Original Research Article

STABILITY OF COLLINEAR LIBRATION POINTS IN THE PHOTOGRAVITAIONAL ELLIPTIC RESTRICTED THREE BODY PROBLEM (ER3BP) UNDER THE EFFECTS OF PR-DRAG FORCE AND TRIAXIALITY ENCLOSED BY A BELT

Abstract

The paper examines the effect of radiation pressure, Poynting-Robertson Drag (Pr-drag) force and triaxiality of two stars (primaries) surrounded by a belt (circumbinary disc) on the positions and stability of a third body of an infinestisimal mass in the neighborhood of collinear libration points in the framework of elliptic restricted three body problem (ER3BP).The solutions to the location of collinear libration points L_i (i =1,2,3) were found.The solutions obtained were investigated numerically and graphically using radiating binary system (FL virginis and Procyon).The positions and stability of the libration points are affected by trixiality,Pr-drag force and the gravitational pontetial from the belt. The collinear libration points were found to be unstable.

Keywords:

ER3BP;Triaxiality;Radiation; Circumbinarydisc;Stability;Collinear,gravitational pontential..

Introduction

The exact solution of the two body **pr**oblem have been found,but the solution to the three body problem is unobtainable when the three bodies are heavenly bodies for example earth,moon and sun.In order to obtain a close form solution Euler in 1765 introduced the restricted three body problem,a simplification of the three body problem where one of the bodies is assumed to have a negligible mass and moves in thesame plane defined by the two revolving bodies around their common centre of mass.An example of the restricted three body problem is earth,moon and an artificial satellite.

During the classical era of investigation of the restricted three body problem Euler in 1765 and Lagrange in 1772 obtained a number of particular solutions from the rotating frame where the infinitesimal mass has zero velocity and acceleration. These solutions correspond to equilibrium positions, which are five in the R3BP and are called Lagrangian equilibrium points; three of them are called collinear equilibrium points namely: L_1,L_2, L_3 and the remaining two are triangular equilibrium points named Langragian equilibrium points (L_4 and L_5).The collinear equilibrium points lie on the line joining the primaries.

For example in the earth –sun system in which the earth orbits the sun, the three collinear points lie on the line joining the earth and the sun. L_1 is between sun and the earth, L_2 in the same direction as the earth, L_3 is opposite the earth. The collinear points are unstable and any object such as satellite placed at this points cannot remain in position except some thrusters are attached.This has allowed SOHO's satellite to be kept at L_1 to monitor the sun and NASA satellite called WMAP to be stationed at $L_{2.}$ If trusthers are not attached,several forces such as gravitational,corolis force and centrifugal forces can drift the satellite from position.

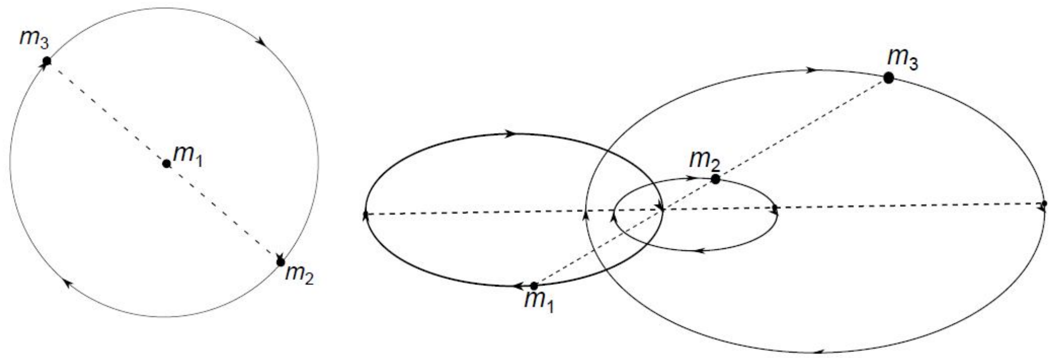


Figure 1: The Euler solution: the three bodies remain collinear at all times, in elliptical orbits around the centre of mass. Left: all masses equal. Right: unequal masses

Apart from these forces other forces such as radiation pressure force,main body shapes,circumbinary disc and P-R drag force can affect the position of the third body at equilibrium points.According to Singh [1] gravitational force alone cannot be considered in studying the dynamics of a stellar system. For example, gravity is not the major force present when a star collides with a gas and dust particles,but the repelling forces of radiation pressure Radzievskii [2].These forces have substantial effect on the stability of equilibrium points.In particular, is the radiation pressure force which is the major force acting on planetary objects after gravity.The classical R3BP is inadequate in describing the dynamics of a particle emitting radiation therefore in other to account for these forces in the equations of motion the classical potential function was amended to admit it.Thus the problem becomes generalized. Kumar and Ishwar [3] included radiation pressure in their study.Singh [1] investigated the triangular libration points under small perturbations of the Corolis and centrifugal forces, triaxiality and radiation pressure of the primaries.

The generalization attracted a lot of researchers to study the effect of the pertubations on the location and stability of R3BP under different characterizations. The perbubations changes the position of the third-body (infinitesimal mass) at equilibrium points slightly, such that the particles' resultant motion may lead to a rapid departure from the vicinity of these points; if this occurs, such a point is said to be unstable. If however, the body returns to its original position after mere oscillations, such a position of equilibrium point is said to be stable.

The effect of triaxiality of the bigger primary and oblateness of the companion on the location and stability of the collinear equilibrium points in ER3BP have been studied by Singh and Umar [4].They observed that the positions and stability of the collinear libration points are affected by the perturbations in addition to the eccentricity and semi-major axis of the primaries orbits and are unstable. Recently, Vicent et.al [5] shows that pertubations can lead to increase in number of collinear points.They obtained four additional collinear equilibrium points L_{ni} = 1, 2, 3, 4, in addition to the three Eulerian points L_{ni} = 1, 2, 3, of the classical case, making up a total of seven collinear points.Out of the four collinear equilibrium points two:L_{n1} and L_{n2} are due to the effect of potential from the belt, while L_{n3} and L_{n4} arise from the effect of triaxiality-in a generalized restricted three-body problem (R3BP) with an oblate infinitesimal body and triaxial-radiating primaries

In studying the elliptical restricted three body problem Danby [6] used numerical integration to obtain the linear stability of the position of equilibrium points. He used the mass value μ and eccentricity e to obtain a stability diagram in the μ-e plane. In their study Kumar and Narayan [7] examined the effects of photogravitation and oblateness of the primaries on the existence and stability of third body around the collinear point L_1 and they concluded that the third body oscillation around L_1 is unstable.Sultan,et al.[8] investigated a test particle in the vicinity of collinear equilibrium points under the influence of triaxial primaries using a series form approach to obtain the location of collinear equilibrium point and they were all found to be unstable.The same results were obtained by Singh and Tyokyaa ,2017) when they studied the stability of collinear points in ER3BP with oblateness up to zonal harmonic $J_{4.}$.

