Original Research Article

Inference on the Parameters of Zubair-Exponential Distribution with Application to Survival Times of Guinea Pigs

Abstract

In this paper, we derived a sub-model of Zubair-G familiy of distribution named Zubair-Exponential distribution with two parameters. Simulation of the Estimates of the parameters based on some classical methods are obtained. The likelihood equations and the maximum likelihood estimator as well as asymptotic confidence interval are derived. Bayes estimates with the estimates of the associated greatest posterior density credible interval are derived using squared error Loss (SEL),Linear-Exponential (LINEX) and Generalized Entropy Loss (GEL) functions. Using the Metropolis-Hasting algorithm and the method of Markov Chain Monte Carlo (MCMC), estimates of Bayes are summarized. To determine the performance of the estimates, a Monte Carlo simulation study is carried out and maximum likelihood estimates, their standard errors and measures of fitness using real data on survival times of Guinea pigs are obtained. The proposed distribution has a better fit based on Akaike Information criterion (AIC) and the Bayesian Information criterion (BIC).

Keywords— Zubair-Exponential distribution, Bayesian estimation, Maximum likelihood estimation, Markov Chain Monte Carlo, Squared Error Loss function, Linear-Exponential loss function, Generalized Entropy loss function

1 Introduction

To achieve a better fit, economy of parameter, flexibility and tractability of model, various authors have explored avenues of providing compact distributions that model lifetime data. Among them are Zubair [1], Onyekwere and Obulezi [2], Onyekwere et al [3]. Some authors have further studied various classical methods of estimation. Dey [4] studied the maximum likelihood estimation procedure together with Maximum product of spacing, Cramer von mises, Least squares, weighted least squares, Anderson Darling, Right-Tailed Anderson Darling and Percentile estimation methods. Recently, there are more advances in Bayesian estimation procedures. However, in Bayesian estimation, there are not many approaches as there are for classical estimation. The loss functions used in Bayesian estimation are generally grouped into symmetric and asymmetric functions. The Squared Error Loss function (SEL) is popular symmetric loss function because it attaches equal loss weight to over-and under-estimation. Next, is the Linear Exponential (LINEX) loss function which is asymmetric was proposed by Varian [5] and the Generalized Entropy Loss (GEL) function.

This work is motivated by the Zubair-G family of distributions proposed by Zubair [1]. The baseline here being the well-known Exponential distribution which compares favourable with the sub-model Zubair-Weibull distribution, an extension studied by Zubair [1].

The rest of the paper is organized as follows; the probability density and the associated functions of Zubair-Exponential distribution is presented in section 2, in section we derive the mathematical properties. Classical estimation of the parameters is carried out in section 4 while the Bayesian estimation is studied in section 5. In the end, section 6 is dedicated to simulation study and real life data analysis with conclusion following it.

2 Zubair-Exponential Distribution

Let $X \sim Zubair - G(\alpha, \xi)$ due to Zubair [1] then using Exponential distribution as a baseline distribution the probability density function (pdf), cumulative distribution function (cdf), survival function (sf) and hazard rate function (hrf), reversed hazard rate function (rhrf) and Odd function (chrf) of Zubair-Exponential distribution are respectively

$$f(x;\alpha,\theta)=\frac{2\alpha\theta \mathrm{e}^{-\theta x}(1-\mathrm{e}^{-\theta x})\mathrm{e}^{\alpha(1-\mathrm{e}^{-\theta x})^2}}{\mathrm{e}^{\alpha}-1};\quad x,\quad \alpha,\quad \theta>0 \tag{1}$$

$$F(x;\alpha,\theta)=\frac{\mathrm{e}^{\alpha(1-\mathrm{e}^{-\theta x})^2}-1}{\mathrm{e}^{\alpha}-1} \tag{2}$$

$$S(x;\alpha,\theta)=\frac{\mathrm{e}^{\alpha}-\mathrm{e}^{\alpha(1-\mathrm{e}^{-\theta x})^2}}{\mathrm{e}^{\alpha}-1} \tag{3}$$

$$h(x;\alpha,\theta)=\frac{2\alpha\theta \mathrm{e}^{-\theta x}(1-\mathrm{e}^{-\theta x})\mathrm{e}^{\alpha(1-\mathrm{e}^{-\theta x})^2}}{\mathrm{e}^{\alpha}-\mathrm{e}^{\alpha(1-\mathrm{e}^{-\theta x})^2}} \tag{4}$$

$$rh(x;\alpha,\theta)=\frac{2\alpha\theta \mathrm{e}^{-\theta x}(1-\mathrm{e}^{-\theta x})\mathrm{e}^{\alpha(1-\mathrm{e}^{-\theta x})^2}}{\mathrm{e}^{\alpha(1-\mathrm{e}^{-\theta x})^2}-1} \tag{5}$$

and

$$O(x;\alpha,\theta)=\frac{\mathrm{e}^{\alpha}(1-\mathrm{e}^{-\theta x})^2-1}{\mathrm{e}^{\alpha}-\mathrm{e}^{\alpha(1-\mathrm{e}^{-\theta x})^2}} \tag{6}$$

