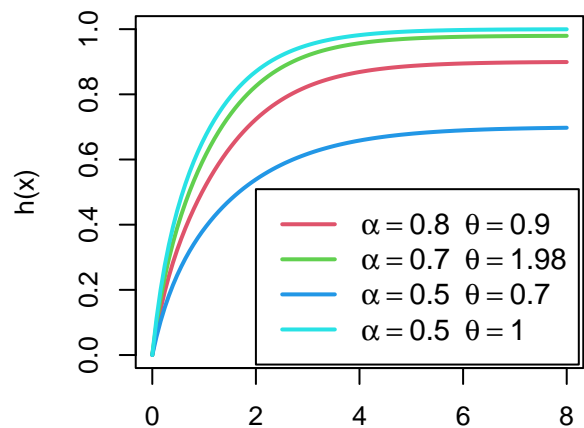


fig 2d: hazard function of Zubair-Exponential dist

Figure 2: Survival and Hazard rate function of Zubair-Exponential distribution

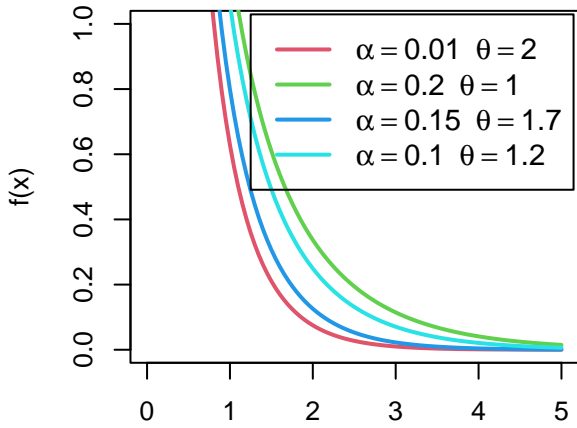


fig 1a: rhrf of Zubair-Exponential dist

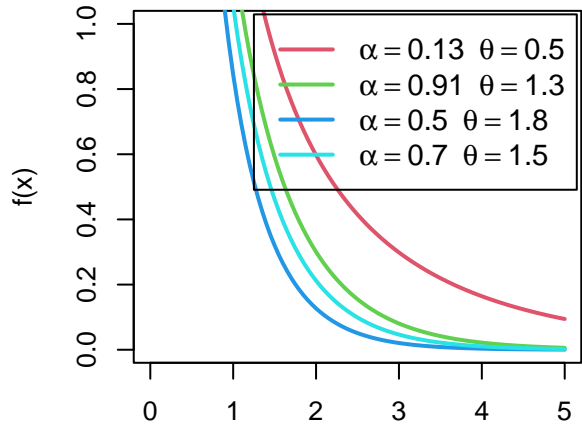


fig 1b: rhrf of Zubair-Exponential distribution

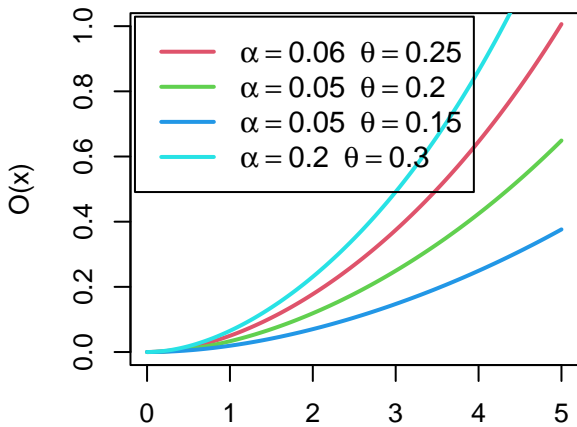


fig 1c: Odd func. Zubair-Exponential distribution

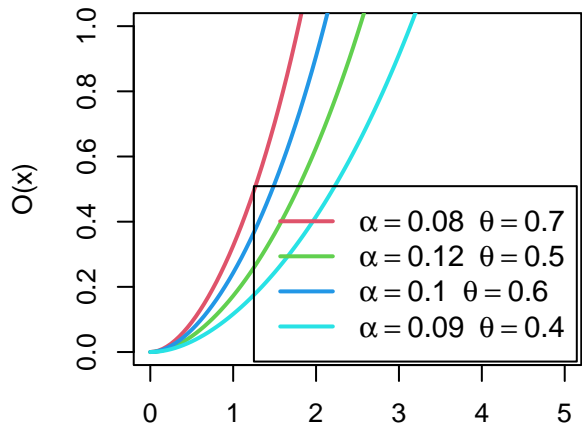


fig 1d: Odd func. of Zubair-Exponential distribution

Figure 3: Reversed hazard rate and Odd functions of Zubair-Exponential distribution

3 Mathematical Properties

In this section, some basic mathematical properties are derived

3.1 Moments

The r^{th} non-central moment of $X \sim$ Zubair-Exponential $(x; \alpha, \theta)$ is

$$\mu_r = \int_0^{\infty} x^r f(x; \alpha, \theta) dx \quad (7)$$

Zubair [1] further expressed the moment as

$$\mu_r' = 2 \sum_{i=0}^{\infty} \frac{\alpha^{i+1}}{(\alpha - 1)i!} \eta_{r;2i+1} \quad (8)$$

where

$$\eta_{r;2i+1} = \int_0^{\infty} x^r \theta e^{-\theta x} (1 - e^{-\theta x})^{2i+1} dx = \theta \sum_{j=0}^{\infty} \binom{2i+1}{j} \frac{\Gamma(r+1)}{(\theta(j+1))^{r+1}} \quad (9)$$

Therefore, the closed-form of the r^{th} non-central moment is

$$\mu_r' = \frac{2}{\theta^r} \sum_{i,j,r=0}^{\infty} \frac{\alpha^{i+1}}{(\alpha - 1)i!} \binom{2i+1}{j} \frac{\Gamma(r+1)}{(j+1)^{r+1}} \quad (10)$$

The mean is

$$\mu = \frac{2}{\theta} \sum_{i,j=0}^{\infty} \frac{\alpha^{i+1}}{(\alpha - 1)i!} \binom{2i+1}{j} \frac{1}{(j+1)^2} \quad (11)$$

The 2nd, 3rd and 4th non-central moments are

$$\mu_2 = \frac{6}{\theta^2} \sum_{i,j=0}^{\infty} \frac{\alpha^{i+1}}{(\alpha - 1)i!} \binom{2i+1}{j} \frac{1}{(j+1)^3} \quad (12)$$

$$\mu_3 = \frac{12}{\theta^3} \sum_{i,j=0}^{\infty} \frac{\alpha^{i+1}}{(\alpha - 1)i!} \binom{2i+1}{j} \frac{1}{(j+1)^4} \quad (13)$$

and

$$\mu_4 = \frac{24}{\theta^4} \sum_{i,j=0}^{\infty} \frac{\alpha^{i+1}}{(\alpha - 1)i!} \binom{2i+1}{j} \frac{1}{(j+1)^5} \quad (14)$$

The moment generating function is

$$M_X(t) = 2 \sum_{i,r=0}^{\infty} \frac{t^r \alpha^{i+1}}{(\alpha - 1)r!i!} \eta_{r;2i+1} \quad (15)$$

Substitute equation 9 in 15 to have

$$M_X(t) = \frac{2}{\theta^r} \sum_{i,j,r=0}^{\infty} \frac{t^r \alpha^{i+1}}{(\alpha - 1)r!i!} \binom{2i+1}{j} \frac{\Gamma(r+1)}{(j+1)^{r+1}} \quad (16)$$

3.2 Order Statistics

Let X_1, X_2, \dots, X_n be a random sample of size n drawn from the Zubair-Exponential (α, θ) . Let $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be **that** associated order statistics, then the density of $X_{r:n}$ where $r \in (1, 2, \dots, n)$ is obtained as

$$f_{r:n}(x) = \frac{f(x, \alpha, \theta)}{\mathcal{B}(r, n-r+1)} \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i [F(x, \alpha, \theta)]^{i+r-1} \quad (17)$$

where $f(x, \alpha, \theta)$ and $F(x, \alpha, \theta)$ are the pdf and cdf defined in equations 1 and 2 respectively. Hence, making substitution yields the density of the order statistics

$$f_{r:n}(x) = \frac{2\alpha\theta e^{-\theta x} (1 - e^{-\theta x}) e^{\alpha(1 - e^{-\theta x})^2}}{(\alpha - 1)\mathcal{B}(r, n-r+1)} \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i \left[\frac{e^{\alpha(1 - e^{-\theta x})^2} - 1}{\alpha - 1} \right]^{i+r-1} \quad (18)$$

3.3 Quantile Function

Let $X \sim$ Zubair-Exponential (x, α, θ) , the quantile function x_q is given as

$$x_q = -\frac{1}{\theta} \left\{ \ln \left[1 - \left\{ \frac{\ln(qe^\alpha - q + 1)}{\alpha} \right\}^{\frac{1}{2}} \right] \right\} \quad (19)$$

4 Classical Estimation of Zubair-Exponential Distribution

In this section, we consider some classical and Bayesian methods of estimation.