The Poynting-Robertson (P-R) effect also called Poynting-Robertson (P-R) Drag force was named after .John Henry Poynting and Howard Percy Robertson. (Poynting ,1903) described the effect based on a theory that supersede the theory of relativity.Later, (Robertson ,1937) described the effect in terms of general relativity.The P-R Drag force is a component of radiation pressure and is tangential to the grain's motion.It is an effective force that opposes the direction of the dust grain's motion and causes a drop in the grain's angular momentum. An investigation by (Tajudeen et.al.,2021) reveals that, though the radiation pressure, oblateness and centrifugal perturbation decrease region of stability when motion is stable, however, they are not the influential forces of instability but the P-R drag force.He showed that in the region when motion around the triangular points are stable an inclusion of the P-R drag of the bigger primary even by an almost negligible value of 1.04548*10-9 overrides other effect and changes stability to instability.

Several authors have examined the nature of stability of collinear points in the framework of ER3BP using different assumptions. For instance, (Alzahrani et al ,2017) considered the primaries to be an irregular asteroid,while the third body moves in their gravitational field; necessary and sufficient condition were obtained for finding the three collinear points and they proved the existence of these points and triangular equilibrium points.In their model (Aliroma et al ,2019) showed that the stability region could depend mainly on the eccentricity of the orbits in addition to considered pertubations.Using Vinti`s method with an oblate-spheroidal coordinate system to describe the orbital motion and finding x-corodinates in the form of series solution (Chakraborty et al,2016) studied the linear stability of the collinear points using two binary system Luyten-726 and Kruger-60 and found them to be unstable.

In this work we wish to study the locations and stability of collinear equilibrium points, when both primaries are triaxial and radiating with gravitational potential from the belt with the smaller primary having an effective P-R Drag force using two radiating binary stars (FL virginis and Procyon).FL virginis is a binary star system that consist of two red dwarf stars at a distance of approximately 14.2 light years away from the sun with moderate eccentric orbit, while Procyon is the brightest star in the constellation minor and usually the eight brightest star, in the night sky consisting of a white hued main-sequence star and a faint white dwarf companion.The paper is an extension of the works of Singh and Isah (2021). It is organized as follows: section 1 is introduction, the equations of motion are presented in section 2, we introduce computations relating to the location of equilibrium points and their stability together with the perturbations in section 3, while in section 4 we give the numerical explanation of our analysis and finally discussion and conclusion are contained in section 5.

2. Equation of motion

The equations of motion of the infinitesimal mass in the three-dimensional restricted three-body problem with the origin at the center of mass, in a barycentric rotating (also called synodical) coordinate system under the gravitational influence of two radiating triaxial bodies with the P–R drag present and surrounded by circumbinary disc have the form:

$$\Omega_\xi=(1-e^2)^{-\frac{1}{2}}\Bigg[\xi-\frac{1}{n^2}\Bigg\{\frac{(1-\mu)(\xi+\mu)}{{r_1}^3}q_1+\frac{3(1-\mu)(\xi+\mu)(2\sigma_1-\sigma_2)}{2{r_1}^5}q_1-$$

$$\frac{15(1-\mu)(\xi+\mu)(\sigma_1-\sigma_2)}{2{r_1}^7}q_1\eta^2-\frac{15(1-\mu)(\xi+\mu)\sigma_1}{2{r_1}^7}q_1\zeta^2+\frac{\mu(\xi+\mu-1)}{{r_2}^3}q_2+$$

$$\frac{3\mu(\xi+\mu-1)(2\sigma_3-\sigma_4)}{2{r_2}^5}q_2-\frac{15\mu(\xi+\mu-1)(\sigma_3-\sigma_4)}{2{r_2}^7}q_2\eta^2-\frac{15\mu(\xi+\mu-1)\sigma_3}{2{r_2}^7}q_2\zeta^2+\frac{M_b\xi}{(r^2+T^2)^{\frac{3}{2}}}\Bigg\}-$$

$$\frac{W_2}{n^2{r_2}^2}\Bigg\{\frac{\{(\xi+\mu-1)\{(\xi+\mu-1)\xi'+\eta\eta'+\zeta\zeta'\}\}}{{r_2}^2}+\xi'-n\eta\Bigg\}\Bigg] \qquad (2)$$

$$\Omega_\eta = (1-e^2)^{-\frac{1}{2}}\left[\eta\left\{1-\frac{1}{n^2}\left(\frac{(1-\mu)}{{r_1}^3}q_1 + \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2{r_1}^5}q_1 + \frac{3(1-\mu)(\sigma_1-\sigma_2)}{{r_1}^5}q_1 - \frac{15(1-\mu)(\xi+\mu)(\sigma_1-\sigma_2)}{2{r_1}^7}q_1\eta^2 - \frac{15(1-\mu)\sigma_1}{2{r_1}^7}q_1\zeta^2 + \frac{\mu}{{r_2}^3}q_2 + \frac{3\mu(2\sigma_3-\sigma_4)}{2{r_2}^5}q_2 + \frac{3\mu(\sigma_3-\sigma_4)}{{r_2}^5}q_2 - \frac{15\mu(\sigma_3-\sigma_4)}{2{r_2}^7}q_2\eta^2 - \frac{15\mu\sigma_3}{2{r_2}^7}q_2\zeta^2 + \frac{M_b}{(r^2+T^2)^{\frac{3}{2}}}\right)\right\} - \frac{W_2}{n^2{r_2}^2}\left\{\frac{\eta\{(\xi+\mu-1)\xi'+\eta\eta'+\zeta\zeta'\}}{{r_2}^2} + \eta' + n(\xi+\mu-1)\right\}\right] \quad (3)$$

$$\Omega_\zeta = (1-e^2)^{-\frac{1}{2}}\left[-\frac{\zeta}{n^2}\left(\frac{(1-\mu)}{{r_1}^3}q_1 + \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2{r_1}^5}q_1 + \frac{3(1-\mu)\sigma_1}{{r_1}^5}q_1 - \frac{15(1-\mu)(\sigma_1-\sigma_2)}{2{r_1}^7}q_1\eta^2 - \frac{15(1-\mu)\sigma_1}{2{r_1}^7}q_1\zeta^2 + \frac{\mu}{{r_2}^3}q_2 + \frac{3\mu(2\sigma_3-\sigma_4)}{2{r_2}^5}q_2 + \frac{3\mu\sigma_3}{{r_2}^5}q_2 - \frac{15\mu(\sigma_3-\sigma_4)}{2{r_2}^7}q_2\eta^2 - \frac{15\mu\sigma_3}{2{r_2}^7}q_2\zeta^2\right) - \frac{W_2}{n^2{r_2}^2}\left\{\frac{\zeta\{(\xi+\mu-1)\xi'+\eta\eta'+\zeta\zeta'\}}{{r_2}^2} + \zeta'\right\}\right] \quad (4)$$

$$r_1^2 = (\xi+\mu)^2 + \eta^2 + \zeta^2, \quad r_2^2 = (\xi+\mu-1)^2 + \eta^2 + \zeta^2 \quad (5)$$

$$n^2 = \frac{1}{a}\left[1 + \frac{3}{2}e^2 + \frac{3}{2}(2\sigma_1-\sigma_2) + \frac{3}{2}(2\sigma_3-\sigma_4) + \frac{2M_b r_c}{\left(r_c^2+T^2\right)^{3/2}}\right] \quad (6)$$