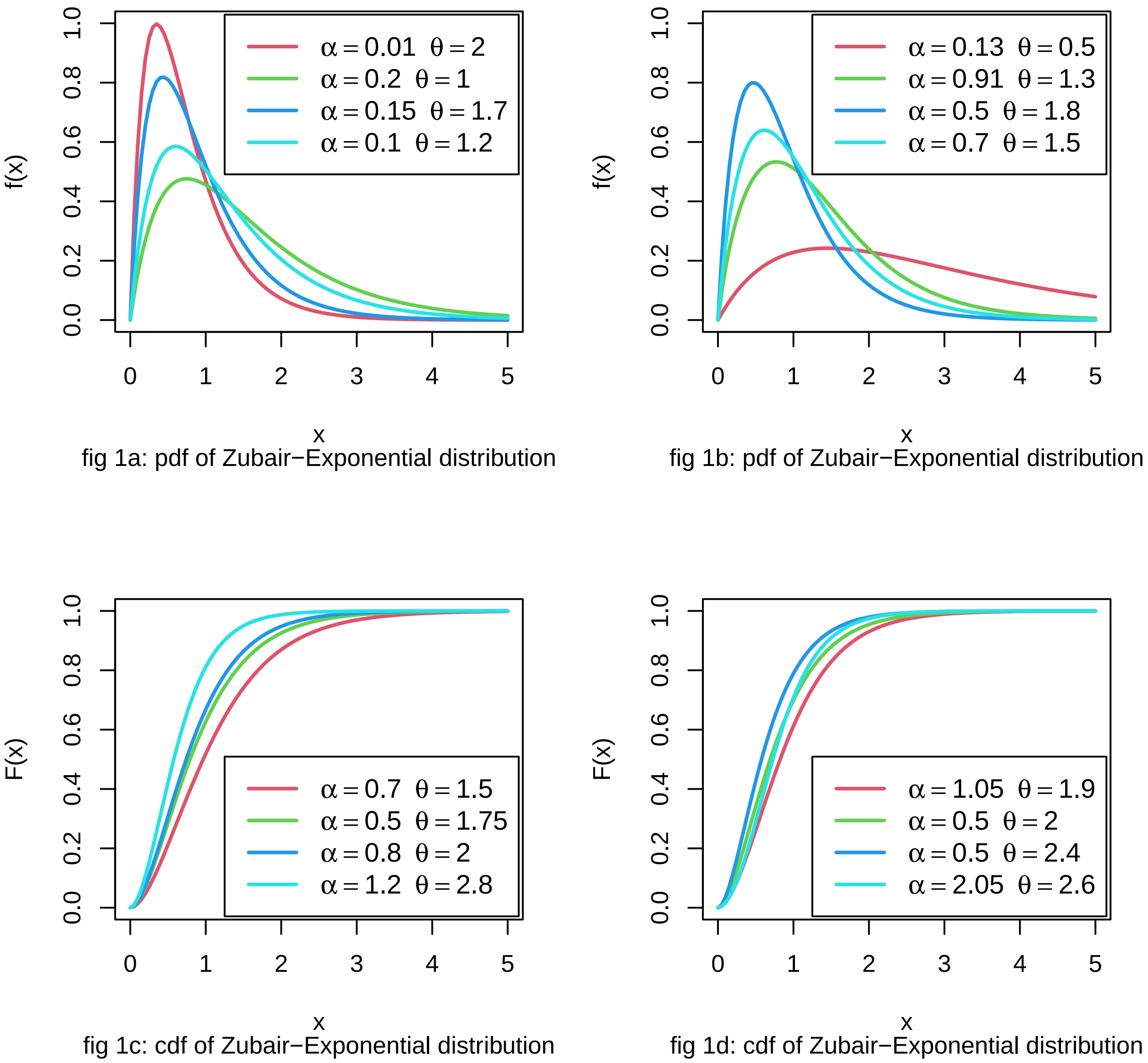

fig 1a: pdf of Zubair−Exponential distribution

fig 1b: pdf of Zubair−Exponential distribution

fig 1c: cdf of Zubair−Exponential distribution

fig 1d: cdf of Zubair−Exponential distribution

Figure 1: pdf and cdf of Zubair-Exponential distribution

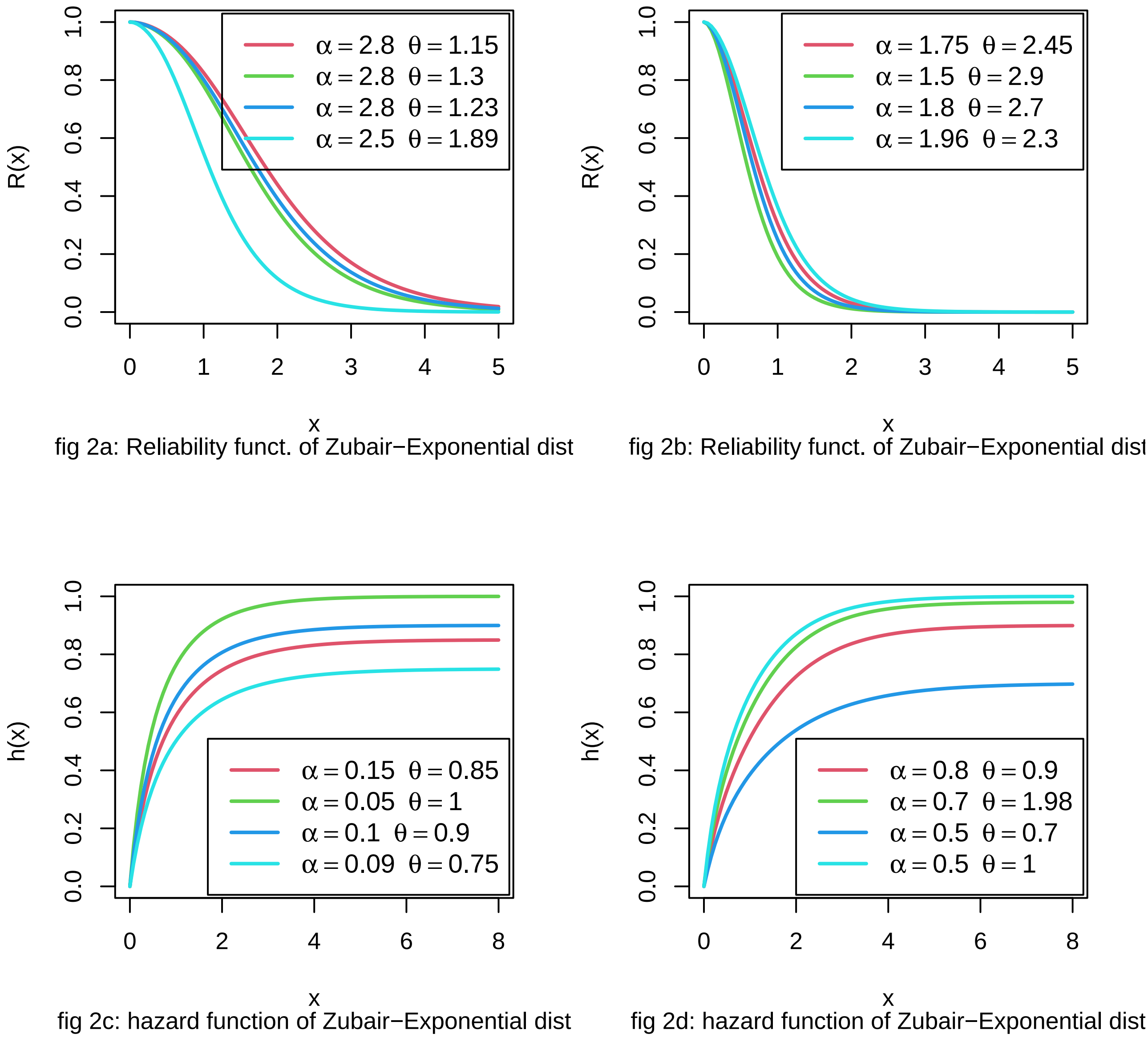

Figure 2: Survival and Hazard rate function of Zubair-Exponential distribution

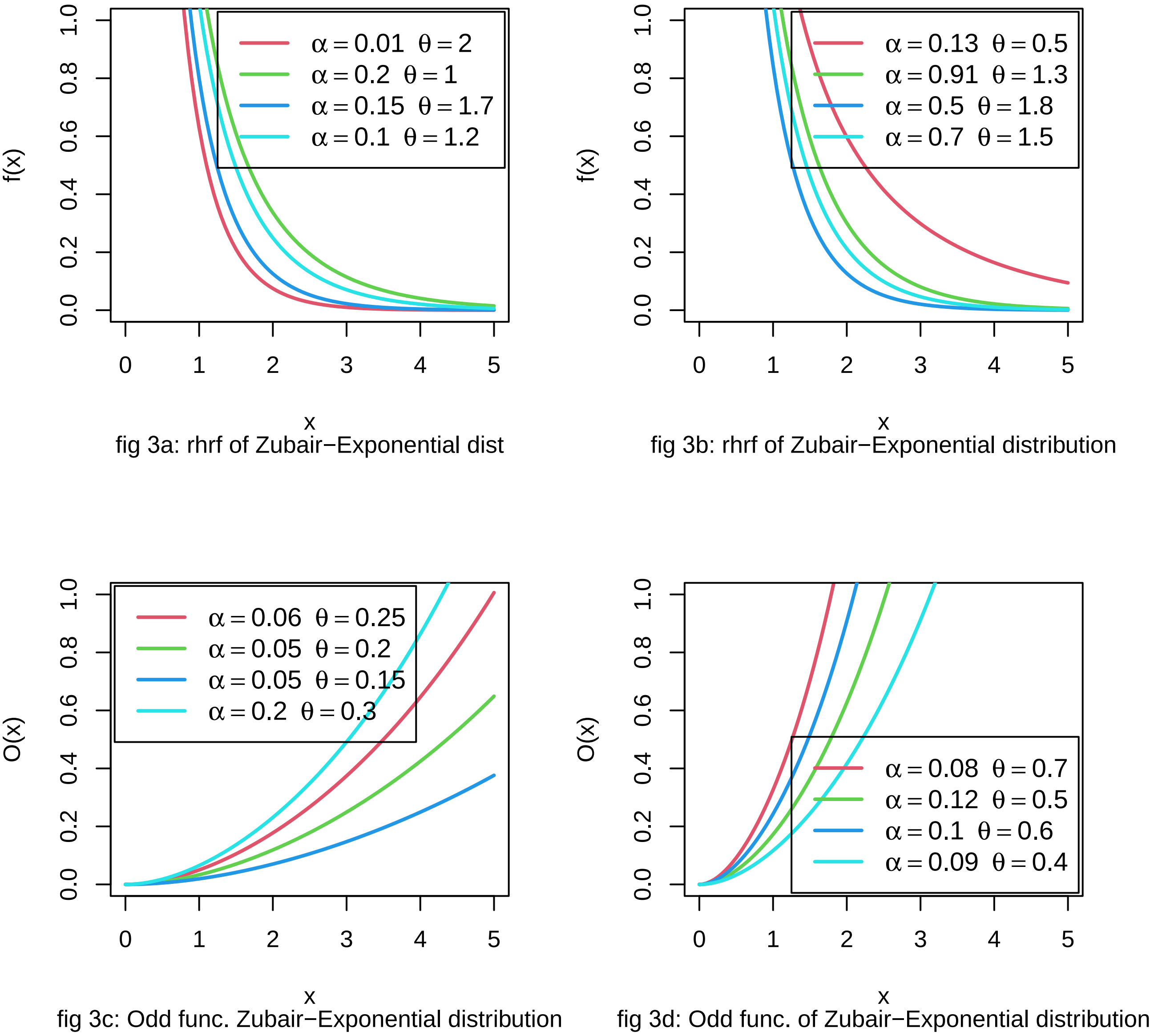

Figure 3: Reversed hazard rate and Odd functions of Zubair-Exponential distribution

3 Mathematical Properties

In this section, some basic mathematical properties are derived

3.1 Moments

The r^{th} non-central moment of $X \sim$ Zubair-Exponential $(x;\alpha,\theta)$ is

$$\mu_r = \int_0^\infty x^r f(x;\alpha,\theta)dx \tag{7}$$

Zubair [1] further expressed the moment as

$$\mu_r^{'} = 2\sum_{i=0}^{\infty} \frac{\alpha^{i+1}}{(\mathrm{e}^\alpha - 1)i!}\eta_{r;2i+1} \tag{8}$$

where

$$\eta_{r;2i+1} = \int_0^\infty x^r \theta \mathrm{e}^{-\theta x}(1-\mathrm{e}^{-\theta x})^{2i+1}dx = \theta\sum_{j=0}^{\infty}\binom{2i+1}{j}\frac{\Gamma(r+1)}{(\theta(j+1))^{r+1}} \tag{9}$$

Therefore , the closed-form of the r^{th} non-central moment is

$$\mu_r^{'} = \frac{2}{\theta^r}\sum_{i,j,r=0}^{\infty}\frac{\alpha^{i+1}}{(\mathrm{e}^\alpha - 1)i!}\binom{2i+1}{j}\frac{\Gamma(r+1)}{(j+1)^{r+1}} \tag{10}$$

The mean is

$$\mu = \frac{2}{\theta}\sum_{i,j=0}^{\infty}\frac{\alpha^{i+1}}{(\mathrm{e}^\alpha - 1)i!}\binom{2i+1}{j}\frac{1}{(j+1)^2} \tag{11}$$

The 2^{nd},3^{rd} and 4^{th} non-central moments are

$$\mu_2 = \frac{6}{\theta^2}\sum_{i,j=0}^{\infty}\frac{\alpha^{i+1}}{(\mathrm{e}^\alpha - 1)i!}\binom{2i+1}{j}\frac{1}{(j+1)^3} \tag{12}$$

$$\mu_3 = \frac{12}{\theta^3}\sum_{i,j=0}^{\infty}\frac{\alpha^{i+1}}{(\mathrm{e}^\alpha - 1)i!}\binom{2i+1}{j}\frac{1}{(j+1)^4} \tag{13}$$

and

$$\mu_4 = \frac{24}{\theta^4}\sum_{i,j=0}^{\infty}\frac{\alpha^{i+1}}{(\mathrm{e}^\alpha - 1)i!}\binom{2i+1}{j}\frac{1}{(j+1)^5} \tag{14}$$