4.1 Maximum Likelihood Estimation of the Zubair-Exponential Distribution Parameters

Let (X_1, X_2, \dots, X_n) be n random samples drawn from Zubair-Exponential distribution, then the log-likelihood function is given as

$$\log L(x, \alpha, \theta) = -n \log(e^\alpha - 1) + n \log 2 + n \log \alpha + \sum_{i=1}^n \log(\theta e^{-\theta x}) + \sum_{i=1}^n \log(1 - e^{-\theta x}) + \alpha \sum_{i=1}^n (1 - e^{-\theta x})^2 \quad (20)$$

Taking a partial differentiation with respect to α and θ and equating the results to zero yield

$$n \left(\frac{1}{1 - e^{-\alpha}} - \frac{1}{\alpha} \right) = \sum_{i=1}^n (1 - e^{-\theta x})^2 \quad (21)$$

$$\frac{n}{\theta} - n\theta + \theta \sum_{i=1}^n \frac{1}{e^{\theta x} - 1} + 2\theta \alpha \sum_{i=1}^n e^{-\theta x} (1 - e^{-\theta x}) = 0 \quad (22)$$

The two non-closed form solutions are implemented in R using Newton-Raphson's iterative algorithm

4.2 Least Squares Estimation (LSE)

Let (X_1, X_2, \dots, X_n) be a random sample of size n drawn from a Zubair-Exponential distribution, then if we take the expectation of the cdf and maximize it, we arrive at the ordinary least squares estimates of the parameters α and θ

$$L(\alpha, \theta) = \operatorname{argmin}_{(\alpha, \theta)} \sum_{i=1}^n \left[F(x_{i:n} | \lambda, \theta) - \frac{i}{n+1} \right]^2. \quad (23)$$

The estimates are obtained by solving the following non-linear equations

$$\sum_{i=1}^n \left[F(x_{i:n} | \alpha, \theta) - \frac{i}{n+1} \right]^2 \Delta_1(x_{i:n} | \alpha, \theta) = 0 \quad (24)$$

$$\sum_{i=1}^n \left[F(x_{i:n} | \alpha, \theta) - \frac{i}{n+1} \right]^2 \Delta_2(x_{i:n} | \alpha, \theta) = 0 \quad (25)$$

where

$$\Delta_1(x_{i:n} | \alpha, \theta) = \frac{(1 - e^{-\theta x})^2 (e^\alpha - 1) - e^\alpha (e^{\alpha(1 - e^{-\theta x})^2} - 1)}{(e^\alpha - 1)^2} \quad (26)$$

$$\Delta_2(x_{i:n} | \alpha, \theta) = \frac{2\alpha \theta e^{-\theta x} (1 - e^{-\theta x}) e^{\alpha(1 - e^{-\theta x})^2}}{e^\alpha - 1} \quad (27)$$

4.3 Weighted Least Squares Estimation (WLSE)

The weighted least squares estimates $\hat{\alpha}_{WLSE}$ and $\hat{\theta}_{WLSE}$ of Zubair-Exponential distribution parameters α and θ are obtained by minimizing the function $W(\alpha, \theta)$ with respect to α and θ

$$W(\alpha, \theta) = \operatorname{argmin}_{(\alpha, \theta)} \sum_{i=1}^n \frac{(n+1)^2 (n+2)}{i(n-i+1)} \left[F(x_{i:n} | \theta) - \frac{i}{n+1} \right]^2. \quad (28)$$

Solving the following non-linear equation yields the estimate

$$\sum_{i=1}^n \frac{(n+1)^2 (n+2)}{i(n-i+1)} \left[F(x_{i:n} | \theta) - \frac{i}{n+1} \right] \Delta_1(x_{i:n} | \theta) = 0 \quad (29)$$

$$\sum_{i=1}^n \frac{(n+1)^2 (n+2)}{i(n-i+1)} \left[F(x_{i:n} | \theta) - \frac{i}{n+1} \right] \Delta_2(x_{i:n} | \theta) = 0 \quad (30)$$

where $\Delta_1(x_{i:n} | \alpha, \theta)$ and $\Delta_2(x_{i:n} | \alpha, \theta)$ is as defined in (26) and (27) respectively.

4.4 Maximum Product Spacing Estimators (MPSE)

A good substitute for the greatest likelihood approach is the maximum product spacing method, which approximates the Kullback-Leibler information measure. Let us now suppose that the data are ordered in an increasing manner. Then, the maximum product spacing for the Zubair-Exponential distribution is given as follows

$$K - L(\alpha, \theta | x_i) = \left(\prod_{i=1}^{n+1} D_i(x_i, \alpha, \theta) \right)^{\frac{1}{n+1}}, \quad (31)$$

where $D_i(x_i, \alpha, \theta) = F(x_i; \alpha, \theta) - F(x_{i-1}; \alpha, \theta)$, $i = 1, 2, 3, \dots, n$

Similarly, one can also choose to maximize the function

$$H(\alpha, \theta) = \frac{1}{n+1} \sum_{i=1}^{n+1} \ln D_i(\alpha, \theta). \quad (32)$$

By taking the first derivative of the function $H(\theta)$ with respect to α and θ , and solving the resulting nonlinear equations, at $\frac{\partial H(\phi)}{\partial \alpha} = 0$ and $\frac{\partial H(\phi)}{\partial \theta} = 0$, where $\phi = (\alpha, \theta)$, we obtain the value of the parameter estimates.

4.5 Cramér-von-Mises Estimation (CVME)

The Cramér-von-Mises estimates $\hat{\alpha}_{CVME}$, and $\hat{\theta}_{CVME}$ of the Zubair-Exponential distribution parameters α , and θ are obtained by minimizing the function $C(\alpha, \theta)$ with respect to α , and θ

$$C(\alpha, \theta) = \arg \min_{(\alpha, \theta)} \left\{ \frac{1}{12n} + \sum_{i=1}^n \left[F(x_{i:n} | \alpha, \theta) - \frac{2i-1}{2n} \right]^2 \right\}. \quad (33)$$

The estimates are obtained by solving the following non-linear equations

$$\begin{aligned} \sum_{i=1}^n \left(F(x_{i:n} | \alpha, \theta) - \frac{2i-1}{2n} \right) \Delta_1(x_{i:n} | \alpha, \theta) &= 0 \\ \sum_{i=1}^n \left(F(x_{i:n} | \alpha, \theta) - \frac{2i-1}{2n} \right) \Delta_2(x_{i:n} | \alpha, \theta) &= 0 \end{aligned} \quad (34)$$

where $\Delta_1(x_{i:n} | \alpha, \theta)$ and $\Delta_2(x_{i:n} | \alpha, \theta)$ is as defined in (26) and (27) respectively.

4.6 Anderson-Darling Estimation (ADE)

The Anderson-Darling estimates $\hat{\alpha}_{ADE}$, and $\hat{\theta}_{ADE}$ of the Zubair-Exponential distribution parameters α and θ are obtained by minimizing the function $A(\alpha, \theta)$ with respect to α and θ

$$A(\alpha, \theta) = \arg \min_{(\alpha, \theta)} \sum_{i=1}^n (2i-1) \left\{ \ln F(x_{i:n} | \alpha, \theta) + \ln \left[1 - F(x_{n+1-i:n} | \alpha, \theta) \right] \right\}. \quad (35)$$

The estimates are obtained by solving the following sets of non-linear equations

$$\begin{aligned} \sum_{i=1}^n (2i-1) \left[\frac{\Delta_1(x_{i:n} | \alpha, \theta)}{F(x_{i:n} | \alpha, \theta)} - \frac{\Delta_1(x_{n+1-i:n} | \alpha, \theta)}{1 - F(x_{n+1-i:n} | \alpha, \theta)} \right] &= 0 \\ \sum_{i=1}^n (2i-1) \left[\frac{\Delta_2(x_{i:n} | \alpha, \theta)}{F(x_{i:n} | \alpha, \theta)} - \frac{\Delta_2(x_{n+1-i:n} | \alpha, \theta)}{1 - F(x_{n+1-i:n} | \alpha, \theta)} \right] &= 0 \end{aligned} \quad (36)$$

where $\Delta_1(x_{i:n} | \alpha, \theta)$ and $\Delta_2(x_{i:n} | \alpha, \theta)$ is as defined in (26) and (27) respectively.

4.7 Right-Tailed Anderson-Darling Estimation (RTADE)

The Right-Tailed Anderson-Darling estimates $\hat{\alpha}_{RTADE}$ and $\hat{\theta}_{RTADE}$ of the Zubair-Exponential distribution parameters α and θ are obtained by minimizing the function $R(\alpha, \theta)$ with respect to α and θ

$$R(\alpha, \theta) = \arg \min_{(\alpha, \theta)} \left\{ \frac{n}{2} - 2 \sum_{i=1}^n F(x_{i:n} | \alpha, \theta) - \frac{1}{n} \sum_{i=1}^n (2i-1) \ln \left[1 - F(x_{n+1-i:n} | \alpha, \theta) \right] \right\}. \quad (37)$$

The estimates can be obtained by solving the following set of non-linear equations

$$\begin{aligned} -2 \sum_{i=1}^n \frac{\Delta_1(x_{i:n} | \alpha, \theta)}{F(x_{i:n} | \alpha, \theta)} + \frac{1}{n} \sum_{i=1}^n (2i-1) \left[\frac{\Delta_1(x_{n+1-i:n} | \alpha, \theta)}{1 - F(x_{n+1-i:n} | \alpha, \theta)} \right] &= 0 \\ -2 \sum_{i=1}^n \frac{\Delta_2(x_{i:n} | \alpha, \theta)}{F(x_{i:n} | \alpha, \theta)} + \frac{1}{n} \sum_{i=1}^n (2i-1) \left[\frac{\Delta_2(x_{n+1-i:n} | \alpha, \theta)}{1 - F(x_{n+1-i:n} | \alpha, \theta)} \right] &= 0 \end{aligned} \quad (38)$$

where $\Delta_1(x_{i:n} | \alpha, \theta)$ and $\Delta_2(x_{i:n} | \alpha, \theta)$ is as defined in (26) and (27) respectively. The estimates given in (15), (27), (26), (28), (30), (34), (36) are obtained using `optim()` function in R with the Newton-Raphson iterative algorithm.