$\sigma_1 = \frac{a^2-c^2}{5R^2}$, $\sigma_2 = \frac{a^2-c^2}{5R^2}$, $\sigma_3 = \frac{a'^2-c'^2}{5R^2}$, $\sigma_4 = \frac{a'^2-c'^2}{5R^2}$, $\mu = \frac{M_2}{M_1+M_2} \leq \frac{1}{2}$, $\sigma_i << 1$, $(i=1,2,3,4)$

where μ is the mass parameter, n is the mean motion of the primaries of masses M_1 and M_2 respectively, q_i ($i=1,2$) are their radiation factors, r_1 and r_2 represent distances of the third body from the primaries, σ_1 and σ_2 denote the triaxiality of the bigger primary, while σ_3 and σ_4 denote the triaxiality of the smaller primary.

The lengths of the axes are denoted by a, b, c for the bigger primary and a', b', c' for the smaller primary, a is the semi-major axis of the orbits of the primaries and e the enccentricity. M_b<<1 is the total mass of the belt, $r = \sqrt{\xi^2 + \eta^2}$ is the radial distance of the third body from the origin. $T = A + B$, A and B are the parameters which determine the density profile of the belt (Miyamoto and Nagai,1975;Jiang and Yeh,2003; Kushvah,2008).The parameter B controls the size of the core of the density profile and is known as the core parameter,r_c is the radial distance of the third body from the collinear point under consideration. $W_2 = \frac{\mu(1-q_2)}{c_d}$ denotes the P-R drag of the smaller primary and C_d is the dimensionless speed of light.The configuration of the problem is shown below:

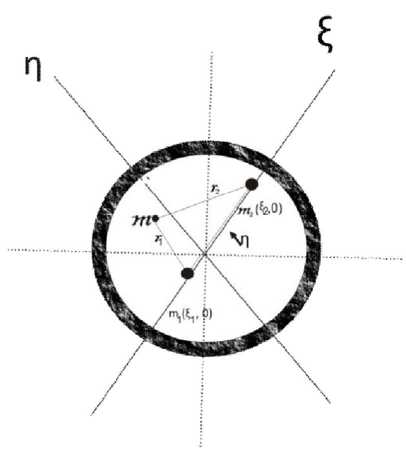


Figure 1: The configuration of the problem

3. Locations and stability of the Collinear Libration Points

At equilibrium points $\xi' = \eta' = \xi'' = \eta'' = \zeta' = \zeta'' = 0$.Therefore,equilibrium points lie in the ξη-plane and are the solutions of equations $\Omega_\xi = \Omega_\eta = 0, \zeta = 0$

Since collinear points lie on the ξ- axis it implies that η = 0, using Equation (5) in above equations and avoiding situations $\xi = 1 - \mu$ and $-\mu$ we have:

$$n^2\xi - \frac{(1-\mu)(\xi+\mu)q_1}{|\xi+\mu|^3} - \frac{3(1-\mu)(\xi+\mu)(2\sigma_1-\sigma_2)q_1}{2|\xi+\mu|^5} - \frac{\mu(\xi+\mu-1)q_2}{|\xi+\mu-1|^3} - \frac{3\mu(\xi+\mu-1)(2\sigma_3-\sigma_4)q_2}{2|\xi+\mu-1|^5} - \frac{Mb\xi}{(r^2+T^2)^{3}/_{2}} = 0 \quad (7)$$

Now we consider,

$$\xi_2 - \xi_1 = 1, \xi_1 = -\mu, \xi_2 = 1 - \mu \quad (8)$$

Rewiting equation (7) using equation (8) we obtain:

$$n^2\xi - \frac{(1-\mu)q_1}{|\xi-\xi_1|^2} - \frac{3(1-\mu)(2\sigma_1-\sigma_2)q_1}{2|\xi+\xi_1|^4} - \frac{\mu q_2}{|\xi-\xi_2|^2} - \frac{3\mu(2\sigma_3-\sigma_4)q_2}{2|\xi-\xi_2|^4} - \frac{Mb\xi}{(r^2+T^2)^{3}/_{2}} = 0 \quad (9)$$

In order to locate the collinear points $L_{1,2,3}$ we divide the orbital plane into three parts $\xi > \xi_2$, $\xi_1 < \xi < \xi_2$ and $\xi_1 > \xi$ with respect to the primaries.

3.1 Position of L_1 ($\xi > \xi_2$)

Let the collinear libration point L_1 be on the right hand side of the smaller primary at a distance ρ from it on the ξ- axis

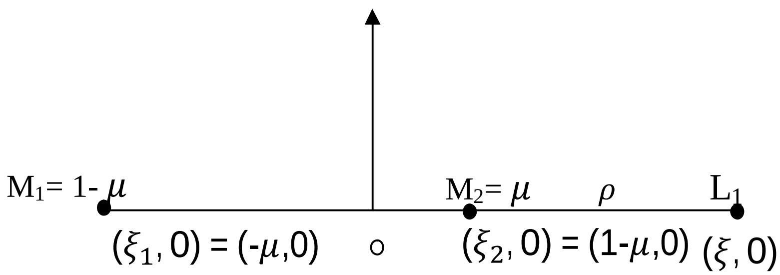


Fig. 2: Position of collinear libration point L_1

In the interval ($\xi > \xi_2$) we let $\xi - \xi_2 = \rho$, $\xi - \xi_1 = 1 + \rho \Rightarrow \xi = 1 + \rho + \xi_1$, $\xi_2 - \xi_1 = 1$

$\Rightarrow \xi = 1 + \rho - \mu$, $\quad r_1 = 1 + \rho, r_2 = \rho$ (10)