The moment generating function is

$$M_X(t) = 2\sum_{i,r=0}^{\infty}\frac{t^r\alpha^{i+1}}{(\mathrm{e}^\alpha - 1)r!i!}\eta_{r,2i+1} \tag{15}$$

Substitute equation 9 in 15 to have

$$M_X(t) = \frac{2}{\theta^r}\sum_{i,j,r=0}^{\infty}\frac{t^r\alpha^{i+1}}{(\mathrm{e}^\alpha - 1)r!i!}\binom{2i+1}{j}\frac{\Gamma(r+1)}{(j+1)^{r+1}} \tag{16}$$

3.2 Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample of size n drawn from the Zubair-Exponential (α,θ). Let $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be thet associated order statistics, then the density of $X_{r:n}$ where $r \in (1,2,...,n)$ is obtained as

$$f_{r:n}(x) = \frac{f(x,\alpha,\theta)}{\mathscr{B}(r,n-r+1)}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i[F(x,\alpha,\theta)]^{i+r-1} \tag{17}$$

where $f(x,\alpha,\theta)$ and $F(x,\alpha,\theta)$ are the pdf and cdf defined in equations 1 and 2 respectively. Hence, making substitution yields the density of the order statistics

$$f_{r:n}(x) = \frac{2\alpha\theta\mathrm{e}^{-\theta x}(1-\mathrm{e}^{-\theta x})\mathrm{e}^{\alpha(1-\mathrm{e}^{-\theta x})^2}}{(\mathrm{e}^\alpha - 1)\mathscr{B}(r,n-r+1)}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i\left[\frac{\mathrm{e}^{\alpha(1-\mathrm{e}^{-\theta x})^2}-1}{\mathrm{e}^\alpha - 1}\right]^{i+r-1} \tag{18}$$

3.3 Quantile Function

Let $X \sim$ Zubair-Exponential (x,α,θ), the quantile function x_q is given as

$$x_q = -\frac{1}{\theta}\left\{\ln\left[1-\left\{\frac{\ln(q\mathrm{e}^\alpha - q + 1}{\alpha}\right\}^{\frac{1}{2}}\right]\right\} \tag{19}$$

4 Classical Estimation of Zubair-Exponential Distribution

In this section, we consider some classical and Bayesian methods of estimation.

4.1 Maximum Likelihood Estimation of the Zubair-Exponential Distribution Parameters

Let $\left(X_1, X_2, ..., X_n\right)$ be n random samples drawn from Zubair-Exponential distribution, then the log-likelihood function is given as

$$logL(x,\alpha,\theta) = -n\log(e^{\alpha}-1) + n\log 2 + n\log\alpha + \sum_{i=1}^{n}\log(\theta e^{-\theta x}) + \sum_{i=1}^{n}\log(1-e^{-\theta x}) + \alpha\sum_{i=1}^{n}(1-e^{-\theta x})^2 \tag{20}$$

Taking a partial differentiation with respect to α and θ and equating the results to zero yield

$$n\left(\frac{1}{1-e^{-\alpha}} - \frac{1}{\alpha}\right) = \sum_{i=1}^{n}(1-e^{-\theta x})^2 \tag{21}$$

$$\frac{n}{\theta} - n\theta + \theta\sum_{i=1}^{n}\frac{1}{e^{\theta x}-1} + 2\theta\alpha\sum_{i=1}^{n}e^{-\theta x}(1-e^{-\theta x}) = 0 \tag{22}$$

The two non-closed form solutions are implemented in R using Newton-Raphson's iterative algorithm

4.2 Least Squares Estimation (LSE)

Let $(X_1, X_2, ..., X_n)$ be a random sample of size n drawn from a Zubair-Exponential distribution, then if we take the expectation of the cdf and maximize it, we arrive at the ordinary least squares estimates of the parameters α and θ

$$L(\alpha,\theta) = \underset{(\alpha,\theta)}{\arg\min}\sum_{i=1}^{n}\left[F(x_{i:n}|\lambda,\theta) - \frac{i}{n+1}\right]^2. \tag{23}$$

The estimates are obtained by solving the following non-linear equations

$$\sum_{i=1}^{n}\left[F(x_{i:n}|\alpha,\theta) - \frac{i}{n+1}\right]^2\Delta_1(x_{i:n}|\alpha,\theta) = 0 \tag{24}$$

$$\sum_{i=1}^{n}\left[F(x_{i:n}|\alpha,\theta) - \frac{i}{n+1}\right]^2\Delta_2(x_{i:n}|\alpha,\theta) = 0 \tag{25}$$

where

$$\Delta_1(x_{i:n}|\alpha,\theta) = \frac{(1-e^{-\theta x})^2(e^{\alpha}-1) - e^{\alpha}(e^{\alpha(1-e^{-\theta x})^2}-1)}{(e^{\alpha}-1)^2} \tag{26}$$

$$\Delta_2(x_{i:n}|\alpha,\theta) = \frac{2\alpha\theta e^{-\theta x}(1-e^{-\theta x})e^{\alpha(1-e^{-\theta x})^2}}{e^{\alpha}-1} \tag{27}$$

4.3 Weighted Least Squares Estimation (WLSE)

The weighted least squares estimates $\hat{\alpha}_{WLSE}$ and $\hat{\theta}_{WLSE}$ of Zubair-Exponential distribution parameters α and θ are obtained by minimizing the function $W(\alpha,\theta)$ with respect to α and θ

$$W(\alpha,\theta) = \underset{(\alpha,\theta)}{\arg\min}\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}|\theta) - \frac{i}{n+1}\right]^2. \tag{28}$$

Solving the following non-linear equation yields the estimate

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}|\theta) - \frac{i}{n+1}\right]\Delta_1(x_{i:n}|\theta) = 0 \tag{29}$$

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}|\theta) - \frac{i}{n+1}\right]\Delta_2(x_{i:n}|\theta) = 0 \tag{30}$$

where $\Delta_1(x.|\alpha,\theta)$ and $\Delta_2(x.|\alpha,\theta)$ is as defined in (26) and (27) respectively.

4.4 Maximum Product Spacing Estimators (MPSE)

A good substitute for the greatest likelihood approach is the maximum product spacing method, which approximates the Kullback-Leibler information measure. Let us now suppose that the data are ordered in an increasing manner. Then, the maximum product spacing for the Zubair-Exponential distribution is given as follows

$$K-L(\alpha,\theta|x_i)=\left(\prod_{i=1}^{n+1}D_l(x_i,\alpha,\theta)\right)^{\frac{1}{n+1}}, \tag{31}$$

where $D_l(x_i,\alpha,\theta)=F(x_i;\alpha,\theta)-F(x_{i-1};\alpha,\theta)$, $i=1,2,3,...,n$

Similarly, one can also choose to maximize the function

$$H(\alpha,\theta)=\frac{1}{n+1}\sum_{i=1}^{n+1}\ln D_i(\alpha,\theta). \tag{32}$$

By taking the first derivative of the function $H(\vartheta)$ with respect to α and θ, and solving the resulting nonlinear equations, at $\frac{\partial H(\phi)}{\partial\alpha}=0$ and $\frac{\partial H(\phi)}{\partial\theta}=0$, where $\phi=(\alpha,\theta)$, we obtain the value of the parameter estimates.

4.5 Cramér-von-Mises Estimation (CVME)

The Cramér-von-Mises estimates $\hat{\alpha}_{CVME}$, and $\hat{\theta}_{CVME}$ of the Zubair-Exponential distribution parameters α, and θ are obtained by minimizing the function $C(\alpha,\theta)$ with respect to α, and θ

$$C(\alpha,\theta)=\underset{(\alpha,\theta)}{\arg\min}\left\{\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{i:n}|\alpha,\theta)-\frac{2i-1}{2n}\right]^2\right\}. \tag{33}$$

The estimates are obtained by solving the following non-linear equations

$$\begin{aligned}\sum_{i=1}^{n}\left(F(x_{i:n}|\alpha,\theta)-\frac{2i-1}{2n}\right)\Delta_1(x_{i:n}|\alpha,\theta)=0\\ \sum_{i=1}^{n}\left(F(x_{i:n}|\alpha,\theta)-\frac{2i-1}{2n}\right)\Delta_2(x_{i:n}|\alpha,\theta)=0\end{aligned} \tag{34}$$

where $\Delta_1(x.|\alpha,\theta)$ and $\Delta_2(x.|\alpha,\theta)$ is as defined in (26) and (27) respectively.

4.6 Anderson-Darling Estimation (ADE)

The Anderson-Darling estimates $\hat{\alpha}_{ADE}$, and $\hat{\theta}_{ADE}$ of the Zubair-Exponential distribution parameters α and θ are obtained by minimizing the function $A(\alpha,\theta)$ with respect to α and θ

$$A(\alpha,\theta)=\underset{(\alpha,\theta)}{\arg\min}\sum_{i=1}^{n}(2i-1)\left\{\ln F(x_{i:n}|\alpha,\theta)+\ln\left[1-F(x_{n+1-i:n}|\alpha,\theta)\right]\right\}. \tag{35}$$

The estimates are obtained by solving the following sets of non-linear equations

$$\begin{aligned}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_1(x_{i:n}|\alpha,\theta)}{F(x_{i:n}|,\theta)}-\frac{\Delta_1(x_{n+1-i:n}|\alpha,\theta)}{1-F(x_{n+1-i:n}|\alpha,\theta)}\right]=0\\ \sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_2(x_{i:n}|\alpha,\theta)}{F(x_{i:n}|\alpha,\theta)}-\frac{\Delta_2(x_{n+1-i:n}|\alpha,\theta)}{1-F(x_{n+1-i:n}|\alpha,\theta)}\right]=0\end{aligned} \tag{36}$$

where $\Delta_1(x.|\alpha,\theta)$ and $\Delta_2(x.|\alpha,\theta)$ is as defined in (26) and (27) respectively.