5 Bayesian Inference on the Zubair-Exponential Distribution Parameters

This section deals with the Bayesian estimate (BE) of the unknown parameters of the Zubair-Exponential distribution. For Bayesian parameter estimation, many loss functions, including squared error, LINEX, and generalized entropy loss functions, can be taken into consideration. We can consider applying independent gamma priors for the parameters α and θ with pdfs in the parameter prior distributions of Zubair-Exponential as follows;

$$\begin{aligned}\pi_1(\alpha) &\propto \alpha^{s_1-1} e^{-q_1\alpha} &> 0, s_1 > 0, q_1 > 0, \\ \pi_2(\theta) &\propto \theta^{s_2-1} e^{-q_2\theta} &> 0, s_2 > 0, q_2 > 0,\end{aligned}\quad (39)$$

where the hyper-parameters $s_j, q_j, j = 1, 2$ are selected to reflect the prior knowledge about the unknown parameters. The joint prior for $\phi = (\alpha, \theta)$ is given by

$$\begin{aligned}\pi(\phi) &= \pi_1(\alpha)\pi_2(\theta) \\ \pi(\phi) &\propto \alpha^{s_1-1}\theta^{s_2-1}e^{\{-q_1\alpha-q_2\theta\}}.\end{aligned}\quad (40)$$

The corresponding posterior density given the observed data $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is given by:

$$\pi(\phi | \mathbf{x}) = \frac{\pi(\phi)\ell(\phi)}{\int_{\phi} \pi(\phi)\ell(\phi)d\phi},$$

Which implies that the posterior density function is:

$$\pi(\phi | \mathbf{x}) \propto \frac{2^n \alpha^{n+s_1-1} \theta^{n+s_2-1}}{(e^\alpha - 1)^n} e^{-\theta \sum x - q_1 \alpha - q_2 \theta + \alpha \sum (1 - e^{-\theta x})^2} \prod_{i=1}^n (1 - e^{-\theta x}) \quad (41)$$

Given any function, such as $l(\phi)$ under the squared error loss (SEL) function, the Bayes estimator is given by

$$\hat{\phi}_{BE_{SEL}} = E[l(\phi) | \mathbf{x}] = \int_{\phi} l(\phi) \pi(\phi | x) d\phi. \quad (42)$$

The SEL impacts underestimation and overestimation equally because it has an asymmetric loss function. In many actual situations, both underestimation and overestimation can have serious implications. A proposed LINEX loss can be made in certain instances as an alternative to the SE loss given by

$$(l(\phi), \hat{l}(\phi)) = e^{\{\hat{l}(\phi) - l(\phi)\}} - v(\hat{l}(\phi) - l(\phi)) - 1.$$

where $v \neq 0$ is a shape parameter. Here $v > 1$ suggests that an overestimation is more serious than an underestimation, and vice versa for $v < 0$. Further v approaching zero replicates the SE loss function itself. One may refer to Varian [5] and Doostparast *et al.* [10] for more details in this regard. The BE of $l(\phi)$ under this loss can be derived as

$$\hat{\phi}_{BE_{LINEX}} = E[e^{\{-vl(\phi)\}} | \mathbf{x}] = -\frac{1}{v} \log \left[\int_{\phi} e^{\{-vl(\phi)\}} \pi(\phi | x) d\phi \right]. \quad (43)$$

Additionally, we take into account the general entropy loss (GEL) function suggested by Calabria and Pulcini [11], which is defined as follows.

$$(l(\phi), \hat{l}(\phi)) = \left(\frac{\hat{l}(\phi)}{l(\phi)} \right)^{\tau} - \tau \log \left(\frac{\hat{l}(\phi)}{l(\phi)} \right) - 1,$$

where the shape parameter $\tau \neq 0$ denotes a departure from symmetry. It views overestimation as more significant than underestimating when $\tau > 0$ and the opposite is true when $\tau < 0$. Given is the Bayes estimator with regard to the GE loss function.

$$\hat{\phi}_{BE_{GEL}} = [E((l(\phi))^{-\tau} | \mathbf{x})]^{-1/\tau} = \left[\int_{\phi} (l(\phi))^{-\tau} \pi(\phi | x) d\phi \right]^{-1/\tau}. \quad (44)$$

The estimations produced by (42), (43), and (44) can be seen to not be able to be transformed into closed-form expressions. We then use the Markov chain Monte Carlo (MCMC) approach to generate posterior samples and arrive at suitable BEs.

5.1 Markov Chain Monte Carlo

A general simulation technique for computing posterior quantities of interest and sampling from posterior distributions is the MCMC technique. Read Ravenzwaaij *et al.* [12] for further details on MCMC. In fact, using a kernel estimate of the posterior distribution and the MCMC samples, it is possible to properly quantify the posterior uncertainty with regard to the parameter ϕ .

The basis of MCMC algorithms is the concept of a discrete-time evolving Markov chain. As a stochastic process, a Markov chain

$$\phi^{(0)}, \phi^{(1)}, \phi^{(2)}, \dots$$

Here, the random variable $\phi^{(i)}$ has values that are contained in a "state space." The state space, or the process's state at time i , remains constant across time i . The following Markov property applies to Markov chains: Only via the current state $\phi^{(i)}$ does the distribution of the next state $\phi^{(i+1)}$ depend on the past $\phi^{(0)}, \phi^{(1)}, \dots, \phi^{(i)}$. The homogeneous Markov chains employed in MCMC approaches allow the conditional distribution of $\phi^{(i+1)}$ given $\phi^{(i)}$ to be independent of the index i . MCMC sample selection from a distribution:

1. Starting with an initial guess: simply one possible value that could be taken as coming from the distribution.
2. creating a number of fresh samples from this first hunch. Two steps are used to create each new sample:
 - A modest random perturbation is added to the most recent sample to create a proposal for the new sample.
 - Acceptance: Whether the new suggestion is rejected or accepted as the new sample (in which case the old sample is retained).

There are various techniques for integrating randomness to produce proposals, as well as numerous methods for acceptance and rejection, including Gibbs sampling and the Metropolis-Hastings algorithm.

5.2 Metropolis-Hasting Algorithm

A proposed distribution and the beginning values of the unknown parameters ϕ must be defined in order to carry out the MH algorithm for the Zubair-Exponential distribution. A multivariate normal distribution, defined as $q(\phi' | \phi) \equiv N_3(\phi, S_\phi)$, will be taken into account for the proposal distribution, where S_ϕ denotes the variance-covariance matrix. It's possible to make unfavorable observations, which is unacceptable. The MLEs for ϕ are taken into account for the starting values, i.e., $\phi^{(0)} = \hat{\phi}_{MLE}$. The Fisher information matrix $I(\cdot)$ is used to pick S_ϕ , which is thought to be the asymptotic variance-covariance matrix $I^{-1}(\hat{\phi}_{MLE})$. It is noted that the choice of S_ϕ , which affects the MH algorithm's acceptance rate, is a critical decision.

In this regard, the MH algorithm's procedures for taking a sample from the provided posterior density (referred to as "(41)") are as follows:

Step 1. Set initial value of ϕ as $\phi^{(0)} = (\hat{\alpha}_{MLE}, \hat{\theta}_{MLE})$.

Step 2. For $i = 1, 2, \dots, M$ repeat the following steps:

- 2.1: Set $\phi = \phi^{(i-1)}$.
- 2.2: Generate a new candidate parameter value δ from $N_3(\log \phi, S_\phi)$.
- 2.3: Set $\theta' = \exp(\delta)$.
- 2.4: Calculate $\beta = \frac{\pi(\phi' | x)}{\pi(\phi | x)}$, where $\pi(\cdot)$ is the posterior density in (41).
- 2.5: Generate a sample u from the uniform $U(0, 1)$ distribution.
- 2.6: Accept or reject the new candidate θ'

$$\begin{cases} \text{If } u \leq \beta & \text{set } \phi^{(i)} = \phi' \\ \text{otherwise} & \text{set } \phi^{(i)} = \phi. \end{cases}$$

Finally, part of the initial samples can be eliminated (burn-in) from the random samples of size M derived from the posterior density, and the remaining samples can then be used to calculate Bayes estimates. Using MCMC under the SEL, LINEX, and GEL functions, the BEs of $\phi^{(i)} = (\alpha^{(i)}, \theta^{(i)})$ can be calculated as follows.

$$\hat{\phi}_{BE_{SEL}} = \frac{1}{M - l_B} \sum_{i=l_B}^M \phi^{(i)}, \quad (45)$$

$$\hat{\phi}_{BE_{LINEX}} = -\frac{1}{v} \log \left[\frac{1}{M - l_B} \sum_{i=l_B}^M e^{\{-v\phi^{(i)}\}} \right], \quad (46)$$

$$\hat{\phi}_{BE_{GEL}} = \left[\frac{1}{M - l_B} \sum_{i=l_B}^M (\phi^{(i)})^{-\tau} \right]^{-1/\tau}, \quad (47)$$

where l_B represents the number of burn-in samples.

6 Applications

In this section, we consider both simulation and real life data study.