Thus by substituting equation (10) in the equation (9) we have:

$$2n^2(1+\rho-\mu)(1+\rho)^4\rho^4 q_1 - 2(1-\mu)(1+\rho)^2\rho^4 q_1 - 3(1-\mu)(2\sigma_1 - \sigma_2)\rho^4 q_1 - 2\mu(1+\rho)^4\rho^2 q_2 - 3\mu(2\sigma_3-\sigma_4)(1+\rho)^4 q_2 - \frac{Mb(1+\rho-\mu)}{\{(1+\rho-\mu)^2+T^2\}^{\frac{3}{2}}} = 0 \quad (11)$$

After expansion we obtain:

$$2n^2\rho^9 + 2n^2(5-\mu)\rho^8 + 2n^2(2(5-2\mu))\rho^7 + 2(2n^2(5-3\mu) - (q_1 - \mu q_1 + \mu q_2))\rho^6 + \left(\left(2n^2(5-4\mu) - 4(q_1 - \mu q_1 + 2\mu q_2)\right)\rho^5 + (2n^2(1-\mu) - 2(q_1 - \mu q_1 + 6\mu q_2) - 3(2\sigma_1 q_1 - \sigma_2 q_1) + 3(2\mu\sigma_1 q_1 - \mu\sigma_2 q_1) - 3(2\mu\sigma_3 - \mu\sigma_4)q_2\right)\rho^4 - 4(\mu q_2 + 6\mu\sigma_3 q_2 - 3\mu\sigma_4 q_2)\rho^3 - 2(\mu q_2 + 18\mu\sigma_3 q_2 - 9\mu\sigma_4 q_2)\rho^2 - 3(8\mu\sigma_3 q_2 - 4\mu\sigma_4 q_2)\rho - (3\mu(2\sigma_3 - \mu\sigma_4)q_2 - \frac{Mb(1+\rho-\mu)}{\{(1+\rho-\mu)^2+T^2\}^{\frac{3}{2}}} = 0 \quad (12)$$

3.2 Position of L$_2$ ($\xi_1 < \xi < \xi_2$)

Let the collinear libration point L$_2$ be on the left hand side of the smaller primary at a distanceρ from it on the ξ- axis

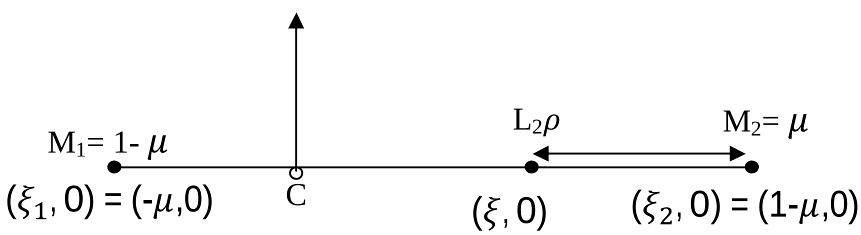


Fig. 3: Position of collinear libration point L$_2$

In the interval L$_2$ ($\xi_1 < \xi < \xi_2$), we let $\xi_2 - \xi = \rho; \xi - \xi_1 = 1 - \rho \Rightarrow \xi = 1 - \rho - \mu$ and $r_1 = 1 - \rho, r_2 = \rho$ (13)

Using equation (13) in the equation (9) we get:

$$2n^2(1-\rho-\mu)(1+\rho)^4\rho^4 q_1 - 2(1-\mu)(1-\rho)^2\rho^4 q_1 - 3(1-\mu)(2\sigma_1 - \sigma_2)\rho^4 q_1 + 2\mu(1-\rho)^4\rho^2 q_2 - 3\mu(2\sigma_3-\sigma_4)(1-\rho)^4 q_2 - \frac{Mb(1-\rho-\mu)}{\{(1+\rho-\mu)^2+T^2\}^{\frac{3}{2}}} = 0 \quad (14)$$

Hence expanding equation (14) we obtain:

$$-2n^2\rho^9 + 2n^2(5-\mu)\rho^8 - 2n^2(2(5-2\mu))\rho^7 + 2(2n^2(5-3\mu) - (q_1-\mu q_1 - \mu q_2))\rho^6 + \left(-2n^2(5-4\mu) + 4(q_1-\mu q_1 - 2\mu q_2)\right)\rho^5 + (2n^2(1-\mu) - 2(q_1-\mu q_1-6\mu q_2) - 3(2\sigma_1 q_1 - \sigma_2 q_1) + 3(2\mu\sigma_1 q_1 - \mu\sigma_2 q_1) + 3(2\mu\sigma_3 - \mu\sigma_4)q_2)\rho^4 - \left(4(2\mu q_2 + 6\mu\sigma_3 q_2 - 3\mu\sigma_4 q_2)\right)\rho^3 + \left(2(\mu q_2 + 18\mu\sigma_3 q_2 - 9\mu\sigma_4 q_2)\right)\rho^2 - \left(3(8\mu\sigma_3 q_2 - 4\mu\sigma_4 q_2)\right)\rho + (3\mu(2\sigma_3 - \mu\sigma_4)q_2) - \frac{Mb(1-\rho-\mu)}{\{(1+\rho-\mu)^2+T^2\}^{\frac{3}{2}}} = 0 \quad (15)$$

3.3 Position of L_3 ($\xi_1 > \xi$)

Let the collinear libration point L_3 be on the left hand side of the bigger primary at a distance

1-ρ from it on the ξ - axis.

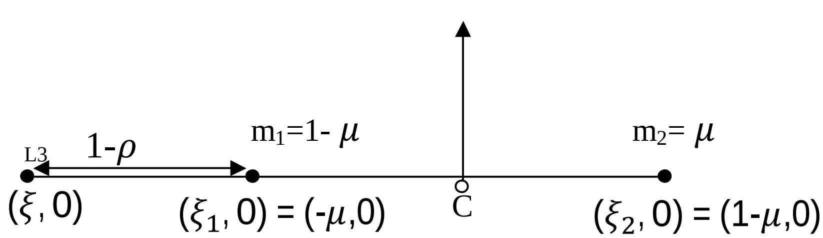


Figure 4: Position of collinear libration point L_3

Finally, in the interval $(\xi_1 > \xi)$ we let

$\xi_1 - \xi = 1 - \rho$; $\xi_2 - \xi = 2 - \rho$, and $r_1 = 1 - \rho$; $r_2 = 2 - \rho$, $\xi = -1 - \mu + \rho$ (16)

Using equation (16) in the equation (9) we get:

$$2n^2(\rho - 1 - \mu)(1-\rho)^4(2-\rho)^4 q_1 + 2(1-\mu)(1-\rho)^2(2-\rho)^4 q_1 + 3(1-\mu)(2\sigma_1 - \sigma_2)(2-\rho)^4 q_1 + 2\mu(1-\rho)^4(2-\rho)^2 q_2 - 3\mu(2\sigma_3 - \sigma_4)(1-\rho)^4 q_2 - \frac{Mb(-1-\mu+\rho)}{\{(-1-\mu+\rho)^2+T^2\}^{\frac{3}{2}}} = 0 \tag{17}$$