4.7 Right-Tailed Anderson-Darling Estimation (RTADE)

The Right-Tailed Anderson-Darling estimates $\hat{\alpha}_{RTADE}$ and $\hat{\theta}_{RTADE}$ of the Zubair-Exponential distribution parameters α and θ are obtained by minimizing the function $R(\alpha,\theta)$ with respect to α and θ

$$R(\alpha,\theta)=\underset{(\alpha,\theta)}{\arg\min}\left\{\frac{n}{2}-2\sum_{i=1}^{n}F(x_{i:n}|\alpha,\theta)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\ln\left[1-F(x_{n+1-i:n}|\alpha,\theta)\right]\right\}. \tag{37}$$

The estimates can be obtained by solving the following set of non-linear equations

$$\begin{aligned}-2\sum_{i=1}^{n}\frac{\Delta_1(x_{i:n}|\alpha,\theta)}{F(x_{i:n}|\alpha,\theta)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_1(x_{n+1-i:n}|\alpha,\theta)}{1-F(x_{n+1-i:n}|\alpha,\theta)}\right]=0\\ -2\sum_{i=1}^{n}\frac{\Delta_2(x_{i:n}|\alpha,\theta)}{F(x_{i:n}|\alpha,\theta)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_2(x_{n+1-i:n}|\alpha,\theta)}{1-F(x_{n+1-i:n}|\alpha,\theta)}\right]=0\end{aligned} \tag{38}$$

where $\Delta_1(x.|\alpha,\theta)$ and $\Delta_2(x.|\alpha,\theta)$ is as defined in (26) and (27) respectively. The estimates given in (15), (27), (26), (28), (30), (34), (36) are obtained using **optim()** function in R with the Newton-Raphson iterative algorithm.

5 Bayesian Inference on the Zubair-Exponential Distribution Parameters

This section deals with the Bayesian estimate (BE) of the unknown parameters of the Zubair-Exponential distribution. For Bayesian parameter estimation, many loss functions, including squared error, LINEX, and generalized entropy loss functions, can be taken into consideration. We can consider applying independent gamma priors for the parameters α and θ with pdfs in the parameter prior distributions of Zubair-Exponential as follows;

$$\begin{aligned} \pi_1(\alpha) &\propto \alpha^{s_1-1}\mathrm{e}^{-q_1\alpha} && > 0, s_1 > 0, q_1 > 0, \\ \pi_2(\theta) &\propto \theta^{s_2-1}\mathrm{e}^{-q_2\theta} && \theta > 0, s_2 > 0, q_2 > 0, \end{aligned} \tag{39}$$

where the hyper-parameters $s_j, q_j, j = 1,2$ are selected to reflect the prior knowledge about the unknown parameters. The joint prior for $\phi = (\alpha, \theta)$ is given by

$$\begin{aligned} \pi(\phi) &= \pi_1(\alpha)\pi_2(\theta) \\ \pi(\phi) &\propto \alpha^{s_1-1}\theta^{s_2-1}\mathrm{e}^{\{-q_1\alpha-q_2\theta\}}. \end{aligned} \tag{40}$$

The corresponding posterior density given the observed data $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is given by:

$$\pi(\phi \mid \mathbf{x}) = \frac{\pi(\phi)\ell(\phi)}{\int_\phi \pi(\phi)\ell(\phi)d\phi},$$

Which implies that the posterior density function is:

$$\pi(\phi \mid \mathbf{x}) \propto \frac{2^n \alpha^{n+s_1-1}\theta^{n+s_2-1}}{(\mathrm{e}^\alpha - 1)^n}\mathrm{e}^{-\theta\sum x - q_1\alpha - q_2\theta + \alpha\sum(1-\mathrm{e}^{-\theta x})^2}\prod_{i=1}^{n}(1-\mathrm{e}^{-\theta x}) \tag{41}$$

Given any function, such as $l(\phi)$ under the squared error loss (SEL) function, the Bayes estimator is given by

$$\hat{\phi}_{BE_{SEL}} = E\left[l(\phi)|\mathbf{x}\right] = \int_\phi l(\phi)\pi(\phi|x)d\phi. \tag{42}$$

The SEL impacts underestimation and overestimation equally because it has an asymmetric loss function. In many actual situations, both underestimation and overestimation can have serious implications. A proposed LINEX loss can be made in certain instances as an alternative to the SE loss given by

$$\left(l(\phi), \hat{l}(\phi)\right) = \mathrm{e}^{\{\hat{l}(\phi)-l(\phi)\}} - v\left(\hat{l}(\phi) - l(\phi)\right) - 1.$$

where $v \neq 0$ is a shape parameter. Here $v > 1$ suggests that an overestimation is more serious than an underestimation, and vice versa for $v < 0$. Further v approaching zero replicates the SE loss function itself. One may refer to Varian [5] and Doostparast *et al.* [6] for more details in this regard. The BE of $l(\phi)$ under this loss can be derived as

$$\hat{\phi}_{BE_{LINEX}} = E\left[\mathrm{e}^{\{-vl(\phi)\}}|\mathbf{x}\right] = -\frac{1}{v}\log\left[\int_\phi \mathrm{e}^{\{-vl(\phi)\}}\pi(\phi|x)d\phi\right]. \tag{43}$$

Additionally, we take into account the general entropy loss (GEL) function suggested by Calabria and Pulcini [7], which is defined as follows.

$$\left(l(\phi), \hat{l}(\phi)\right) = \left(\frac{\hat{l}(\phi)}{l(\phi)}\right)^\tau - \tau\log\left(\frac{\hat{l}(\phi)}{l(\phi)}\right) - 1,$$

where the shape parameter $\tau \neq 0$ denotes a departure from symmetry. It views overestimation as more significant than underestimating when $\tau > 0$ and the opposite is true when $\tau < 0$. Given is the Bayes estimator with regard to the GE loss function.

$$\hat{\phi}_{BE_{GEL}} = \left[E\left(\left(l(\phi)\right)^{-\tau}|\mathbf{x}\right)\right]^{-1/\tau} = \left[\int_\phi \left(l(\phi)\right)^{-\tau}\pi(\phi|x)d\phi\right]^{-1/\tau}. \tag{44}$$

The estimations produced by (42), (43), and (44) can be seen to not be able to be transformed into closed-form expressions. We then use the Markov chain Monte Carlo (MCMC) approach to generate posterior samples and arrive at suitable BEs.

5.1 Markov Chain Monte Carlo

A general simulation technique for computing posterior quantities of interest and sampling from posterior distributions is the MCMC technique. Read Ravenzwaaij *et al.* [8] for further details on MCMC. In fact, using a kernel estimate of the posterior distribution and the MCMC samples, it is possible to properly quantify the posterior uncertainty with regard to the parameter ϕ.

The basis of MCMC algorithms is the concept of a discrete-time evolving Markov chain. As a stochastic process, a Markov chain

$$\phi^{(0)}, \phi^{(1)}, \phi^{(2)}, \ldots$$

.

Here, the random variable $\phi^{(i)}$ has values that are contained in a "state space." The state space, or the process's state at time i, remains constant across time i. The following Markov property applies to Markov chains: Only via the current state $\phi^{(i)}$ does the distribution of the next state $\phi^(i+1)$ depend on the past $\phi^{(0)}, \phi^{(1)}, \ldots, \phi^{(i)}$. The homogeneous Markov chains employed in MCMC approaches allow the conditional distribution of $\phi^{(i+1)}$ given $\phi^{(i)}$ to be independent of the index i. MCMC sample selection from a distribution:

1. Starting with an initial guess: simply one possible value that could be taken as coming from the distribution.
2. creating a number of fresh samples from this first hunch. Two steps are used to create each new sample:
   - A modest random perturbation is added to the most recent sample to create a proposal for the new sample.
   - Acceptance: Whether the new suggestion is rejected or accepted as the new sample (in which case the old sample is retained).

There are various techniques for integrating randomness to produce proposals, as well as numerous methods for acceptance and rejection, including Gibbs sampling and the Metropolis-Hastings algorithm.

5.2 Metropolis-Hasting Algorithm

A proposed distribution and the beginning values of the unknown parameters ϕ must be defined in order to carry out the MH algorithm for the Zubair-Exponential distribution. A multivariate normal distribution, defined as $q\left(\phi^{'}|\phi\right) \equiv N_3(\phi, S_\phi)$, will be taken into account for the proposal distribution, where S_ϕ denotes the variance-covariance matrix. It's possible to make unfavorable observations, which is unacceptable. The MLEs for ϕ are taken into account for the starting values, i.e., $\phi^{(0)} = \hat{\phi}_{MLE}$. The Fisher information matrix $I(.)$ is used to pick S_ϕ, which is thought to be the asymptotic variance-covariance matrix $I^{-1}\left(\hat{\phi}_{MLE}\right)$. It is noted that the choice of S_ϕ, which affects the MH algorithm's acceptance rate, is a critical decision.

In this regard, the MH algorithm's procedures for taking a sample from the provided posterior density (referred to as "(41)") are as follows:

Step 1. Set initial value of ϕ as $\phi^{(0)} = \left(\hat{\alpha}_{MLE}, \hat{\theta}_{MLE}\right)$.