6.1 Simulation study

In this subsection, we simulate data for the Zubair-Exponential to compare the performance of the Non-Bayesian estimation methods discussed in the previous section. we generate 1000 data from the Zubair-Exponential distribution by considering the initial parameter values as

- $\alpha = 0.15$ and $\theta = 0.25$
- $\alpha = 0.10$ and $\theta = 0.50$
- $\alpha = 0.05$ and $\theta = 0.50$
- $\alpha = 0.25$ and $\theta = 1.25$
- $\alpha = 1.0$ and $\theta = 0.50$
- $\alpha = 1.50$ and $\theta = 1.20$
- $\alpha = 0.25$ and $\theta = 2.0$
- $\alpha = 0.15$ and $\theta = 1.0$

and sample sizes $n = 25, 50, 75, 100$. For each estimate $\hat{\phi} = (\hat{\alpha}, \hat{\theta})$, we compute the Bias and Root Mean Squared Error(RMSE) respectively as

$$Bias(\hat{\phi}) = \frac{1}{B} \sum_{i=1}^B (\hat{\phi}_i - \phi),$$

and

$$RMSE(\hat{\phi}) = \sqrt{\frac{1}{B} \sum_{i=1}^B (\hat{\phi}_i - \phi)^2}.$$

For the Non-Bayesian procedure, we employed the Newton-Raphson algorithm for finding the desired estimates. For the Bayesian procedure, BEs using MCMC by utilizing the MH algorithm under informative prior are computed. For the informative prior, we assumed all hyperparameters of gamma distributions are equal to twice of parameter values. These values are then plugged-in to calculate the desired estimates. While utilizing the MH algorithm, the MLEs take into account initial guess values. In the end, 2000 burn-in samples are discarded among the overall 10000 samples generated from the posterior density and subsequently obtained BEs under different loss functions, namely: SEL, LINEX at $v = -1.5, 1.5$, and finally GEL at $\tau = -0.5, 0.5$. For each approach, we determine Bias and RMSE.

Table 1: Average estimated Biases and RMSEs of different estimation methods for Zubair-Exponential distribution at different sample sizes n and different values of the parameters ($\alpha = 0.15$, $\theta = 0.25$).

| Method | | $n = 25$ | | $n = 50$ | | $n = 75$ | | $n = 100$ | |
|---------------|----------|----------|---------|----------|---------|----------|---------|-----------|---------|
| | | Bias | RMSE | Bias | RMSE | Bias | RMSE | Bias | RMSE |
| MLE | α | 0.27572 | 3.55456 | 0.00449 | 2.58690 | 0.02469 | 1.66180 | 0.05772 | 1.53109 |
| | θ | 0.02616 | 0.00952 | 0.00960 | 0.00539 | 0.00562 | 0.00324 | 0.00165 | 0.00291 |
| MPSE | α | 0.80541 | 4.36457 | 0.83695 | 4.11626 | 0.75338 | 3.36524 | 0.67598 | 3.09018 |
| | θ | 0.02732 | 0.00852 | 0.02911 | 0.00662 | 0.02686 | 0.00489 | 0.02515 | 0.00455 |
| LSE | α | 0.09472 | 3.15273 | 0.11413 | 2.06325 | 0.20930 | 1.54713 | 0.16962 | 1.36258 |
| | θ | 0.00503 | 0.00951 | 0.00073 | 0.00562 | 0.00554 | 0.00382 | 0.00561 | 0.00346 |
| WLSE | α | 0.11416 | 3.21506 | 0.07756 | 2.10771 | 0.13185 | 1.46836 | 0.11426 | 1.30819 |
| | θ | 0.00476 | 0.00930 | 0.00283 | 0.00547 | 0.00156 | 0.00351 | 0.00280 | 0.00314 |
| CVME | α | 0.42522 | 3.76136 | 0.19392 | 2.07264 | 0.00073 | 1.50594 | 0.00148 | 1.29131 |
| | θ | 0.03309 | 0.01215 | 0.01739 | 0.00620 | 0.00603 | 0.00404 | 0.00338 | 0.00349 |
| ADE | α | 0.03753 | 3.22576 | 0.04870 | 2.11722 | 0.06481 | 1.26523 | 0.06990 | 1.17150 |
| | θ | 0.01290 | 0.00893 | 0.00474 | 0.00507 | 0.00158 | 0.00310 | 0.00078 | 0.00283 |
| RTADE | α | 0.03955 | 4.02506 | 0.12369 | 2.47219 | 0.10998 | 1.50134 | 0.08989 | 1.34895 |
| | θ | 0.00757 | 0.00955 | 0.00073 | 0.00537 | 0.00090 | 0.00325 | 0.00199 | 0.00298 |
| BE_{SEL} | α | 1.47577 | 2.67869 | 1.49266 | 2.65383 | 1.52892 | 2.70194 | 1.56658 | 2.78276 |
| | θ | 0.10940 | 0.01712 | 0.10444 | 0.01347 | 0.10275 | 0.01228 | 0.10270 | 0.01189 |
| BE_{Linex1} | α | 1.47374 | 2.66492 | 1.51114 | 2.71647 | 1.55077 | 2.77507 | 1.58985 | 2.86181 |
| | θ | 0.10992 | 0.01726 | 0.10476 | 0.01354 | 0.10298 | 0.01233 | 0.10289 | 0.01193 |
| BE_{Linex2} | α | 1.44788 | 2.57392 | 1.45973 | 2.53554 | 1.50431 | 2.61877 | 1.53966 | 2.69010 |
| | θ | 0.10888 | 0.01698 | 0.10413 | 0.01339 | 0.10252 | 0.01222 | 0.10251 | 0.01185 |
| BE_{GEL1} | α | 1.46077 | 2.63211 | 1.47284 | 2.59008 | 1.51485 | 2.66130 | 1.54949 | 2.72868 |
| | θ | 0.10800 | 0.01679 | 0.10357 | 0.01328 | 0.10210 | 0.01214 | 0.10217 | 0.01178 |
| BE_{GEL2} | α | 1.42046 | 2.50171 | 1.43422 | 2.46963 | 1.48041 | 2.55588 | 1.51721 | 2.62985 |
| | θ | 0.10519 | 0.01614 | 0.10183 | 0.01290 | 0.10082 | 0.01187 | 0.10111 | 0.01156 |

Table 2: Average estimated Biases and RMSEs of different estimation methods for Zubair-Exponential distribution at different sample sizes n and different values of the parameters ($\alpha = 0.10$, $\theta = 0.50$).

| Method | | $n = 25$ | | $n = 50$ | | $n = 75$ | | $n = 100$ | |
|---------------|----------|----------|---------|----------|---------|----------|---------|-----------|---------|
| | | Bias | RMSE | Bias | RMSE | Bias | RMSE | Bias | RMSE |
| MLE | α | 0.24880 | 3.77480 | 0.02210 | 2.19270 | 0.07960 | 1.72840 | 0.10890 | 1.60496 |
| | θ | 0.04980 | 0.03780 | 0.01390 | 0.01850 | 0.00380 | 0.01360 | 0.00040 | 0.01178 |
| MPSE | α | 0.71260 | 4.06370 | 0.85390 | 3.80540 | 0.76120 | 3.17540 | 0.77300 | 3.36579 |
| | θ | 0.04890 | 0.03200 | 0.06260 | 0.02410 | 0.05800 | 0.01990 | 0.05740 | 0.01915 |
| LSE | α | 0.03750 | 3.20750 | 0.26070 | 1.88430 | 0.27720 | 1.62880 | 0.21380 | 1.38076 |
| | θ | 0.01560 | 0.03980 | 0.01650 | 0.01860 | 0.02000 | 0.01540 | 0.01490 | 0.01351 |
| WLSE | α | 0.05620 | 3.35080 | 0.23770 | 2.03860 | 0.21470 | 1.60340 | 0.16310 | 1.35124 |
| | θ | 0.01500 | 0.03920 | 0.01220 | 0.01890 | 0.01350 | 0.01430 | 0.00950 | 0.01245 |
| CVME | α | 0.46550 | 3.87480 | 0.01290 | 1.93230 | 0.05250 | 1.53100 | 0.04860 | 1.31904 |
| | θ | 0.07220 | 0.05250 | 0.01400 | 0.02060 | 0.00420 | 0.01560 | 0.00310 | 0.01363 |
| ADE | α | 0.00100 | 3.49240 | 0.16860 | 2.01110 | 0.17670 | 1.55650 | 0.13730 | 1.30385 |
| | θ | 0.02290 | 0.03610 | 0.00420 | 0.01800 | 0.00900 | 0.01360 | 0.00650 | 0.01179 |
| RTADE | α | 0.00304 | 4.11808 | 0.16508 | 2.36558 | 0.21145 | 1.76838 | 0.12724 | 1.33593 |
| | θ | 0.01782 | 0.03778 | 0.00549 | 0.01994 | 0.01311 | 0.01470 | 0.00715 | 0.01169 |
| BE_{SEL} | α | 1.14634 | 1.54647 | 1.18340 | 1.60165 | 1.24588 | 1.72728 | 1.28227 | 1.79314 |
| | θ | 0.21452 | 0.06674 | 0.20905 | 0.05386 | 0.20656 | 0.04950 | 0.20712 | 0.04829 |
| BE_{Linex1} | α | 1.15123 | 1.56236 | 1.17837 | 1.58725 | 1.25111 | 1.74162 | 1.29122 | 1.81726 |
| | θ | 0.21661 | 0.06786 | 0.21033 | 0.05447 | 0.20748 | 0.04992 | 0.20788 | 0.04863 |
| BE_{Linex2} | α | 1.14091 | 1.52973 | 1.17754 | 1.58547 | 1.23279 | 1.69146 | 1.28067 | 1.79108 |
| | θ | 0.21245 | 0.06566 | 0.20778 | 0.05327 | 0.20563 | 0.04909 | 0.20636 | 0.04795 |
| BE_{GEL1} | α | 1.13590 | 1.52094 | 1.17215 | 1.57461 | 1.23353 | 1.69659 | 1.27496 | 1.77689 |
| | θ | 0.21170 | 0.06543 | 0.20730 | 0.05309 | 0.20527 | 0.04895 | 0.20605 | 0.04783 |
| BE_{GEL2} | α | 1.12243 | 1.49157 | 1.15726 | 1.54260 | 1.20934 | 1.63795 | 1.26082 | 1.74585 |
| | θ | 0.20606 | 0.06287 | 0.20380 | 0.05158 | 0.20269 | 0.04786 | 0.20393 | 0.04693 |