Expanding equation (20) we get:

$$+2n^2\rho^9 + 2n^2(-13-\mu)\rho^8 + 2n^2(2(37+6\mu))\rho^7 + 2(-2n^2(121-31\mu) + (q_1 - \mu q_1 + \mu q_2))\rho^6 + 2\left(n^2(501-180\mu) - 2(4q_1 - 5\mu q_1 + 4\mu q_2)\right)\rho^5 + \left(-2n^2(681-321\mu) + 2(40q_1 - 41\mu q_1 - 26\mu q_2) + 3(2\sigma_1 q_1 - \sigma_2 q_1 - 2\mu\sigma_1 q_1 + \mu\sigma_2 q_1 + 2\mu\sigma_3 q_2 - \mu\sigma_4 q_2)\right)\rho^4 + (2n^2(680+360\mu) - 88(2q_1 - 2\mu q_1 + \mu q_2) + 12(-4\sigma_1 q_1 + 2\sigma_2 q_1 + 4\mu\sigma_1 q_1 - 2\mu\sigma_1 q_1 - 2\mu\sigma_3 q_2 + \mu\sigma_4 q_2))\rho^3 + \left(-2n^2(344+248\mu) + 8(26q_1 - 26\mu q_1 - 26\mu q_1 + 18\sigma_1 q_1 - 9\sigma_2 q_1 - 18\mu\sigma_1 q_1 + 9\mu\sigma_2 q_1) + 2(41\mu q_2 + 18\mu\sigma_3 q_2 - 9\mu\sigma_4 q_2)\right)\rho^2 + (2n^2(112+96\mu) - 8(-16q_1 + 16\mu q_1 - 24\sigma_1 q_1 + 12\sigma_2 q_1 + 24\mu\sigma_1 q_1 - 12\mu\sigma_2 q_1 - 5\mu q_2 - 3\mu\sigma_3 q_2) - 12\mu\sigma_4 q_2)\big)\rho^1 + \left(2n^2(-16+16\mu) + 8(4q_1 - 4\mu q_1 - 6\sigma_2 q_1 - 12\mu\sigma_1 q_1 + 6\mu\sigma_2 q_1 + q_2) + 3(2\mu\sigma_3 q_2 - \mu\sigma_4 q_2)\right) - \frac{Mb(-1-\mu+\rho)}{\{(-1-\mu+\rho)^2+T^2\}^{\frac{3}{2}}} = 0$$

(18)

We shall further solve Eqs.(12),(15) and (18) numerically for the real values of ρ.Then using its values we shall find the positions of $L_{1,2,3.}$

3.4 Stability of the collinear equilibrium points

We use the characteristic equation of the system as given by Singh and Isah (2021), to determine the stability of the collinear libration point L_i (i=1,2,3) which is:

$$\lambda^4 - \left(\Omega^0{}_{\xi\xi} + \Omega^0{}_{\eta\eta} - 4\right)\lambda^2 + \Omega^0{}_{\xi\xi}\Omega^0{}_{\eta\eta} - \left(\Omega^0{}_{\xi\eta}\right)^2 = 0 \qquad (19)$$

The second partial derivative of equation (2), with $\eta = 0$ can be written as:

$$\Omega^0{}_{\xi\xi} = (1-e^2)^{-1/2}\Bigg[1 + \frac{2}{n^2}\left\{\frac{(1-\mu)q_1}{|\xi+\mu|^3} + \frac{3(1-\mu)(2\sigma_1-\sigma_2)q_1}{|\xi+\mu|^5} + \frac{\mu q_2}{|\xi+\mu-1|^3} + \frac{3\mu(2\sigma_3-\sigma_4)q_2}{|\xi+\mu-1|^5} - \frac{Mb}{(r^2+T^2)^{3/2}} + \frac{Mb\xi^2}{(r^2+T^2)^{5/2}}\right\}\Bigg]$$

$$\Omega^0{}_{\eta\eta} = (1-e^2)^{-1/2}\left[1 - \frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{|\xi+\mu|^3} + \frac{3(1-\mu)(4\sigma_1-3\sigma_2)q_4}{2|\xi+\mu|^5} + \frac{\mu q_2}{|\xi+\mu-1|^3} + \frac{3\mu(4\sigma_3-3\sigma_4)q_1}{2|\xi+\mu-1|^5} + \frac{Mb}{(r_c{}^2+T^2)^{3/2}}\right\} - \frac{4W_2\eta(\xi+\mu-1)}{r_2{}^3 n}\right] \qquad (20)$$

$$\Omega^0{}_{\zeta\eta} = \Omega^0{}_{\eta\zeta} = 0 \qquad (21)$$

It is obvious that $\Omega^0{}_{\xi\xi} > 0$ (22)

3.4.1 Stability of collinear point L_1 in the interval $\xi > \xi_2$)

In the first interval we have $\xi > \xi_2$)

$r_1 = (\xi + \mu) \Longrightarrow \xi = (r_1 - \mu)$ and $r_2 = (\xi + \mu - 1)$ (23)

Substituting equation (23) in the equation (7), we get:

$$\frac{(1-\mu)}{r_1{}^2} = n^2\xi - \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1{}^4} - \frac{\mu q_2}{r_2{}^2} - \frac{3\mu(2\sigma_3-\sigma_4)q_1}{2r_2{}^4} + \frac{Mb(r_1-\mu)}{(r_c{}^2+T^2)^{3/2}} \qquad (24)$$

Using equation (24) in the second equation of (20), we obtain

$$\Omega^0{}_{\eta\eta} = (1-e^2)^{-1/2}\left[1 - \frac{1}{n^2}\left\{\frac{1}{r_1}\left(n^2\xi - \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1{}^4} - \frac{\mu q_2}{r_2{}^2} - \frac{3\mu(2\sigma_3-\sigma_4)q_2}{2r_2{}^4} + \frac{Mb(r_1-\mu)}{(r_c{}^2+T^2)^{3/2}}\right) + \frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2r_1{}^5} + \frac{\mu q_2}{r_2{}^3} + \frac{3\mu(4\sigma_3-3\sigma_4)q_2}{2r_2{}^5} + \frac{M_b}{(r_c{}^2+T^2)^{3/2}}\right\} + \frac{4W_2\eta(\xi+\mu-1)}{r_2{}^3 n}\right]$$

(25)