Step 2. For $i = 1, 2, \ldots, M$ repeat the following steps:

2.1: Set $\phi = \phi^{(i-1)}$.

2.2: Generate a new candidate parameter value δ from $N_3\left(\log\phi, S_\phi\right)$.

2.3: Set $\theta^{'} = \exp(\delta)$.

2.4: Calculate $\beta = \frac{\pi(\phi^{'}|x)}{\pi(\phi|x)}$, where $\pi(\cdot)$ is the posterior density in (41).

2.5: Generate a sample u from the uniform $U(0,1)$ distribution.

2.6: Accept or reject the new candidate $\theta^{'}$

$$\begin{cases} \text{If} \quad u \leq \beta \quad \text{set} \quad \phi^{(i)} = \phi^{'} \\ \text{otherwise} \quad \text{set} \quad \phi^{(i)} = \phi. \end{cases}$$

Finally, part of the initial samples can be eliminated (burn-in) from the random samples of size M derived from the posterior density, and the remaining samples can then be used to calculate Bayes estimates. Using MCMC under the SEL, LINEX, and GEL functions, the BEs of $\phi^{(i)} = \left(\alpha^{(i)}, \theta^{(i)}\right)$ can be calculated as follows.

$$\hat{\phi}_{BE_{SEL}} = \frac{1}{M - l_B} \sum_{i=l_B}^{M} \phi^{(i)}, \tag{45}$$

$$\hat{\phi}_{BE_{LINEX}} = -\frac{1}{v} \log\left[\frac{1}{M - l_B} \sum_{i=l_B}^{M} e^{\{-v\phi^{(i)}\}}\right], \tag{46}$$

$$\hat{\phi}_{BE_{GEL}} = \left[\frac{1}{M - l_B} \sum_{i=l_B}^{M} \left(\phi^{(i)}\right)^{-\tau}\right]^{-1/\tau}, \tag{47}$$

where l_B represents the number of burn-in samples.

6 Applications

In this section, we consider both simulation and real life data study.

6.1 Simulation study

In this subsection, we simulate data for the Zubair-Exponential to compare the performance of the Non-Bayesian estimation methods discussed in the previous section. we generate 1000 data from the Zubair-Exponential distribution by considering the initial parameter values as

- $\alpha = 0.15$ and $\theta = 0.25$
- $\alpha = 0.10$ and $\theta = 0.50$
- $\alpha = 0.05$ and $\theta = 0.50$
- $\alpha = 0.25$ and $\theta = 1.25$
- $\alpha = 1.0$ and $\theta = 0.50$
- $\alpha = 1.50$ and $\theta = 1.20$
- $\alpha = 0.25$ and $\theta = 2.0$
- $\alpha = 0.15$ and $\theta = 1.0$

and sample sizes $n = 25, 50, 75, 100$. For each estimate $\hat{\phi} = (\hat{\alpha}, \hat{\theta})$, we compute the Bias and Root Mean Squared Error(RMSE) respectively as

$$Bias(\hat{\phi}) = \frac{1}{B}\sum_{i=1}^{B}(\hat{\phi}_i - \phi),$$

and

$$RMSE(\hat{\phi}) = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(\hat{\phi}_i - \phi)^2}.$$

For the Non-Bayesian procedure, we employed the Newton-Raphson algorithm for finding the desired estimates. For the Bayesian procedure, BEs using MCMC by utilizing the MH algorithm under informative prior are computed. For the informative prior, we assumed all hyperparameters of gamma distributions are equal to twice of parameter values. These values are then plugged-in to calculate the desired estimates. While utilizing the MH algorithm, the MLEs take into account initial guess values. In the end, 2000 burn-in samples are discarded among the overall 10000 samples generated from the posterior density and subsequently obtained BEs under different loss functions, namely: SEL, LINEX at $v = -1.5, 1.5$, and finally GEL at $\tau = -0.5, 0.5$. For each approach, we determine Bias and RMSE.

Table 1: Average estimated Biases and RMSEs of different estimation methods for Zubair-Exponential distribution at different sample sizes n and different values of the parameters ($\alpha = 0.15,\quad \theta = 0.25$).

Method		$n = 25$		$n = 50$		$n = 75$		$n = 100$	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	α	0.27572	3.55456	0.00449	2.58690	0.02469	1.66180	0.05772	1.53109
	θ	0.02616	0.00952	0.00960	0.00539	0.00562	0.00324	0.00165	0.00291
MPSE	α	0.80541	4.36457	0.83695	4.11626	0.75338	3.36524	0.67598	3.09018
	θ	0.02732	0.00852	0.02911	0.00662	0.02686	0.00489	0.02515	0.00455
LSE	α	0.09472	3.15273	0.11413	2.06325	0.20930	1.54713	0.16962	1.36258
	θ	0.00503	0.00951	0.00073	0.00562	0.00554	0.00382	0.00561	0.00346
WLSE	α	0.11416	3.21506	0.07756	2.10771	0.13185	1.46836	0.11426	1.30819
	θ	0.00476	0.00930	0.00283	0.00547	0.00156	0.00351	0.00280	0.00314
CVME	α	0.42522	3.76136	0.19392	2.07264	0.00073	1.50594	0.00148	1.29131
	θ	0.03309	0.01215	0.01739	0.00620	0.00603	0.00404	0.00338	0.00349
ADE	α	0.03753	3.22576	0.04870	2.11722	0.06481	1.26523	0.06990	1.17150
	θ	0.01290	0.00893	0.00474	0.00507	0.00158	0.00310	0.00078	0.00283
RTADE	α	0.03955	4.02506	0.12369	2.47219	0.10998	1.50134	0.08989	1.34895
	θ	0.00757	0.00955	0.00073	0.00537	0.00090	0.00325	0.00199	0.00298
BE_{SEL}	α	1.47577	2.67869	1.49266	2.65383	1.52892	2.70194	1.56658	2.78276
	θ	0.10940	0.01712	0.10444	0.01347	0.10275	0.01228	0.10270	0.01189
BE_{Linex1}	α	1.47374	2.66492	1.51114	2.71647	1.55077	2.77507	1.58985	2.86181
	θ	0.10992	0.01726	0.10476	0.01354	0.10298	0.01233	0.10289	0.01193
BE_{Linex2}	α	1.44788	2.57392	1.45973	2.53554	1.50431	2.61877	1.53966	2.69010
	θ	0.10888	0.01698	0.10413	0.01339	0.10252	0.01222	0.10251	0.01185
BE_{GEL1}	α	1.46077	2.63211	1.47284	2.59008	1.51485	2.66130	1.54949	2.72868
	θ	0.10800	0.01679	0.10357	0.01328	0.10210	0.01214	0.10217	0.01178
BE_{GEL2}	α	1.42046	2.50171	1.43422	2.46963	1.48041	2.55588	1.51721	2.62985
	θ	0.10519	0.01614	0.10183	0.01290	0.10082	0.01187	0.10111	0.01156

Table 2: Average estimated Biases and RMSEs of different estimation methods for Zubair-Exponential distribution at different sample sizes n and different values of the parameters ($\alpha = 0.10, \quad \theta = 0.50$).

Method		$n = 25$		$n = 50$		$n = 75$		$n = 100$	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	α	0.24880	3.77480	0.02210	2.19270	0.07960	1.72840	0.10890	1.60496
	θ	0.04980	0.03780	0.01390	0.01850	0.00380	0.01360	0.00040	0.01178
MPSE	α	0.71260	4.06370	0.85390	3.80540	0.76120	3.17540	0.77300	3.36579
	θ	0.04890	0.03200	0.06260	0.02410	0.05800	0.01990	0.05740	0.01915
LSE	α	0.03750	3.20750	0.26070	1.88430	0.27720	1.62880	0.21380	1.38076
	θ	0.01560	0.03980	0.01650	0.01860	0.02000	0.01540	0.01490	0.01351
WLSE	α	0.05620	3.35080	0.23770	2.03860	0.21470	1.60340	0.16310	1.35124
	θ	0.01500	0.03920	0.01220	0.01890	0.01350	0.01430	0.00950	0.01245
CVME	α	0.46550	3.87480	0.01290	1.93230	0.05250	1.53100	0.04860	1.31904
	θ	0.07220	0.05250	0.01400	0.02060	0.00420	0.01560	0.00310	0.01363
ADE	α	0.00100	3.49240	0.16860	2.01110	0.17670	1.55650	0.13730	1.30385
	θ	0.02290	0.03610	0.00420	0.01800	0.00900	0.01360	0.00650	0.01179
RTADE	α	0.00304	4.11808	0.16508	2.36558	0.21145	1.76838	0.12724	1.33593
	θ	0.01782	0.03778	0.00549	0.01994	0.01311	0.01470	0.00715	0.01169
BE_{SEL}	α	1.14634	1.54647	1.18340	1.60165	1.24588	1.72728	1.28227	1.79314
	θ	0.21452	0.06674	0.20905	0.05386	0.20656	0.04950	0.20712	0.04829
BE_{Linex1}	α	1.15123	1.56236	1.17837	1.58725	1.25111	1.74162	1.29122	1.81726
	θ	0.21661	0.06786	0.21033	0.05447	0.20748	0.04992	0.20788	0.04863
BE_{Linex2}	α	1.14091	1.52973	1.17754	1.58547	1.23279	1.69146	1.28067	1.79108
	θ	0.21245	0.06566	0.20778	0.05327	0.20563	0.04909	0.20636	0.04795
BE_{GEL1}	α	1.13590	1.52094	1.17215	1.57461	1.23353	1.69659	1.27496	1.77689
	θ	0.21170	0.06543	0.20730	0.05309	0.20527	0.04895	0.20605	0.04783
BE_{GEL2}	α	1.12243	1.49157	1.15726	1.54260	1.20934	1.63795	1.26082	1.74585
	θ	0.20606	0.06287	0.20380	0.05158	0.20269	0.04786	0.20393	0.04693

Table 3: Average estimated Biases and RMSEs of different estimation methods for Zubair-Exponential distribution at different sample sizes n and different values of the parameters ($\alpha = 0.05, \quad \theta = 0.50$).