Table 3: Average estimated Biases and RMSEs of different estimation methods for Zubair-Exponential distribution at different sample sizes n and different values of the parameters ($\alpha = 0.05$, $\theta = 0.50$).

| Method | | $n = 25$ | | $n = 50$ | | $n = 75$ | | $n = 100$ | |
|---------------|----------|----------|---------|----------|---------|----------|---------|-----------|---------|
| | | Bias | RMSE | Bias | RMSE | Bias | RMSE | Bias | RMSE |
| MLE | α | 0.23956 | 3.55575 | 0.04685 | 2.27915 | 0.12818 | 2.19896 | 0.19170 | 1.75152 |
| | θ | 0.04686 | 0.03584 | 0.02177 | 0.02021 | 0.00535 | 0.01613 | 0.00556 | 0.01225 |
| MPSE | α | 0.81745 | 4.36545 | 0.74691 | 3.59691 | 0.81122 | 3.69056 | 0.84419 | 3.43437 |
| | θ | 0.05776 | 0.03279 | 0.05280 | 0.02414 | 0.05652 | 0.02175 | 0.06227 | 0.01938 |
| LSE | α | 0.13143 | 3.07292 | 0.22056 | 2.07073 | 0.16544 | 1.52782 | 0.23630 | 1.35456 |
| | θ | 0.00386 | 0.03418 | 0.01022 | 0.02129 | 0.00548 | 0.01543 | 0.01487 | 0.01372 |
| WLSE | α | 0.16675 | 3.29244 | 0.15492 | 2.09898 | 0.15592 | 1.65819 | 0.20215 | 1.31406 |
| | θ | 0.00180 | 0.03459 | 0.00248 | 0.02045 | 0.00331 | 0.01522 | 0.01129 | 0.01245 |
| CVME | α | 0.33792 | 3.86754 | 0.08047 | 2.07249 | 0.04108 | 1.51376 | 0.07842 | 1.31993 |
| | θ | 0.05699 | 0.04612 | 0.02254 | 0.02322 | 0.01776 | 0.01658 | 0.00262 | 0.01408 |
| ADE | α | 0.07233 | 3.44263 | 0.10827 | 2.13325 | 0.15686 | 1.75606 | 0.17152 | 1.24431 |
| | θ | 0.01454 | 0.03402 | 0.00338 | 0.01980 | 0.00186 | 0.01500 | 0.00812 | 0.01153 |
| RTADE | α | 0.09259 | 4.14665 | 0.09467 | 2.23493 | 0.19432 | 1.99400 | 0.17154 | 1.29276 |
| | θ | 0.00763 | 0.03702 | 0.00239 | 0.02077 | 0.00652 | 0.01573 | 0.00968 | 0.01156 |
| BE_{SEL} | α | 0.59743 | 0.42680 | 0.65856 | 0.48960 | 0.67698 | 0.50511 | 0.71922 | 0.54888 |
| | θ | 0.22035 | 0.07003 | 0.21403 | 0.05623 | 0.21153 | 0.05172 | 0.21184 | 0.05038 |
| BE_{Linex1} | α | 0.59438 | 0.42435 | 0.65450 | 0.48508 | 0.68032 | 0.51063 | 0.72080 | 0.55179 |
| | θ | 0.22248 | 0.07119 | 0.21533 | 0.05686 | 0.21247 | 0.05215 | 0.21261 | 0.05074 |
| BE_{Linex2} | α | 0.60496 | 0.43521 | 0.65988 | 0.49011 | 0.67226 | 0.49782 | 0.71442 | 0.54132 |
| | θ | 0.21824 | 0.06889 | 0.21275 | 0.05561 | 0.21060 | 0.05130 | 0.21107 | 0.05003 |
| BE_{GEL1} | α | 0.60247 | 0.43253 | 0.65729 | 0.48721 | 0.66935 | 0.49450 | 0.71175 | 0.53816 |
| | θ | 0.21750 | 0.06866 | 0.21226 | 0.05544 | 0.21023 | 0.05116 | 0.21077 | 0.04991 |
| BE_{GEL2} | α | 0.60760 | 0.43699 | 0.65536 | 0.48420 | 0.66828 | 0.49403 | 0.70753 | 0.53346 |
| | θ | 0.21181 | 0.06600 | 0.20873 | 0.05387 | 0.20764 | 0.05003 | 0.20863 | 0.04899 |

Table 4: Average estimated Biases and RMSEs of different estimation methods for Zubair-Exponential distribution at different sample sizes n and different values of the parameters ($\alpha = 0.25$, $\theta = 1.25$).

| Method | | $n = 25$ | | $n = 50$ | | $n = 75$ | | $n = 100$ | |
|---------------|----------|----------|---------|----------|---------|----------|---------|-----------|---------|
| | | Bias | RMSE | Bias | RMSE | Bias | RMSE | Bias | RMSE |
| MLE | α | 0.28374 | 3.84590 | 0.04361 | 2.13066 | 0.07628 | 1.84194 | 0.06554 | 1.35021 |
| | θ | 0.12903 | 0.22525 | 0.04893 | 0.10716 | 0.01508 | 0.08572 | 0.00931 | 0.06351 |
| MPSE | α | 0.76168 | 4.47452 | 0.78095 | 3.76563 | 0.74262 | 3.32011 | 0.64883 | 2.78396 |
| | θ | 0.12565 | 0.19943 | 0.13724 | 0.14086 | 0.13475 | 0.12136 | 0.11873 | 0.09984 |
| LSE | α | 0.10095 | 3.34588 | 0.18848 | 1.83325 | 0.23320 | 1.61072 | 0.20582 | 1.26131 |
| | θ | 0.01643 | 0.21215 | 0.02314 | 0.11549 | 0.03182 | 0.09927 | 0.03340 | 0.07617 |
| WLSE | α | 0.06597 | 3.33432 | 0.12468 | 1.79492 | 0.18754 | 1.66275 | 0.14482 | 1.20079 |
| | θ | 0.02795 | 0.20672 | 0.00613 | 0.10843 | 0.01862 | 0.09531 | 0.01784 | 0.06811 |
| CVME | α | 0.45133 | 3.86780 | 0.08537 | 1.87402 | 0.02873 | 1.55920 | 0.04116 | 1.18808 |
| | θ | 0.16337 | 0.27196 | 0.05232 | 0.12870 | 0.02339 | 0.10273 | 0.01102 | 0.07662 |
| ADE | α | 0.02313 | 3.58853 | 0.07249 | 1.79504 | 0.12598 | 1.51210 | 0.11513 | 1.13417 |
| | θ | 0.05943 | 0.20833 | 0.01000 | 0.10442 | 0.00449 | 0.08683 | 0.00998 | 0.06385 |
| RTADE | α | 0.01521 | 4.50222 | 0.05539 | 2.04169 | 0.14034 | 1.62592 | 0.13159 | 1.24889 |
| | θ | 0.04687 | 0.22684 | 0.00890 | 0.10821 | 0.01193 | 0.08751 | 0.01601 | 0.06561 |
| BE_{SEL} | α | 1.54787 | 3.14515 | 1.47495 | 2.72216 | 1.47576 | 2.60471 | 1.49436 | 2.59759 |
| | θ | 0.45050 | 0.30722 | 0.45307 | 0.26050 | 0.45628 | 0.24522 | 0.46199 | 0.24353 |
| BE_{Linex1} | α | 1.59812 | 3.35532 | 1.51909 | 2.89408 | 1.51445 | 2.74638 | 1.52978 | 2.72317 |
| | θ | 0.46254 | 0.32093 | 0.46053 | 0.26838 | 0.46173 | 0.25067 | 0.46652 | 0.24805 |
| BE_{Linex2} | α | 1.49340 | 2.91531 | 1.43395 | 2.56969 | 1.43937 | 2.47634 | 1.46081 | 2.48224 |
| | θ | 0.43871 | 0.29422 | 0.44574 | 0.25297 | 0.45087 | 0.23989 | 0.45749 | 0.23910 |
| BE_{GEL1} | α | 1.52073 | 3.04608 | 1.45258 | 2.64971 | 1.45533 | 2.54096 | 1.47534 | 2.53908 |
| | θ | 0.44370 | 0.30053 | 0.44883 | 0.25641 | 0.45313 | 0.24224 | 0.45938 | 0.24105 |
| BE_{GEL2} | α | 1.46807 | 2.86087 | 1.40886 | 2.51209 | 1.41524 | 2.41896 | 1.43794 | 2.42657 |
| | θ | 0.43011 | 0.28753 | 0.44035 | 0.24840 | 0.44685 | 0.23636 | 0.45417 | 0.23613 |

Table 5: Average estimated Biases and RMSEs of different estimation methods for Zubair-Exponential distribution at different sample sizes n and different values of the parameters ($\alpha = 1.0$, $\theta = 0.50$).

