

An Error Analysis of Implicit Finite Difference Method with Mamadu-Njoseh Basis Functions for Time Fractional Telegraph Equation

ABSTRACT

In this paper, we proposed and analyzed the error estimate of an implicit finite difference method with Mamadu-Njoseh as basis functions for time fractional telegraph equation. To enhance the efficacy of the method we first transform the Caputo type fractional derivative into Riemann-Liouville derivatives. The error analysis of the method is stated and proven. Also, the optimal results for scalars unknown in L_∞ norm were derived for the two-dimensional case. Numerical illustrations are presented to test the reliability of the analytical and computed results. The resulting numerical evidence shows that the proposed method convergences more rapidly than the standard finite difference method. MAPLE 18 is used for all mathematical procedures in this paper.

Keywords: Riemann–Lionville derivatives, Quadrature formula, Orthogonal collocation method, Mamadu-Njoseh polynomials, Sobolev space, Finite difference method.

1. INTRODUCTION

The telegraph equation (otherwise known as the transmission line model) is a coupled partial differential equation that models the flow of voltage and current on a transmission line in time and distance. The equation was designed by Oliver Heaviside in 1876 in the course of developing the transmission line model. The equation has revolved, over the years with direct applications to transmission lines involving all frequencies, such as telephone lines, radio frequency, telegraph wires, power line and wire radio antenna [1].

In view of Wang *et. al.* [2], a typical time fractional telegraph equation is given as

$${}_0^C D_t^\beta u(x, t) + u_t(x, t) - u_{xx}(x, t) = g(x, t), x \in [0, T], t > 0, \quad (1.1)$$

with the initial conditions

$$\left. \begin{aligned} u(x, 0) &= u_0(x) \\ u_t(x, 0) &= u_1(x) \end{aligned} \right\} x \in [0, T], \quad (1.2)$$

and boundary conditions

$$\left. \begin{aligned} u(0, t) &= 0 \\ u(T, t) &= 0 \end{aligned} \right\} t > 0, x \in [0, T] \quad (1.3)$$

where $1 < \beta < 2$ and $g(x, t)$ is the source term.

In recent literature, there exist some numerical techniques for solving partial differential equations (PDEs), which include the spectral method [3], finite volume method [4], finite difference method [5], mixed finite element method [6], finite element method [7] and H^1 –

39 Galerkin mixed finite element method [2]. Yazdani *et. al.* [8] studied the numerical solution
40 of space fractional advection-diffusion equation adopting the finite volume-element method.
41 The work expressed the fractional derivative composition in the Grunwald-Letnikov form.
42 The convergence and stability of the method was also studied which resulted to the
43 conclusion that full discretization is possible and stable in as much as the mesh graded size
44 is sufficiently small. In like fashion, Hao *et. al.* [9] considered the Galerkin finite element
45 method (GFEM) for the solution of two-sided one-dimensional diffusion equation with
46 variable coefficients. They reformulated the governing problem into a low-ordered term that
47 is fractional by mere introduction of an extra parameter. It was argued that the GFEM is far
48 superior to that of the Petro-Galerkin method in the sense that the GFEM can easily be
49 extended to three-dimensional variable coefficients.

50
51 However, the stability and convergence analysis of this method was not treated.
52 Superconvergence of the finite element method as applied to time-fractional diffusion
53 equation (TFDE) governed by a time-space diffusivity was studied by An [10]. Weak
54 singularity of the model problem was studied at $t = 0$. Also, the fully discrete scheme on a
55 bounded mesh, and fully discrete conforming finite element method was investigated. The
56 author in conclusion remarked that superconvergence is achievable if temporary mesh pints
57 are set at $r \geq (2 - \alpha)/\alpha$, where r is graded mesh size and $0.5 < \alpha < 1$. Liu *et. al.*, [6] also
58 studied the numerical solution of time-fractional partial differential equations. The mixed
59 element method (MEM) was adopted as the numerical solver of the problem. The work of [6]
60 is quiet fascinating in the sense that the Caputo fractional derivative was discretized in time
61 via the two-step method (otherwise, finite difference method), and spatial direction was
62 discretized using the mixed finite element method.

63
64 There are few works existing in literature on the numerical methods of the telegraph
65 equation. For instance, Wang *et. al.* [2] applied the H^1 – Galerkin mixed finite element
66 method (H^1 -GMFEM) for the solution of time fractional telegraph equation. In line with the
67 approach of Liu *et. al.* [6], the authors also fully discretized in time the Caputo fractional
68 derivative using the finite difference method, and discretized in space using the H^1 -GMFEM.
69 For more on this, see Wei *et. al.* [11], and Zhao and Li [7].