Method		$n = 25$		$n = 50$		$n = 75$		$n = 100$	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	α	0.23956	3.55575	0.04685	2.27915	0.12818	2.19896	0.19170	1.75152
	θ	0.04686	0.03584	0.02177	0.02021	0.00535	0.01613	0.00556	0.01225
MPSE	α	0.81745	4.36545	0.74691	3.59691	0.81122	3.69056	0.84419	3.43437
	θ	0.05776	0.03279	0.05280	0.02414	0.05652	0.02175	0.06227	0.01938
LSE	α	0.13143	3.07292	0.22056	2.07073	0.16544	1.52782	0.23630	1.35456
	θ	0.00386	0.03418	0.01022	0.02129	0.00548	0.01543	0.01487	0.01372
WLSE	α	0.16675	3.29244	0.15492	2.09898	0.15592	1.65819	0.20215	1.31406
	θ	0.00180	0.03459	0.00248	0.02045	0.00331	0.01522	0.01129	0.01245
CVME	α	0.33792	3.86754	0.08047	2.07249	0.04108	1.51376	0.07842	1.31993
	θ	0.05699	0.04612	0.02254	0.02322	0.01776	0.01658	0.00262	0.01408
ADE	α	0.07233	3.44263	0.10827	2.13325	0.15686	1.75606	0.17152	1.24431
	θ	0.01454	0.03402	0.00338	0.01980	0.00186	0.01500	0.00812	0.01153
RTADE	α	0.09259	4.14665	0.09467	2.23493	0.19432	1.99400	0.17154	1.29276
	θ	0.00763	0.03702	0.00239	0.02077	0.00652	0.01573	0.00968	0.01156
BE_{SEL}	α	0.59743	0.42680	0.65856	0.48960	0.67698	0.50511	0.71922	0.54888
	θ	0.22035	0.07003	0.21403	0.05623	0.21153	0.05172	0.21184	0.05038
BE_{Linex1}	α	0.59438	0.42435	0.65450	0.48508	0.68032	0.51063	0.72080	0.55179
	θ	0.22248	0.07119	0.21533	0.05686	0.21247	0.05215	0.21261	0.05074
BE_{Linex2}	α	0.60496	0.43521	0.65988	0.49011	0.67226	0.49782	0.71442	0.54132
	θ	0.21824	0.06889	0.21275	0.05561	0.21060	0.05130	0.21107	0.05003
BE_{GEL1}	α	0.60247	0.43253	0.65729	0.48721	0.66935	0.49450	0.71175	0.53816
	θ	0.21750	0.06866	0.21226	0.05544	0.21023	0.05116	0.21077	0.04991
BE_{GEL2}	α	0.60760	0.43699	0.65536	0.48420	0.66828	0.49403	0.70753	0.53346
	θ	0.21181	0.06600	0.20873	0.05387	0.20764	0.05003	0.20863	0.04899

Table 4: Average estimated Biases and RMSEs of different estimation methods for Zubair-Exponential distribution at different sample sizes n and different values of the parameters ($\alpha = 0.25, \quad \theta = 1.25$).

Method		$n = 25$		$n = 50$		$n = 75$		$n = 100$	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	α	0.28374	3.84590	0.04361	2.13066	0.07628	1.84194	0.06554	1.35021
	θ	0.12903	0.22525	0.04893	0.10716	0.01508	0.08572	0.00931	0.06351
MPSE	α	0.76168	4.47452	0.78095	3.76563	0.74262	3.32011	0.64883	2.78396
	θ	0.12565	0.19943	0.13724	0.14086	0.13475	0.12136	0.11873	0.09984
LSE	α	0.10095	3.34588	0.18848	1.83325	0.23320	1.61072	0.20582	1.26131
	θ	0.01643	0.21215	0.02314	0.11549	0.03182	0.09927	0.03340	0.07617
WLSE	α	0.06597	3.33432	0.12468	1.79492	0.18754	1.66275	0.14482	1.20079
	θ	0.02795	0.20672	0.00613	0.10843	0.01862	0.09531	0.01784	0.06811
CVME	α	0.45133	3.86780	0.08537	1.87402	0.02873	1.55920	0.04116	1.18808
	θ	0.16337	0.27196	0.05232	0.12870	0.02339	0.10273	0.01102	0.07662
ADE	α	0.02313	3.58853	0.07249	1.79504	0.12598	1.51210	0.11513	1.13417
	θ	0.05943	0.20833	0.01000	0.10442	0.00449	0.08683	0.00998	0.06385
RTADE	α	0.01521	4.50222	0.05539	2.04169	0.14034	1.62592	0.13159	1.24889
	θ	0.04687	0.22684	0.00890	0.10821	0.01193	0.08751	0.01601	0.06561
BE_{SEL}	α	1.54787	3.14515	1.47495	2.72216	1.47576	2.60471	1.49436	2.59759
	θ	0.45050	0.30722	0.45307	0.26050	0.45628	0.24522	0.46199	0.24353
BE_{Linex1}	α	1.59812	3.35532	1.51909	2.89408	1.51445	2.74638	1.52978	2.72317
	θ	0.46254	0.32093	0.46053	0.26838	0.46173	0.25067	0.46652	0.24805
BE_{Linex2}	α	1.49340	2.91531	1.43395	2.56969	1.43937	2.47634	1.46081	2.48224
	θ	0.43871	0.29422	0.44574	0.25297	0.45087	0.23989	0.45749	0.23910
BE_{GEL1}	α	1.52073	3.04608	1.45258	2.64971	1.45533	2.54096	1.47534	2.53908
	θ	0.44370	0.30053	0.44883	0.25641	0.45313	0.24224	0.45938	0.24105
BE_{GEL2}	α	1.46807	2.86087	1.40886	2.51209	1.41524	2.41896	1.43794	2.42657
	θ	0.43011	0.28753	0.44035	0.24840	0.44685	0.23636	0.45417	0.23613

Table 5: Average estimated Biases and RMSEs of different estimation methods for Zubair-Exponential distribution at different sample sizes n and different values of the parameters ($\alpha = 1.0, \quad \theta = 0.50$).

Method		$n = 25$		$n = 50$		$n = 75$		$n = 100$	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	α	0.18112	3.94918	0.02192	2.05581	0.03013	1.12993	0.02399	0.89396
	θ	0.02274	0.02236	0.00418	0.01332	0.00423	0.00804	0.00545	0.00613
MPSE	α	0.91673	5.19381	0.78562	3.93200	0.56720	2.34698	0.38207	1.73467
	θ	0.07004	0.02751	0.05866	0.02036	0.04190	0.01269	0.03000	0.00920
LSE	α	0.33569	4.79695	0.27178	1.89863	0.25151	1.45845	0.12636	0.90983
	θ	0.03050	0.02601	0.02252	0.01544	0.01739	0.01144	0.01083	0.00765
WLSE	α	0.27294	3.99028	0.20148	1.85911	0.17207	1.25679	0.05081	0.75515
	θ	0.02205	0.02456	0.01548	0.01421	0.01012	0.00974	0.00364	0.00632
CVME	α	0.26596	7.28444	0.01176	1.79697	0.04195	1.26604	0.01469	0.86068
	θ	0.02133	0.02762	0.00460	0.01527	0.00249	0.01072	0.00306	0.00742
ADE	α	0.08103	3.41601	0.12261	1.74487	0.13083	1.13829	0.02635	0.75885
	θ	0.00301	0.02188	0.00800	0.01304	0.00626	0.00892	0.00093	0.00619
RTADE	α	0.02573	4.47026	0.11021	2.02857	0.09815	1.20204	0.00299	0.79213
	θ	0.00227	0.02494	0.00830	0.01362	0.00462	0.00893	0.00084	0.00626
BE_{SEL}	α	1.22875	2.42396	1.21473	2.14625	1.25721	2.09113	1.30131	2.13236
	θ	0.14185	0.03271	0.13867	0.02603	0.13969	0.02427	0.14254	0.02418
BE_{Linex1}	α	1.31350	2.75499	1.27939	2.36209	1.31379	2.26945	1.35272	2.29138
	θ	0.14346	0.03328	0.13966	0.02634	0.14044	0.02450	0.14316	0.02437
BE_{Linex2}	α	1.15347	2.15975	1.15518	1.96070	1.20437	1.93373	1.25289	1.98929
	θ	0.14026	0.03216	0.13769	0.02573	0.13895	0.02405	0.14192	0.02400
BE_{GEL1}	α	1.19685	2.33060	1.18830	2.07669	1.23363	2.03017	1.27991	2.07637
	θ	0.13942	0.03197	0.13715	0.02559	0.13853	0.02394	0.14158	0.02390
BE_{GEL2}	α	1.13486	2.15844	1.13653	1.94561	1.18723	1.91416	1.23769	1.96900
	θ	0.13457	0.03052	0.13411	0.02473	0.13622	0.02329	0.13966	0.02335

Table 6: Average estimated Biases and RMSEs of different estimation methods for Zubair-Exponential distribution at different sample sizes n and different values of the parameters ($\alpha = 1.5, \quad \theta = 1.20$).