| Method | | $n = 25$ | | $n = 50$ | | $n = 75$ | | $n = 100$ | |
|---------------|----------|----------|---------|----------|---------|----------|---------|-----------|---------|
| | | Bias | RMSE | Bias | RMSE | Bias | RMSE | Bias | RMSE |
| MLE | α | 0.18112 | 3.94918 | 0.02192 | 2.05581 | 0.03013 | 1.12993 | 0.02399 | 0.89396 |
| | θ | 0.02274 | 0.02236 | 0.00418 | 0.01332 | 0.00423 | 0.00804 | 0.00545 | 0.00613 |
| MPSE | α | 0.91673 | 5.19381 | 0.78562 | 3.93200 | 0.56720 | 2.34698 | 0.38207 | 1.73467 |
| | θ | 0.07004 | 0.02751 | 0.05866 | 0.02036 | 0.04190 | 0.01269 | 0.03000 | 0.00920 |
| LSE | α | 0.33569 | 4.79695 | 0.27178 | 1.89863 | 0.25151 | 1.45845 | 0.12636 | 0.90983 |
| | θ | 0.03050 | 0.02601 | 0.02252 | 0.01544 | 0.01739 | 0.01144 | 0.01083 | 0.00765 |
| WLSE | α | 0.27294 | 3.99028 | 0.20148 | 1.85911 | 0.17207 | 1.25679 | 0.05081 | 0.75515 |
| | θ | 0.02205 | 0.02456 | 0.01548 | 0.01421 | 0.01012 | 0.00974 | 0.00364 | 0.00632 |
| CVME | α | 0.26596 | 7.28444 | 0.01176 | 1.79697 | 0.04195 | 1.26604 | 0.01469 | 0.86068 |
| | θ | 0.02133 | 0.02762 | 0.00460 | 0.01527 | 0.00249 | 0.01072 | 0.00306 | 0.00742 |
| ADE | α | 0.08103 | 3.41601 | 0.12261 | 1.74487 | 0.13083 | 1.13829 | 0.02635 | 0.75885 |
| | θ | 0.00301 | 0.02188 | 0.00800 | 0.01304 | 0.00626 | 0.00892 | 0.00093 | 0.00619 |
| RTADE | α | 0.02573 | 4.47026 | 0.11021 | 2.02857 | 0.09815 | 1.20204 | 0.00299 | 0.79213 |
| | θ | 0.00227 | 0.02494 | 0.00830 | 0.01362 | 0.00462 | 0.00893 | 0.00084 | 0.00626 |
| BE_{SEL} | α | 1.22875 | 2.42396 | 1.21473 | 2.14625 | 1.25721 | 2.09113 | 1.30131 | 2.13236 |
| | θ | 0.14185 | 0.03271 | 0.13867 | 0.02603 | 0.13969 | 0.02427 | 0.14254 | 0.02418 |
| BE_{Linex1} | α | 1.31350 | 2.75499 | 1.27939 | 2.36209 | 1.31379 | 2.26945 | 1.35272 | 2.29138 |
| | θ | 0.14346 | 0.03328 | 0.13966 | 0.02634 | 0.14044 | 0.02450 | 0.14316 | 0.02437 |
| BE_{Linex2} | α | 1.15347 | 2.15975 | 1.15518 | 1.96070 | 1.20437 | 1.93373 | 1.25289 | 1.98929 |
| | θ | 0.14026 | 0.03216 | 0.13769 | 0.02573 | 0.13895 | 0.02405 | 0.14192 | 0.02400 |
| BE_{GEL1} | α | 1.19685 | 2.33060 | 1.18830 | 2.07669 | 1.23363 | 2.03017 | 1.27991 | 2.07637 |
| | θ | 0.13942 | 0.03197 | 0.13715 | 0.02559 | 0.13853 | 0.02394 | 0.14158 | 0.02390 |
| BE_{GEL2} | α | 1.13486 | 2.15844 | 1.13653 | 1.94561 | 1.18723 | 1.91416 | 1.23769 | 1.96900 |
| | θ | 0.13457 | 0.03052 | 0.13411 | 0.02473 | 0.13622 | 0.02329 | 0.13966 | 0.02335 |

Table 6: Average estimated Biases and RMSEs of different estimation methods for Zubair-Exponential distribution at different sample sizes n and different values of the parameters ($\alpha = 1.5$, $\theta = 1.20$).

| Method | | $n = 25$ | | $n = 50$ | | $n = 75$ | | $n = 100$ | |
|---------------|----------|----------|---------|----------|---------|----------|---------|-----------|---------|
| | | Bias | RMSE | Bias | RMSE | Bias | RMSE | Bias | RMSE |
| MLE | α | 0.21152 | 3.35555 | 0.03130 | 1.74988 | 0.05140 | 0.93839 | 0.02471 | 0.81305 |
| | θ | 0.04894 | 0.10561 | 0.01525 | 0.05743 | 0.01586 | 0.03428 | 0.01032 | 0.02821 |
| MPSE | α | 0.95973 | 5.56126 | 0.58826 | 2.83953 | 0.35013 | 1.42070 | 0.30407 | 1.32429 |
| | θ | 0.16585 | 0.14877 | 0.10728 | 0.08104 | 0.06794 | 0.04535 | 0.05701 | 0.03833 |
| LSE | α | 0.32795 | 3.78344 | 0.23492 | 1.82697 | 0.13195 | 1.07146 | 0.11690 | 0.87451 |
| | θ | 0.06895 | 0.12876 | 0.04746 | 0.07230 | 0.02809 | 0.04494 | 0.02444 | 0.03619 |
| WLSE | α | 0.22478 | 3.52674 | 0.14124 | 1.61434 | 0.06285 | 0.90917 | 0.03956 | 0.69030 |
| | θ | 0.04641 | 0.11736 | 0.02761 | 0.06351 | 0.01202 | 0.03769 | 0.00748 | 0.02893 |
| CVME | α | 0.25878 | 3.94911 | 0.05234 | 1.66635 | 0.04395 | 1.03957 | 0.01794 | 0.82135 |
| | θ | 0.04598 | 0.13198 | 0.01199 | 0.06793 | 0.00992 | 0.04380 | 0.00458 | 0.03455 |
| ADE | α | 0.04465 | 3.24963 | 0.07202 | 1.50212 | 0.03188 | 0.87070 | 0.02391 | 0.65740 |
| | θ | 0.00688 | 0.10569 | 0.01220 | 0.05768 | 0.00516 | 0.03579 | 0.00395 | 0.02761 |
| RTADE | α | 0.07750 | 4.86251 | 0.01523 | 1.81904 | 0.03147 | 1.02519 | 0.01391 | 0.77199 |
| | θ | 0.00192 | 0.12403 | 0.00564 | 0.06112 | 0.00407 | 0.03838 | 0.00164 | 0.02906 |
| BE_{SEL} | α | 0.94474 | 1.83007 | 0.99737 | 1.69937 | 1.08796 | 1.73426 | 1.17206 | 1.84599 |
| | θ | 0.23425 | 0.10830 | 0.24520 | 0.09155 | 0.25927 | 0.08987 | 0.27414 | 0.09349 |
| BE_{Linex1} | α | 1.04989 | 2.17655 | 1.07816 | 1.92858 | 1.15837 | 1.92901 | 1.23506 | 2.02266 |
| | θ | 0.24229 | 0.11316 | 0.25025 | 0.09437 | 0.26313 | 0.09204 | 0.27735 | 0.09536 |
| BE_{Linex2} | α | 0.85268 | 1.56954 | 0.92368 | 1.50949 | 1.02260 | 1.56660 | 1.11297 | 1.68999 |
| | θ | 0.22634 | 0.10369 | 0.24021 | 0.08882 | 0.25544 | 0.08775 | 0.27096 | 0.09166 |
| BE_{GEL1} | α | 0.90815 | 1.74628 | 0.96764 | 1.63632 | 1.06211 | 1.67755 | 1.14934 | 1.79318 |
| | θ | 0.22880 | 0.10551 | 0.24176 | 0.08981 | 0.25665 | 0.08849 | 0.27199 | 0.09230 |
| BE_{GEL2} | α | 0.83721 | 1.59573 | 0.90945 | 1.51943 | 1.01120 | 1.57054 | 1.10443 | 1.69231 |
| | θ | 0.21792 | 0.10016 | 0.23487 | 0.08641 | 0.25140 | 0.08579 | 0.26768 | 0.08996 |

Table 7: Average estimated Biases and RMSEs of different estimation methods for Zubair-Exponential distribution at different sample sizes n and different values of the parameters ($\alpha = 0.25$, $\theta = 2.0$).

| Method | | $n = 25$ | | $n = 50$ | | $n = 75$ | | $n = 100$ | |
|---------------|----------|----------|---------|----------|---------|----------|---------|-----------|---------|
| | | Bias | RMSE | Bias | RMSE | Bias | RMSE | Bias | RMSE |
| MLE | α | 0.21482 | 3.35577 | 0.04554 | 2.15588 | 0.07182 | 1.88099 | 0.06817 | 1.38321 |
| | θ | 0.16736 | 0.49641 | 0.04613 | 0.27129 | 0.03695 | 0.23990 | 0.00750 | 0.16067 |
| MPSE | α | 0.87024 | 4.46650 | 0.85628 | 3.81697 | 0.72486 | 3.33901 | 0.64634 | 2.79912 |
| | θ | 0.25005 | 0.48497 | 0.24717 | 0.36602 | 0.19808 | 0.32437 | 0.19508 | 0.25454 |
| LSE | α | 0.20470 | 2.90192 | 0.29232 | 1.96393 | 0.25371 | 1.77811 | 0.20743 | 1.27950 |
| | θ | 0.02692 | 0.49205 | 0.07065 | 0.30897 | 0.04914 | 0.27741 | 0.05843 | 0.19904 |
| WLSE | α | 0.19012 | 3.08809 | 0.24926 | 2.00364 | 0.20814 | 1.74428 | 0.14912 | 1.20909 |
| | θ | 0.01201 | 0.49873 | 0.04915 | 0.29442 | 0.03079 | 0.25609 | 0.03476 | 0.17711 |
| CVME | α | 0.28004 | 3.38317 | 0.01319 | 1.96335 | 0.03684 | 1.67825 | 0.04506 | 1.21044 |
| | θ | 0.17993 | 0.61439 | 0.05106 | 0.33614 | 0.04258 | 0.28299 | 0.01102 | 0.19978 |
| ADE | α | 0.03254 | 3.03561 | 0.18590 | 1.98069 | 0.15240 | 1.64511 | 0.12263 | 1.18370 |
| | θ | 0.05780 | 0.47396 | 0.02042 | 0.27611 | 0.00790 | 0.23992 | 0.02282 | 0.16854 |
| RTADE | α | 0.02352 | 3.73884 | 0.19775 | 2.25501 | 0.17546 | 1.85295 | 0.09130 | 1.09792 |
| | θ | 0.05845 | 0.52111 | 0.03041 | 0.29285 | 0.01985 | 0.25465 | 0.02027 | 0.15811 |
| BE_{SEL} | α | 0.94474 | 1.83007 | 0.99737 | 1.69937 | 1.08796 | 1.73426 | 1.17206 | 1.84599 |
| | θ | 0.23425 | 0.10830 | 0.24520 | 0.09155 | 0.25927 | 0.08987 | 0.27414 | 0.09349 |
| BE_{Linex1} | α | 1.04989 | 2.17655 | 1.07816 | 1.92858 | 1.15837 | 1.92901 | 1.23506 | 2.02266 |
| | θ | 0.24229 | 0.11316 | 0.25025 | 0.09437 | 0.26313 | 0.09204 | 0.27735 | 0.09536 |
| BE_{Linex2} | α | 0.85268 | 1.56954 | 0.92368 | 1.50949 | 1.02260 | 1.56660 | 1.11297 | 1.68999 |
| | θ | 0.22634 | 0.10369 | 0.24021 | 0.08882 | 0.25544 | 0.08775 | 0.27096 | 0.09166 |
| BE_{GEL1} | α | 0.90815 | 1.74628 | 0.96764 | 1.63632 | 1.06211 | 1.67755 | 1.14934 | 1.79318 |
| | θ | 0.22880 | 0.10551 | 0.24176 | 0.08981 | 0.25665 | 0.08849 | 0.27199 | 0.09230 |
| BE_{GEL2} | α | 0.83721 | 1.59573 | 0.90945 | 1.51943 | 1.01120 | 1.57054 | 1.10443 | 1.69231 |
| | θ | 0.21792 | 0.10016 | 0.23487 | 0.08641 | 0.25140 | 0.08579 | 0.26768 | 0.08996 |