Neglecting higher order terms in e^2,a,σ_i (i=1,2,3,4) and M_b we have

$$\Omega^0{}_{\eta\eta} = \frac{\mu}{r_1} + \frac{1}{n^2}\left\{\frac{\mu q_2}{{r_2}^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{3\mu q_2}{2{r_2}^4}\left\{\frac{(4\sigma_3 - 3\sigma_4)}{r_2} - \frac{(2\sigma_3 - \sigma_4)}{r_1}\right\} - \frac{3(1-\mu)q_1}{{r_1}^5}(\sigma_1 - \sigma_2) + \frac{M_b(1+r_1-\mu)}{({r_c}^2+T^2)^{3/2}}\right\} - \frac{4W_2\eta(\xi+\mu-1)}{{r_2}^3 n}$$

Thus $\Omega^0{}_{\eta\eta} < 0$, since $\mu<\frac{1}{2}$, $\sigma_1(i = 1,2,3,4) \ll 1$, $r_1 > 1$, $r_2 < 1$ and $Mb<<1$

Also,for the collinear points lying in the interval , $\xi_1<\xi<\xi_2$ and $\xi_1>\xi$ with respect to their primaries with $\eta=\zeta=0$,we have $\Omega^0{}_{\xi\xi} > 0$, $\Omega^0{}_{\eta\eta} < 0$, $\Omega^0{}_{\zeta\eta} = 0$.

Since $\Omega^0{}_{\xi\xi}\Omega^0{}_{\eta\eta} - (\Omega^0{}_{\xi\eta})^2<0$, the discriminant of equation (19) is positive for all intervals and therefore the characteristic roots can be written as;

$$\lambda_{1,2} = \pm a \ \ and \ \ \lambda_{3,4} = \pm ib \qquad (26)$$

where a and b are real numbers

Therefore we conclude that the collinear libration points are unstable as a result of the characteristic root of (26).

4. Numerical Application

The positions of collinear libration points $L_i(i = 1,2,3)$ and their corresponding characteristic roots for the two binary system FL virginis and Procyon are obtained numerically. The radiation pressure factors q_1 and q_2 of the bigger and the small primary are computed using q=1-(∧×L/rρM) on the basis of Stefan boltzmann'slaw, where M and L are the mass and luminousity of a star respectively, r is the radius and ρ the density of a moving test particle, q is the radiation pressure efficiency of a star, $\wedge=\left(\frac{3}{16\pi CG}\right)$ is a constant. In the centimeter – gram second system of units (C.G.S system) $\wedge = 2.9838 \times 10^{-5}$ .we take $r = 2x10^{-2}$cm and $\rho = 1.4$gcm^{-3} for some dust particle (Singh and Umar,2012). Table 1 contains the numerical data of the binaries and Table 2 shows the effect of triaxiality on the positions of collinear equilibrium points of the two binary systems (FL virginis and Procyon).We present the characteristic roots in Tables 3 and 4.

The effects of the greavitational pontetial from the belt and Pr-drag on the positions of equilibrium points are shown on the graphs in fig 5 and 6.

Table 1: Numerical data for the binary system

Binary system	Masses $M_\odot$		e	a	Luminosity $L_\odot$		Spectral type
	M_1	M_2			$L_{1\odot}$	$L_{2\odot}$	
FL virginis	0.076	0.067	0.3	0.9306''	1.372×10^{-3}	1.078×10^{-3}	M5se/M7
Procyon	1.53	0.617	0.4	4.3''	6.93	0.00049	F5/DA

Sorce:Stellar-DataBase/The American Astronomical Society/Wikepedia

Table 2: The effect of triaxiality on the position of collinear equilibrium points $L_i(i = 1,2,3)$ for the binary system FL virginis and Procyon

For FL virginis M_b=0.01, μ=0.4685,e = 0.3, a = 0.219

For Procyon M_b=0.01, μ=0.2874,e = 0.4, a = 0.714

Binary system	Triaxiality				Location		
FL virginis	σ_1	σ_2	σ_3	σ_4	L_1	L_2	L_3
	0.004	0.02	0.003	0.011	1.187357	0.53983926	-1.4208216
	0.005	0.03	0.004	0.012	1.187463	0.53690786	-1.4205792
	0.0051	0.031	0.0041	0.0121	1.187474	0.53661700	-1.4205549
	0.0052	0.032	0.0042	0.0122	1.187484	0.53631497	-1.4205306
Procyon	0.004	0.02	0.003	0.011	1.441199	0.192524	-1.2733849
	0.005	0.03	0.004	0.012	1.441287	0.178198	-1.2723137
	0.0055	0.035	0.0045	0.0125	1.441296	0.176627	-1.2722063
	0.0056	0.0356	0.0046	0.0126	1.441305	0.175053	-1.2720989

Table 3: The effect of Gravitational pontetial from the belt on the position of collinear equilibrium points $L_i(i = 1,2,3)$ for the binary system FL virginis and Procyon

For Flvirginis M_b=0.01, μ=0.4685,e = 0.3, a=0.219

For Procyon M_b=0.01, μ=0.2874,e = 0.4, a=0.714

Binary system	Radiation		Belt	Locations		
	q_1	q_2	M_b	L_1	L_2	L_3
FL virginis	0.999981	0.999983	0.01	1.187357	0.539839	-1.4208216
			0.001	1.187345	0.539213	-1.4208201
			0.012	1.187333	0.538581	-1.4208186
			0.0122	1.187330	0.538455	-1.4208183
Procyon	0.9952	0.9999	0.01	1.441199	0.192524	-1.2733849
			0.001	1.441178	0.190772	-1.2733801
			0.012	1.441156	0.189048	-1.2733530
			0.0122	1.441152	0.188707	-1.2733743

Table 4: The effect of the Pr-drag force on the position of collinear equilibrium points $L_i(i = 1,2,3)$ for the binary system FL virginis and Procyon

For Flvirginis M_b=0.01,W_2=3.39$\times 10^{-10}$, μ=0.4685,e = 0.3, a=0.219

For Procyon M_b=0.01,W_2=3.39$\times 10^{-10}$, μ=0.2874,e = 0.4, a=0.714

Binary system	Radiation		Pr-drag	Locations		
FL virginis	q_1	q_2	W_2	L_1	L_2	L_3
	0.999981	0.999983	1.39$\times 10^{-10}$,	1.155656	0.649725	-1.5346789
			2.39$\times 10^{-10}$,	1.155634	0.649613	-1.5345278
			3.39$\times 10^{-10}$,	1.155612	0.649599	-1.5344832
			4.39$\times 10^{-10}$,	1.155596	0.649458	-1.5344542
Procyon	0.9952	0.9999	1.39$\times 10^{-10}$,	1.229343	0.346632	-1.1653730
			2.39$\times 10^{-10}$,	1.229339	0.345611	-1.1653701
			3.39$\times 10^{-10}$,	1.229330	0.344597	-1.1653554