Method		$n = 25$		$n = 50$		$n = 75$		$n = 100$	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	α	0.21152	3.35555	0.03130	1.74988	0.05140	0.93839	0.02471	0.81305
	θ	0.04894	0.10561	0.01525	0.05743	0.01586	0.03428	0.01032	0.02821
MPSE	α	0.95973	5.56126	0.58826	2.83953	0.35013	1.42070	0.30407	1.32429
	θ	0.16585	0.14877	0.10728	0.08104	0.06794	0.04535	0.05701	0.03833
LSE	α	0.32795	3.78344	0.23492	1.82697	0.13195	1.07146	0.11690	0.87451
	θ	0.06895	0.12876	0.04746	0.07230	0.02809	0.04494	0.02444	0.03619
WLSE	α	0.22478	3.52674	0.14124	1.61434	0.06285	0.90917	0.03956	0.69030
	θ	0.04641	0.11736	0.02761	0.06351	0.01202	0.03769	0.00748	0.02893
CVME	α	0.25878	3.94911	0.05234	1.66635	0.04395	1.03957	0.01794	0.82135
	θ	0.04598	0.13198	0.01199	0.06793	0.00992	0.04380	0.00458	0.03455
ADE	α	0.04465	3.24963	0.07202	1.50212	0.03188	0.87070	0.02391	0.65740
	θ	0.00688	0.10569	0.01220	0.05768	0.00516	0.03579	0.00395	0.02761
RTADE	α	0.07750	4.86251	0.01523	1.81904	0.03147	1.02519	0.01391	0.77199
	θ	0.00192	0.12403	0.00564	0.06112	0.00407	0.03838	0.00164	0.02906
BE_{SEL}	α	0.94474	1.83007	0.99737	1.69937	1.08796	1.73426	1.17206	1.84599
	θ	0.23425	0.10830	0.24520	0.09155	0.25927	0.08987	0.27414	0.09349
BE_{Linex1}	α	1.04989	2.17655	1.07816	1.92858	1.15837	1.92901	1.23506	2.02266
	θ	0.24229	0.11316	0.25025	0.09437	0.26313	0.09204	0.27735	0.09536
BE_{Linex2}	α	0.85268	1.56954	0.92368	1.50949	1.02260	1.56660	1.11297	1.68999
	θ	0.22634	0.10369	0.24021	0.08882	0.25544	0.08775	0.27096	0.09166
BE_{GEL1}	α	0.90815	1.74628	0.96764	1.63632	1.06211	1.67755	1.14934	1.79318
	θ	0.22880	0.10551	0.24176	0.08981	0.25665	0.08849	0.27199	0.09230
BE_{GEL2}	α	0.83721	1.59573	0.90945	1.51943	1.01120	1.57054	1.10443	1.69231
	θ	0.21792	0.10016	0.23487	0.08641	0.25140	0.08579	0.26768	0.08996

Table 7: Average estimated Biases and RMSEs of different estimation methods for Zubair-Exponential distribution at different sample sizes n and different values of the parameters ($\alpha = 0.25$, $\theta = 2.0$).

Method		$n = 25$		$n = 50$		$n = 75$		$n = 100$	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	α	0.21482	3.35577	0.04554	2.15588	0.07182	1.88099	0.06817	1.38321
	θ	0.16736	0.49641	0.04613	0.27129	0.03695	0.23990	0.00750	0.16067
MPSE	α	0.87024	4.46650	0.85628	3.81697	0.72486	3.33901	0.64634	2.79912
	θ	0.25005	0.48497	0.24717	0.36602	0.19808	0.32437	0.19508	0.25454
LSE	α	0.20470	2.90192	0.29232	1.96393	0.25371	1.77811	0.20743	1.27950
	θ	0.02692	0.49205	0.07065	0.30897	0.04914	0.27741	0.05843	0.19904
WLSE	α	0.19012	3.08809	0.24926	2.00364	0.20814	1.74428	0.14912	1.20909
	θ	0.01201	0.49873	0.04915	0.29442	0.03079	0.25609	0.03476	0.17711
CVME	α	0.28004	3.38317	0.01319	1.96335	0.03684	1.67825	0.04506	1.21044
	θ	0.17993	0.61439	0.05106	0.33614	0.04258	0.28299	0.01102	0.19978
ADE	α	0.03254	3.03561	0.18590	1.98069	0.15240	1.64511	0.12263	1.18370
	θ	0.05780	0.47396	0.02042	0.27611	0.00790	0.23992	0.02282	0.16854
RTADE	α	0.02352	3.73884	0.19775	2.25501	0.17546	1.85295	0.09130	1.09792
	θ	0.05845	0.52111	0.03041	0.29285	0.01985	0.25465	0.02027	0.15811
BE_{SEL}	α	0.94474	1.83007	0.99737	1.69937	1.08796	1.73426	1.17206	1.84599
	θ	0.23425	0.10830	0.24520	0.09155	0.25927	0.08987	0.27414	0.09349
BE_{Linex1}	α	1.04989	2.17655	1.07816	1.92858	1.15837	1.92901	1.23506	2.02266
	θ	0.24229	0.11316	0.25025	0.09437	0.26313	0.09204	0.27735	0.09536
BE_{Linex2}	α	0.85268	1.56954	0.92368	1.50949	1.02260	1.56660	1.11297	1.68999
	θ	0.22634	0.10369	0.24021	0.08882	0.25544	0.08775	0.27096	0.09166
BE_{GEL1}	α	0.90815	1.74628	0.96764	1.63632	1.06211	1.67755	1.14934	1.79318
	θ	0.22880	0.10551	0.24176	0.08981	0.25665	0.08849	0.27199	0.09230
BE_{GEL2}	α	0.83721	1.59573	0.90945	1.51943	1.01120	1.57054	1.10443	1.69231
	θ	0.21792	0.10016	0.23487	0.08641	0.25140	0.08579	0.26768	0.08996

Table 8: Average estimated Biases and RMSEs of different estimation methods for Zubair-Exponential distribution at different sample sizes n and different values of the parameters ($\alpha = 0.15, \quad \theta = 1.0$).

Method		$n = 25$		$n = 50$		$n = 75$		$n = 100$	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	α	0.19297	3.87371	0.02093	2.51168	0.02172	1.93934	0.02719	1.31088
	θ	0.09525	0.15225	0.03092	0.08073	0.02486	0.06182	0.01688	0.04216
MPSE	α	0.84649	4.57048	0.82222	3.94821	0.72547	3.59672	0.64564	2.86972
	θ	0.11076	0.13729	0.11718	0.10014	0.09956	0.08513	0.09108	0.06656
LSE	α	0.20220	3.34815	0.19999	2.06710	0.11982	1.60041	0.20269	1.34178
	θ	0.00362	0.14400	0.01755	0.08538	0.00558	0.06359	0.02242	0.05385
WLSE	α	0.14843	3.37915	0.17335	2.14048	0.10168	1.66194	0.13009	1.20276
	θ	0.01563	0.14329	0.01036	0.08238	0.00071	0.06131	0.00913	0.04646
CVME	α	0.38571	3.72041	0.07238	2.15066	0.09228	1.55408	0.03927	1.28179
	θ	0.12791	0.18417	0.04270	0.09498	0.04011	0.06664	0.01307	0.05477
ADE	α	0.04006	3.50002	0.09615	2.04323	0.08291	1.70671	0.12479	1.26095
	θ	0.04132	0.13696	0.00598	0.07640	0.00696	0.06001	0.00600	0.04565
RTADE	α	0.02870	4.08467	0.09021	2.23040	0.06764	1.86243	0.09472	1.18080
	θ	0.03282	0.14995	0.00233	0.07936	0.00663	0.06274	0.00467	0.04345
BE_{SEL}	α	1.45853	2.62075	1.45230	2.51465	1.48861	2.56392	1.52446	2.64130
	θ	0.39149	0.22653	0.39002	0.18997	0.38959	0.17734	0.39264	0.17453
BE_{Linex1}	α	1.47193	2.66788	1.47463	2.58993	1.51399	2.64896	1.55088	2.73091
	θ	0.39952	0.23442	0.39498	0.19445	0.39320	0.18041	0.39562	0.17705
BE_{Linex2}	α	1.43202	2.52476	1.43488	2.46119	1.46626	2.49272	1.49700	2.54899
	θ	0.38360	0.21897	0.38513	0.18566	0.38601	0.17432	0.38968	0.17205
BE_{GEL1}	α	1.44122	2.56676	1.43840	2.47675	1.47164	2.51334	1.50591	2.58295
	θ	0.38594	0.22180	0.38656	0.18711	0.38704	0.17527	0.39053	0.17282
BE_{GEL2}	α	1.40341	2.45041	1.40852	2.39442	1.44136	2.42861	1.47359	2.48764
	θ	0.37487	0.21259	0.37964	0.18150	0.38193	0.17120	0.38632	0.16944

Table 9: Confidence Intervals for MLEs and Credible Intervals for the Bayesian Estimates using BE_{SEL}, BE_{Linex1}, BE_{Linex2}, BE_{GEL1} & BE_{GEL2}