Table 8: Average estimated Biases and RMSEs of different estimation methods for Zubair-Exponential distribution at different sample sizes n and different values of the parameters ($\alpha = 0.15$, $\theta = 1.0$).

| Method | | $n = 25$ | | $n = 50$ | | $n = 75$ | | $n = 100$ | |
|---------------|----------|----------|---------|----------|---------|----------|---------|-----------|---------|
| | | Bias | RMSE | Bias | RMSE | Bias | RMSE | Bias | RMSE |
| MLE | α | 0.19297 | 3.87371 | 0.02093 | 2.51168 | 0.02172 | 1.93934 | 0.02719 | 1.31088 |
| | θ | 0.09525 | 0.15225 | 0.03092 | 0.08073 | 0.02486 | 0.06182 | 0.01688 | 0.04216 |
| MPSE | α | 0.84649 | 4.57048 | 0.82222 | 3.94821 | 0.72547 | 3.59672 | 0.64564 | 2.86972 |
| | θ | 0.11076 | 0.13729 | 0.11718 | 0.10014 | 0.09956 | 0.08513 | 0.09108 | 0.06656 |
| LSE | α | 0.20220 | 3.34815 | 0.19999 | 2.06710 | 0.11982 | 1.60041 | 0.20269 | 1.34178 |
| | θ | 0.00362 | 0.14400 | 0.01755 | 0.08538 | 0.00558 | 0.06359 | 0.02242 | 0.05385 |
| WLSE | α | 0.14843 | 3.37915 | 0.17335 | 2.14048 | 0.10168 | 1.66194 | 0.13009 | 1.20276 |
| | θ | 0.01563 | 0.14329 | 0.01036 | 0.08238 | 0.00071 | 0.06131 | 0.00913 | 0.04646 |
| CVME | α | 0.38571 | 3.72041 | 0.07238 | 2.15066 | 0.09228 | 1.55408 | 0.03927 | 1.28179 |
| | θ | 0.12791 | 0.18417 | 0.04270 | 0.09498 | 0.04011 | 0.06664 | 0.01307 | 0.05477 |
| ADE | α | 0.04006 | 3.50002 | 0.09615 | 2.04323 | 0.08291 | 1.70671 | 0.12479 | 1.26095 |
| | θ | 0.04132 | 0.13696 | 0.00598 | 0.07640 | 0.00696 | 0.06001 | 0.00600 | 0.04565 |
| RTADE | α | 0.02870 | 4.08467 | 0.09021 | 2.23040 | 0.06764 | 1.86243 | 0.09472 | 1.18080 |
| | θ | 0.03282 | 0.14995 | 0.00233 | 0.07936 | 0.00663 | 0.06274 | 0.00467 | 0.04345 |
| BE_{SEL} | α | 1.45853 | 2.62075 | 1.45230 | 2.51465 | 1.48861 | 2.56392 | 1.52446 | 2.64130 |
| | θ | 0.39149 | 0.22653 | 0.39002 | 0.18997 | 0.38959 | 0.17734 | 0.39264 | 0.17453 |
| BE_{Linex1} | α | 1.47193 | 2.66788 | 1.47463 | 2.58993 | 1.51399 | 2.64896 | 1.55088 | 2.73091 |
| | θ | 0.39952 | 0.23442 | 0.39498 | 0.19445 | 0.39320 | 0.18041 | 0.39562 | 0.17705 |
| BE_{Linex2} | α | 1.43202 | 2.52476 | 1.43488 | 2.46119 | 1.46626 | 2.49272 | 1.49700 | 2.54899 |
| | θ | 0.38360 | 0.21897 | 0.38513 | 0.18566 | 0.38601 | 0.17432 | 0.38968 | 0.17205 |
| BE_{GEL1} | α | 1.44122 | 2.56676 | 1.43840 | 2.47675 | 1.47164 | 2.51334 | 1.50591 | 2.58295 |
| | θ | 0.38594 | 0.22180 | 0.38656 | 0.18711 | 0.38704 | 0.17527 | 0.39053 | 0.17282 |
| BE_{GEL2} | α | 1.40341 | 2.45041 | 1.40852 | 2.39442 | 1.44136 | 2.42861 | 1.47359 | 2.48764 |
| | θ | 0.37487 | 0.21259 | 0.37964 | 0.18150 | 0.38193 | 0.17120 | 0.38632 | 0.16944 |

Table 9: Confidence Intervals for MLEs and Credible Intervals for the Bayesian Estimates using BE_{SEL} , $BE_{Linear1}$, $BE_{Linear2}$, BE_{GEL1} & BE_{GEL2}

| Initial Values | Lower MLE | Upper MLE | Lower BE_{SEL} | Upper BE_{SEL} | Lower $BE_{Linear1}$ | Upper $BE_{Linear1}$ | Lower $BE_{Linear2}$ | Upper $BE_{Linear2}$ | Lower BE_{GEL1} | Upper BE_{GEL1} | Lower BE_{GEL2} | Upper BE_{GEL2} |
|-----------------|-----------|-----------|------------------|------------------|----------------------|----------------------|----------------------|----------------------|-------------------|-------------------|-------------------|-------------------|
| $\alpha = 0.15$ | 0.45359 | 2.99950 | 2.54591 | 0.61432 | 2.97925 | 2.36493 | 0.60819 | 2.85610 | 2.24791 | 0.64150 | 2.78190 | 2.14040 |
| $\theta = 0.25$ | 0.24174 | 0.52151 | 0.27977 | 0.26987 | 0.46131 | 0.19144 | 0.27790 | 0.43287 | 0.15497 | 0.28790 | 0.42426 | 0.13636 |
| $\alpha = 0.10$ | 0.34833 | 1.98825 | 1.63992 | 0.51848 | 1.99931 | 1.48083 | 0.59080 | 1.99491 | 1.40411 | 0.68136 | 1.99080 | 1.30944 |
| $\theta = 0.50$ | 0.47216 | 1.03588 | 0.56372 | 0.53894 | 0.91812 | 0.37918 | 0.56008 | 0.87002 | 0.30994 | 0.56260 | 0.83689 | 0.27429 |
| $\alpha = 0.05$ | 0.13236 | 0.99618 | 0.86382 | 0.18955 | 0.99873 | 0.80917 | 0.28449 | 0.99824 | 0.71376 | 0.42172 | 0.99661 | 0.57489 |
| $\theta = 0.50$ | 0.47622 | 1.05327 | 0.57705 | 0.54308 | 0.92943 | 0.38635 | 0.56233 | 0.87356 | 0.31122 | 0.58488 | 0.86146 | 0.27658 |
| $\alpha = 0.25$ | 0.12657 | 3.32064 | 3.19407 | 0.35274 | 3.14141 | 2.78867 | 0.52666 | 3.04532 | 2.51866 | 0.63052 | 2.90631 | 2.27579 |
| $\theta = 1.25$ | 1.18115 | 2.42399 | 1.24284 | 1.29896 | 2.18152 | 0.88257 | 1.35804 | 2.08793 | 0.72989 | 1.36926 | 2.01669 | 0.64743 |
| $\alpha = 1.0$ | 0.51873 | 4.08811 | 3.56937 | 0.78279 | 3.92454 | 3.14176 | 0.90953 | 3.63155 | 2.72202 | 1.09317 | 3.56610 | 2.47293 |
| $\theta = 0.50$ | 0.44233 | 0.86562 | 0.42330 | 0.49537 | 0.80730 | 0.31194 | 0.51600 | 0.78048 | 0.26448 | 0.52581 | 0.76038 | 0.23457 |
| $\alpha = 1.50$ | 0.51743 | 4.15348 | 3.63605 | 1.05796 | 4.27065 | 3.21269 | 1.11791 | 3.97303 | 2.85512 | 1.43437 | 4.00025 | 2.56588 |
| $\theta = 1.20$ | 1.02530 | 1.89675 | 0.87144 | 1.11208 | 1.79520 | 0.68312 | 1.16524 | 1.74094 | 0.57570 | 1.21481 | 1.72723 | 0.51241 |
| $\alpha = 0.25$ | 0.19481 | 3.24804 | 3.05323 | 0.34383 | 3.05487 | 2.71104 | 0.41576 | 2.85558 | 2.43983 | 0.69355 | 2.90839 | 2.21484 |
| $\theta = 2.0$ | 1.85699 | 3.74149 | 1.88450 | 2.05779 | 3.41504 | 1.35726 | 2.14484 | 3.27088 | 1.12603 | 2.16854 | 3.17947 | 1.01092 |
| $\alpha = 0.15$ | 0.45084 | 2.98673 | 2.53589 | 0.54684 | 2.86244 | 2.31560 | 0.60285 | 2.79490 | 2.19204 | 0.66239 | 2.77334 | 2.11095 |
| $\theta = 1.0$ | 0.92650 | 1.96084 | 1.03434 | 1.05693 | 1.78961 | 0.73268 | 1.09881 | 1.70043 | 0.60162 | 1.14441 | 1.67715 | 0.53274 |