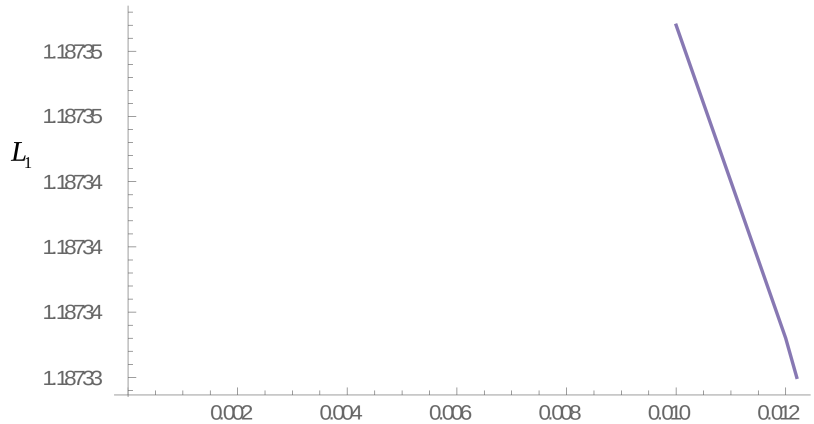
Table 5: The characteristic roots ($\lambda_{1,2}$, $\lambda_{3,4}$) of collinear points for the system FL Virginis

M_b=0.01 ,W_2=3.39$\times 10^{-10}$

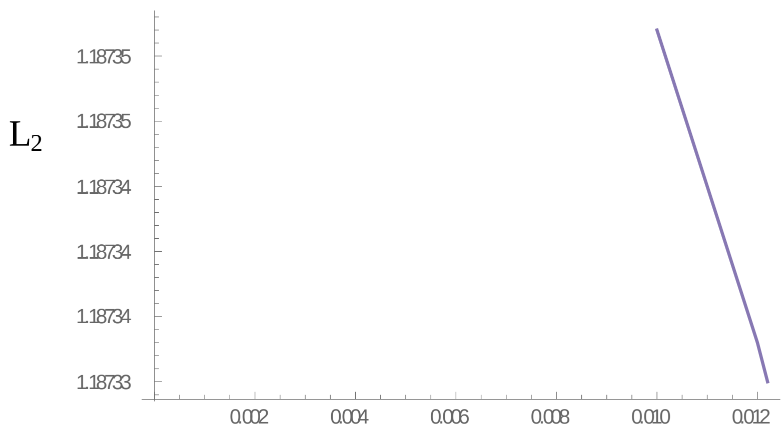
L_1	$\lambda_{1,2}$	$\lambda_{3,4}$	Stability Behaviour	
1.187357	± 0.772008	±1.45379i	Unstable	
1.187463	±0.773183	±1.45204i	Unstable	
1.187474	±0.773299	±1.45187i	“	
1.187484	±0.773418	±1.45168i	“	
L_2	$\lambda_{1,2}$	$\lambda_{3,4}$		
0.52316074	±(5919.01-072.62 i)	±(5919.01+6072.62 i)	Unstable	
0.526092139	±(15670.3-7545.3 i)	±(17545.3+1567.3i)	“	
0.5263883	±(17810.2-0147.9 i)	±(17810.2+20147.9 i)	“	
0.52668503	±(20410.4-3337.2 i)	±(20410.4+23337.2 i)	“	
L_3	$\lambda_{1,2}$	$\lambda_{3,4}$		
-1.4208216	±(74.8461-69.842i)	±(74.8461+69.842i)	Unstable	
-1.4205792	±(71.884-59.9976i)	±(71.884+59.9976i)	“	
-1.4205549	±(71.5622-58.9124i)	±(71.5622+58.9124i)	“	
-1.4205306	±(71.2346-57.8039i)	±(71.2346+57.8039i)	“	

Table 6: The characteristic roots ($\lambda_{1,2}$, $\lambda_{3,4}$) of collinear points for the system Procyon $M_b = 0.01$, $W_2 = 3.39\times 10^{-2}$

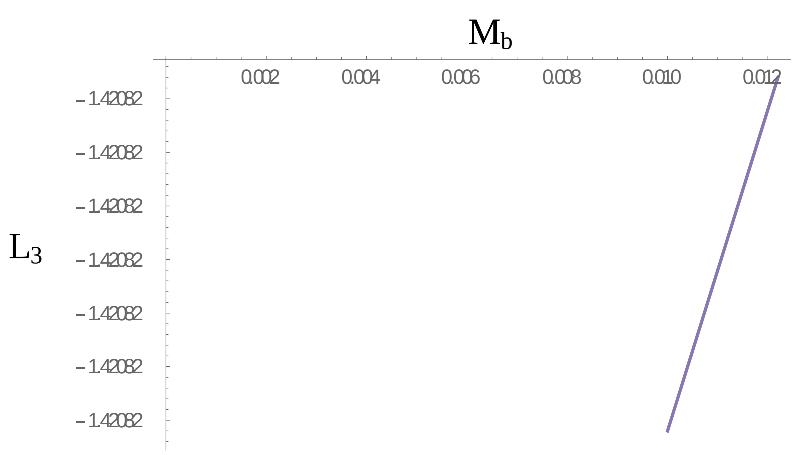
L_1	$\lambda_{1,2}$	$\lambda_{3,4}$	Stability Behaviour
1.441199	±0.838287	1.35774i	Unstable
1.441287	±0.839167	±1.3549i	“
1.441296	±0.839254	±1.35461i	“
1.441305	±0.839342	±1.35432i	“
L_2	$\lambda_{1,2}$	$\lambda_{3,4}$	
0.192524	±(0.699956-0.789718i)	±(0.699956+0.789718i)	Unstable
0.178198	±(0.617039-0.746857i)	±(0.617039+0.746857i)	“
0.176627	±(0.606967-0.741765i)	±(0.606967+0.741765i)	“
0.175053	±(0.596454-0.736477i)	±(0.596454+0.736477i)	“
L_3	$\lambda_{1,2}$	$\lambda_{3,4}$	
-1.2733849	±(0.644039-1.53291i)	±0.644039+1.53291i)	Unstable
-1.2723137	±(411.494-360.325i)	±(411.494+360.325i)	“
-1.2722063	±(406.36-351.777i)	±(406.36+351.777i)	“
-1.2720989	±(401.274-343.264i)	±(401.274+343.264i)	“



L_1
1.18735
1.18735
1.18734
1.18734
1.18734
1.18733
0.002
0.004
0.006
0.008
0.010
0.012



L_2
1.18735
1.18735
1.18734
1.18734
1.18734
1.18733
0.002
0.004
0.006
0.008
0.010
0.012



M_b
0.002
0.004
0.006
0.008
0.010
0.012
L_3
−1.42082
−1.42082
−1.42082
−1.42082
−1.42082
−1.42082
−1.42082