Initial Values	Lower MLE	Upper MLE	Lower BE_{SEL}	Upper BE_{SEL}	Lower BE_{Linex1}	Upper BE_{Linex1}	Lower BE_{Linex2}	Upper BE_{Linex2}	Lower BE_{GEL1}	Upper BE_{GEL1}	Lower BE_{GEL2}	Upper BE_{GEL2}
$\alpha = 0.15$	0.45359	2.99950	2.54591	0.61432	2.97925	2.36493	0.60819	2.85610	2.24791	0.64150	2.78190	2.14040
$\theta = 0.25$	0.24174	0.52151	0.27977	0.26987	0.46131	0.19144	0.27790	0.43287	0.15497	0.28790	0.42426	0.13636
$\alpha = 0.10$	0.34833	1.98825	1.63992	0.51848	1.99931	1.48083	0.59080	1.99491	1.40411	0.68136	1.99080	1.30944
$\theta = 0.50$	0.47216	1.03588	0.56372	0.53894	0.91812	0.37918	0.56008	0.87002	0.30994	0.56260	0.83689	0.27429
$\alpha = 0.05$	0.13236	0.99618	0.86382	0.18955	0.99873	0.80917	0.28449	0.99824	0.71376	0.42172	0.99661	0.57489
$\theta = 0.50$	0.47622	1.05327	0.57705	0.54308	0.92943	0.38635	0.56233	0.87356	0.31122	0.58488	0.86146	0.27658
$\alpha = 0.25$	0.12657	3.32064	3.19407	0.35274	3.14141	2.78867	0.52666	3.04532	2.51866	0.63052	2.90631	2.27579
$\theta = 1.25$	1.18115	2.42399	1.24284	1.29896	2.18152	0.88257	1.35804	2.08793	0.72989	1.36926	2.01669	0.64743
$\alpha = 1.0$	0.51873	4.08811	3.56937	0.78279	3.92454	3.14176	0.90953	3.63155	2.72202	1.09317	3.56610	2.47293
$\theta = 0.50$	0.44233	0.86562	0.42330	0.49537	0.80730	0.31194	0.51600	0.78048	0.26448	0.52581	0.76038	0.23457
$\alpha = 1.50$	0.51743	4.15348	3.63605	1.05796	4.27065	3.21269	1.11791	3.97303	2.85512	1.43437	4.00025	2.56588
$\theta = 1.20$	1.02530	1.89675	0.87144	1.11208	1.79520	0.68312	1.16524	1.74094	0.57570	1.21481	1.72723	0.51241
$\alpha = 0.25$	0.19481	3.24804	3.05323	0.34383	3.05487	2.71104	0.41576	2.85558	2.43983	0.69355	2.90839	2.21484
$\theta = 2.0$	1.85699	3.74149	1.88450	2.05779	3.41504	1.35726	2.14484	3.27088	1.12603	2.16854	3.17947	1.01092
$\alpha = 0.15$	0.45084	2.98673	2.53589	0.54684	2.86244	2.31560	0.60285	2.79490	2.19204	0.66239	2.77334	2.11095
$\theta = 1.0$	0.92650	1.96084	1.03434	1.05693	1.78961	0.73268	1.09881	1.70043	0.60162	1.14441	1.67715	0.53274

The following conclusions can be drawn from the simulation results

1. As sample size increases, all estimators' Bais and RMSE values fall, demonstrating improved accuracy in model parameter estimation.
2. The least biased parameters across all the parameters and various sample sizes are LSE, WLSE, ADE, and RTADE.
3. For all sample sizes, the estimators' bias is positive.
4. BE_{SEL} fluctuates as sample size increases. While the other loss functions considered increase as sample size does.

6.2 Real Data Application

The data represent survival times of guinea pigs injected with different amount of tubercle bacilli studied by Bjerkedal [9]. In this section, the proposed method is illustrated by comparing the goodness of fits of the Z–Exponential (ZE) distribution with the other life models such as Zubair-Weibull (ZW) distribution, Kumaraswamy-Weibull (KW) distribution, Exponentiated-Weibull (EW) distribution and Exponential distribution (ED) distributions by applying real data. The analytical measures of goodness of fit namely the Log-Likelihood (Loglik), Akaike information criterion (AIC), Bayesian information criterion (BIC) and Kolmogorov–Smirnov (K-S) statistics are considered to compare the proposed method with the other fitted models. The decision for a better fit is based on the model with smaller values of these analytical measures.

Table 10: MLEs and Standard Errors of the Parameters of the fitted Distributions using Guinea pigs data

Distribution	θ		α		β		γ	
	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error
ZE	3.1718	0.9269	0.0139	0.0016				
ZW	5.8420	2.2686	0.7768	0.1284	0.0553	0.0432		
KW	0.3609	0.01638	0.0553	0.0065	0.6288	0.0019	0.5197	0.0025
EW	2.6541	1.5362	1.1604	0.3090	112.8544	46.2945		
ED	0.0056	0.0057						

Table 11: The analytical measures of the fitted models for the Guinea pigs data

Distribution	LogLik	AIC	BIC	K-S
ZE	-425.3208	854.6417	857.195	0.089
ZW	-424.4359	854.8717	861.7018	0.0766
KW	-474.6325	957.265	966.3717	0.3854
EW	-425.6656	857.3311	864.1611	0.0891
ED	-444.6156	891.2312	893.5079	0.2959

The proposed model (ZE) appears to have the smallest values of AIC and BIC hence has a better fit compared to the other distributions considered in this study.

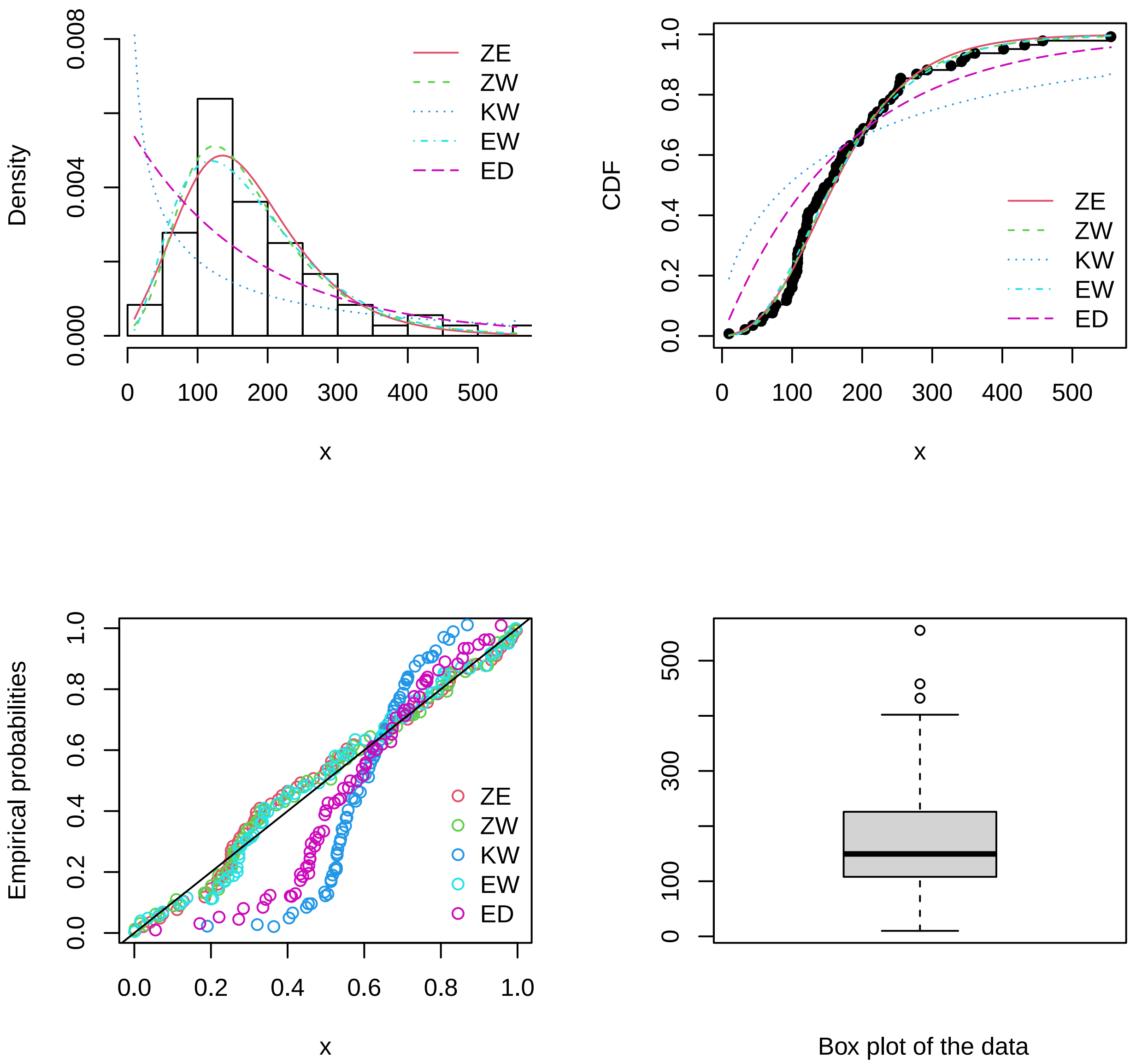

Figure 4: The estimated pdf, cdf, Kaplan-Meier and Box plots of the Zubair-Exponential distribution using the Guinea pigs data

7 Conclusion

A new sub-model of Zubair-G family life distributions, called the Zubair-Exponential distribution is proposed and studied. The mathematical properties of the new distribution including moments, moment generating function, order statistics and quantile function are derived. The quantile function has a closed form solution. The cdf, hazard rate and odd functions are increasing function and therefore can be deployed to study decreasing and increasing lifetime events. The maximum likelihood function, least squares estimate, weighted least squares, cramer von mises, Anderson Darling estimate and right-tailed Anderson Darling estimate are derived and implemented. Bayesian inference with Squared error loss, Linear-Exponential loss and Generalized Entropy loss functions were also studied and simulation of parameters based on the classical and bayesian methods are done. Application to real data using survival times of Guinea pigs was also illustrated. From the analytical measures of fitness and performance (AIC and BIC), the proposed Zubair-Exponential distribution is preferred to the other distributions compared.

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