The following conclusions can be drawn from the simulation results

1. As sample size increases, all estimators' Bias and RMSE values fall, demonstrating improved accuracy in model parameter estimation.
2. The least biased parameters across all the parameters and various sample sizes are LSE, WLSE, ADE, and RTADE.
3. For all sample sizes, the estimators' bias is positive.
4. BE_{SEL} fluctuates as sample size increases. While the other loss functions considered increase as sample size does.

6.2 Real Data Application

The data represent survival times of guinea pigs injected with different amount of tubercle bacilli studied by Bjerkedal [13]. In this section, the proposed method is illustrated by comparing the goodness of fits of the Z-Exponential (ZE) distribution with the other life models such as Zubair-Weibull (ZW) distribution, Kumaraswamy-Weibull (KW) distribution, Exponentiated-Weibull (EW) distribution and Exponential distribution (ED) distributions by applying real data. The analytical measures of goodness of fit namely the Log-Likelihood (Loglik), Akaike information criterion (AIC), Bayesian information criterion (BIC) and Kolmogorov-Smirnov (K-S) statistics are considered to compare the proposed method with the other fitted models. The decision for a better fit is based on the model with smaller values of these analytical measures.

Table 10: MLEs and Standard Errors of the Parameters of the fitted Distributions using Guinea pigs data

| Distribution | α | | θ | | β | | γ | |
|--------------|----------|-----------|----------|-----------|----------|-----------|----------|-----------|
| | Estimate | Std.Error | Estimate | Std.Error | Estimate | Std.Error | Estimate | Std.Error |
| ZE | 3.1718 | 0.9269 | 0.0139 | 0.0016 | | | | |
| ZW | 5.8420 | 2.2686 | 0.7768 | 0.1284 | 0.0553 | 0.0432 | | |
| KW | 0.3609 | 0.01638 | 0.0553 | 0.0065 | 0.6288 | 0.0019 | 0.5197 | 0.0025 |
| EW | 2.6541 | 1.5362 | 1.1604 | 0.3090 | 112.8544 | 46.2945 | | |
| ED | 0.0056 | 0.0057 | | | | | | |

Table 11: The analytical measures of the fitted models for the Guinea pigs data

| Distribution | LogLik | AIC | BIC | K-S |
|--------------|-----------|----------|----------|--------|
| ZE | -425.3208 | 854.6417 | 857.195 | 0.089 |
| ZW | -424.4359 | 854.8717 | 861.7018 | 0.0766 |
| KW | -474.6325 | 957.265 | 966.3717 | 0.3854 |
| EW | -425.6656 | 857.3311 | 864.1611 | 0.0891 |
| ED | -444.6156 | 891.2312 | 893.5079 | 0.2959 |

The proposed model (ZE) appears to have the smallest values of AIC and BIC hence has a better fit compared to the other distributions considered in this study.

Table 12: Classical and Bayes Methods Estimates for the ZE Distribution using the Guinea pigs data

| Method | | α | θ |
|--------|------------|----------|----------|
| MLE | Estimate | 3.17178 | 0.01388 |
| | Std. Error | 0.92693 | 0.00159 |
| MPS | Estimate | 2.71348 | 0.01303 |
| | Std. Error | 0.90672 | 0.00162 |
| LSE | Estimate | 3.04302 | 0.01406 |
| | Std. Error | 4.14189 | 0.00778 |
| WLSE | Estimate | 3.16415 | 0.01406 |
| | Std. Error | 0.18127 | 0.00034 |
| CVME | Estimate | 3.23315 | 0.01440 |
| | Std. Error | 4.24379 | 0.00778 |
| ADE | Estimate | 3.05141 | 0.01386 |
| | Std. Error | 1.56693 | 0.00282 |
| RADE | Estimate | 2.35009 | 0.01272 |
| | Std. Error | 2.26925 | 0.00390 |
| BE | Estimate | 2.78180 | 0.01340 |
| | Std. Error | 0.28363 | 0.00027 |

From the table 12, the α and θ values the Bayesian Estimation (BE) is preferred to other classical approaches because standard errors are less than others.

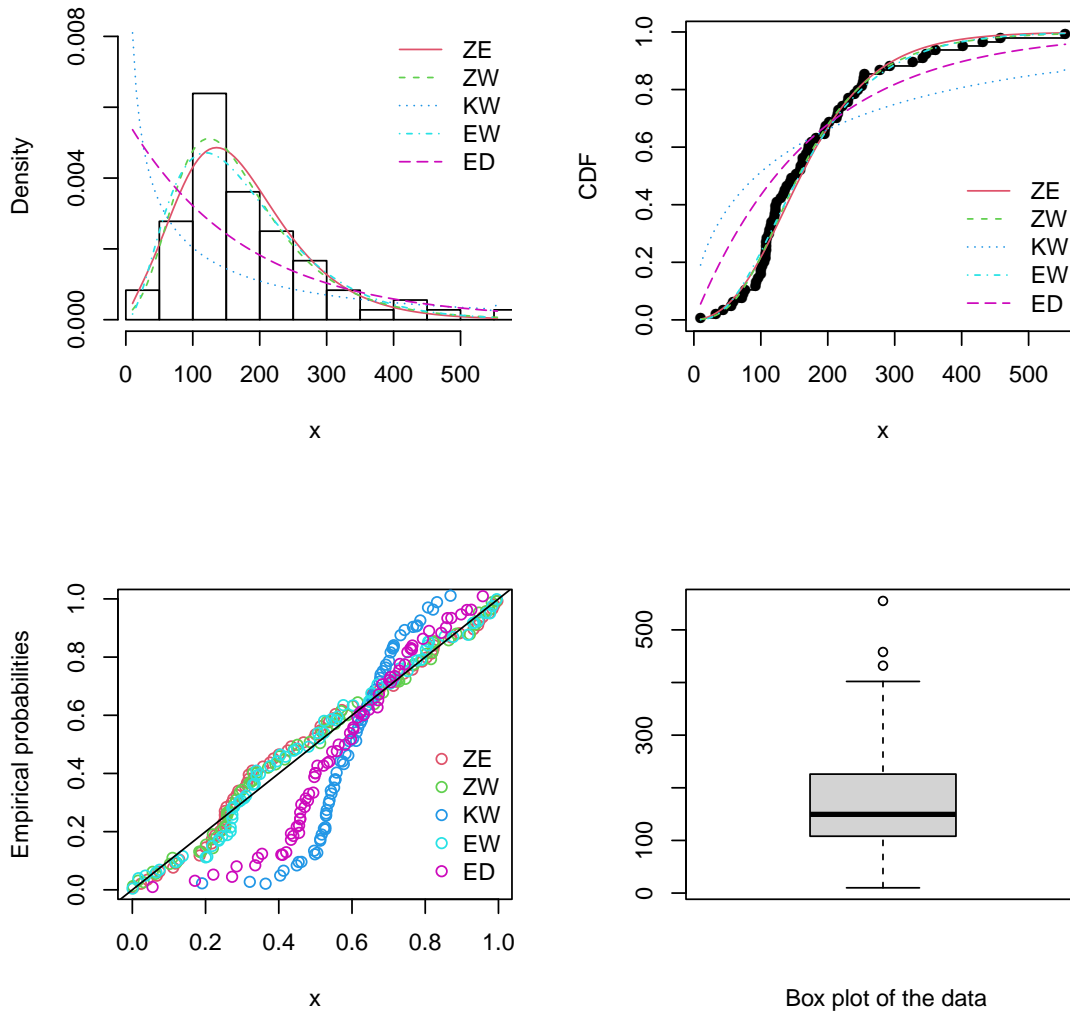


Figure 4: The estimated pdf, cdf, Kaplan-Meier and Box plots of the Zubair-Exponential distribution using the Guinea pigs data

7 Conclusion

A new sub-model of Zubair-G family life distributions, called the Zubair-Exponential distribution is proposed and studied. The mathematical properties of the new distribution including moments, moment generating function, order statistics and quantile function are derived. The quantile function has a closed form solution. The cdf, hazard rate and odd functions are increasing function and therefore can be deployed to study decreasing and increasing lifetime events. The maximum likelihood function, least squares estimate, weighted least squares, cramer von mises, Anderson Darling estimate and right-tailed Anderson Darling estimate are derived and implemented. Bayesian inference with Squared error loss, Linear-Exponential loss and Generalized Entropy loss functions were also studied and simulation of parameters based on the classical and bayesian methods are done. Application to real data using survival times of Guinea pigs was also illustrated. From the analytical measures of fitness and performance (AIC and BIC), the proposed Zubair-Exponential distribution is preferred to the other distributions compared.

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