Fig.5 The effect of pontetial from the Belt and Pr-Drag on $L_{1,2,3}$ of FL virginis

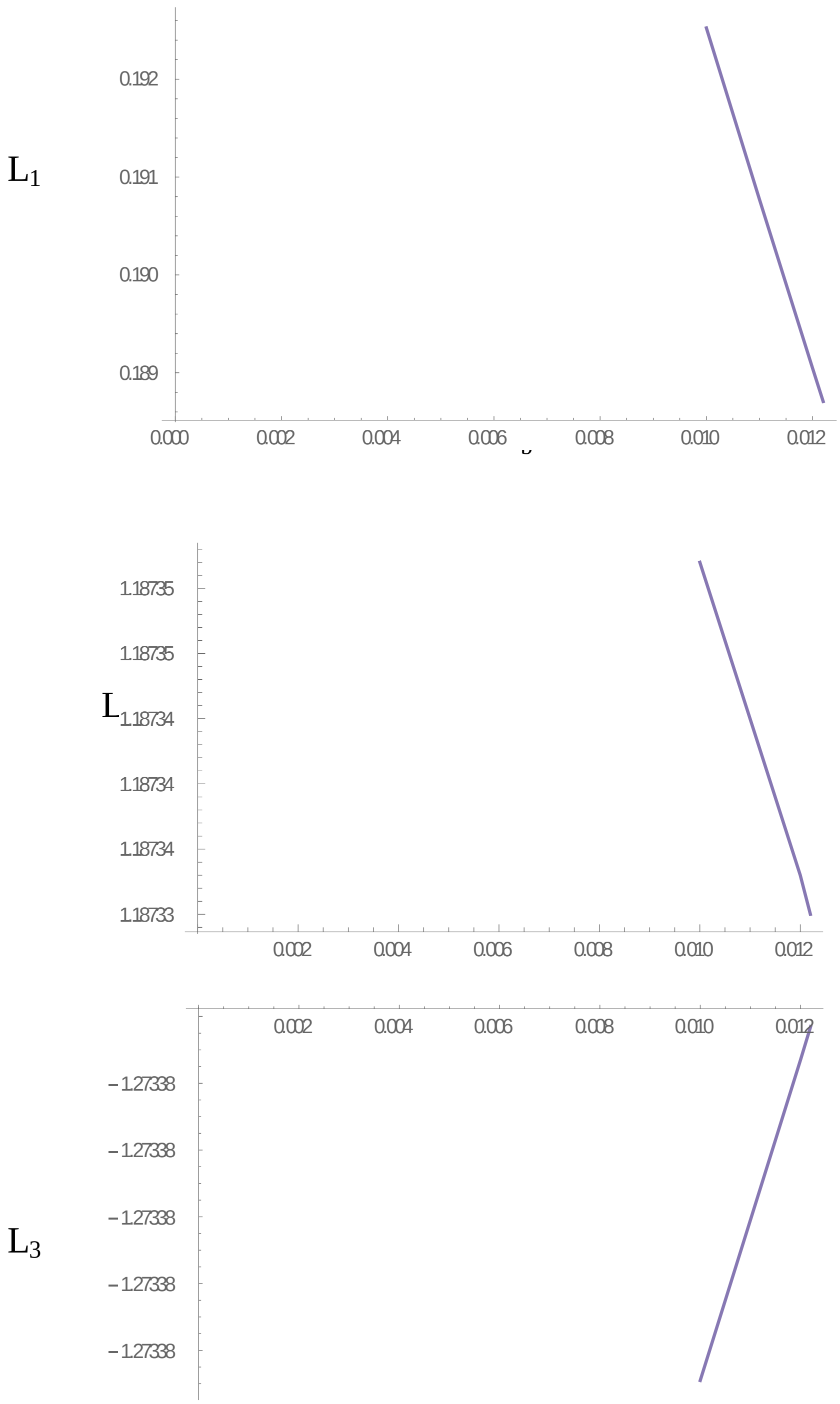


Fig.6 The effect of pontetial from the Belt and PR-Drag on $L_{1,2,3}$ of FL Procyon

5. Discussion and Conclusion

In Tables 2-4 we have used equation (12), (15) and (18) to compute numerically, using the software MATHEMATICA, the positions of the collinear equilibrium points for the increasing values of triaxiality coefficient of the two binaries (FLvirginis and Procyon).We can see from Table 2, that the dynamical effect of triaxiality on the positions of the collinear equilibrium points of FL virginis and procyon are similar because in both cases the collinear points $L_{1,2,3}$ move in the same direction.Both L_1 and L_2 moves towards the smaller primary , while L_3 shifts towards the bigger primary. In Table 3 we increasingly varied the effect of the belt on the collinear points $L_{1,2,3}$ of FL virginis and procyon.Due to increasing values of the belt, L_1 in the two system moves towards the barycentre, L_2 moves towards the smaller primary,while the third collinear point L_3 moves closer to the bigger primary.From Table 4 we can see that the dynamical effect of the Pr-drag force on the positios of collinear equilibrium points are similar that shown in Table 3.

The stability of collinear equilibrium points are obtained by substituting Equations (20) and (21) in (19).The characteristic roots obtained are shown in Table 5 and 6 for the systems FL virginis and Procyon.The characteristic roots obtained in Table 5 and 6 using the eccentricity, semi-major axis and radiation factor of the two binary systems with arbitrarily chosen triaxiaity coefficients,belt and Pr-drag are either positive, negative or complex.The presence of positive real parts in the roots shows that the collinear libration points are unstable.This instability behaviour agrees with those of (Singh and Umar,2014;Kumar and Naraya,2012;Singh and Tokyaa,2017**).**We have also shown graphically, the effect of the belt and Pr-drag on the positions of collinear points (figures 5 and 6) by substutimg the collinear points of the binaries and the increasingly varied values of the belt and Pr-drag into equation (19).The graphs show clearly that the position of collinear points move uniformly with increasing values of the belt and Pr-drag.

We have examined the positions and stability of collinear equilibrium points in the elliptic restricted three body problem under the influence of triaxiality, radiation,Pr-drag and gravitational potential from the belt.We found that the positions and linear stability of the collinear equilibrium points are affected significantly by triaxiality, radiation,Pr-drag and gravitational potential from the belt..The collinear equilibrium points are found to be unstable..).

A practical example of this model is the recently discovered circumbinary disc around the binary star Oph-IRS67AB,located 500 light years away in the ophiuchus constellation,separated from each other by a distance of about 90 A.U.

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