70
71 Mamadu-Njoseh polynomials (MNP) are orthogonal polynomials developed by Njoseh and
72 Mamadu [12] for seeking the approximate solution of many linear and nonlinear problems in
73 the field of applied mathematics. The polynomials were constructed in the interval $[-1,1]$ with
74 respect to the weight function $w(x) = 1 + x^2$. These polynomials so far has contributed so
75 much in seeking the approximate solutions of integral equations, boundary value problems,
76 singular initial value problems in ordinary differential equation, integro-differential equation,
77 delay differential equations. However, these polynomials are implemented through an
78 appropriate numerical scheme as basis functions. For instance, the MNP were adopted by
79 Ahmed and Singh [13] for the solution of integral equation via the Galerkin method. In like
80 manner, Al-Humedi and KadhimMunaty [14] studied comparatively the MNP alongside
81 Chebyshev and Laguerre polynomials for the solution of first kind intergral equation by the
82 spectra petro-Galerkin method. Montazer *et. al.* [15] studied the MNP alongside non-uniform
83 Haar wavelets for the numerical treatment of linear Volterra integral equations.

84 Problems involving fractional order have equally solved applying these polynomials as seen
85 in the literature Xie [16]. Njoseh and Mamadu [17] applied the MNP as trial functions for the
86 solution of fifth order boundary value problems via the power series approximation method.
87 These polynomials were also used by Mamadu and Njoseh [18] for the solution of Votterra
88 integral equation via the Galerkin Method. Mamadu and Njoseh [19] considered the
89 Mamadu-Njoseh polynomials in orthogonal collocation methods for the solution of integro-
90 differential equations. Ogeh and Njoseh [20] constructed a modified variational iteration

91 method for the solution of fifth and sixth order boundary value problems adopting the
92 Mamadu-Njoseh polynomials as trial functions. In like manner, Njoseh and Musa [21]
93 adopted these polynomials for the solution of pantograph-type delay differential equation in a
94 variational iteration approach. Also, Mamadu and Ojarikre [22] proposed a reconstructed
95 Elzaki transform method (RETM) for the solution of delay differential equation using
96 Mamadu-Njoseh polynomials as basis functions. A perturbation by decomposition technique
97 was considered by Mamadu and Tsetimi [23] adopting the MNP as basis functions for the
98 solution of singular initial value problems.

99 This paper will centre on Mamadu-Njoseh polynomials as basis functions in a discretization
100 scheme. Thus, a novel finite difference method with Mamadu-Njoseh basis functions for the
101 time-fractional telegraph equation (1.1) will be proposed. An optimal error analysis for scalar
102 unknowns in L_∞ norm will be established for the equation (1.1).
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107 **2. PRELIMINARIES**

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109 Let equation (1.1) can be transformed into a fractional differential equation with the
110 Riemann – Lionville derivatives [24]

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$$112 \quad {}^R_0D_t^\beta [u - u_0](x, t) + u_t(x, t) - u_{xx}(x, t) = g(x, t). \quad (1.4)$$

113

114 By definition,

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$$116 \quad {}^R_0D_t^\beta (u_0) = \frac{d}{dt} \frac{1}{\gamma(1-\beta)} \int_0^t (t-s)^{-\beta} u_0 ds = \frac{u_0}{\gamma(1-\beta)} \frac{d}{dt} \left(\frac{1}{1-\beta} t^{1-\beta} \right) = \frac{u_0}{\gamma(1-\beta)} t^{-\beta}.$$

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118 Thus,

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$$120 \quad {}^R_0D_t^\beta u(x, t) = \frac{1}{\gamma(-\beta)} \int_0^t (t-s)^{-\beta-1} u(s) ds. \quad (1.5)$$

121 Let $[0, 1]$ be partitioned as $0 = t_0 < t_1 < \dots < t_n = 1$. by using $nt_j = j, j = 1, 2, \dots, n$,
122 we approximate (1.4) in time step as

123

$$124 \quad {}^R_0D_t^\beta u(x, t) = \frac{1}{\gamma(-\beta)} \int_0^{t_j} (t_j - s)^{\beta+1} u(s) ds.$$

125

126 let $t_j = t_j w + s$, we obtain,

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$$128 \quad {}^R_0D_t^\beta u(x, t_j) = \frac{t_j^{-\beta}}{\gamma(-\beta)} \int_0^1 \frac{u(t_j - t_j w) - u(0)}{w^{\beta+1}} dw = \frac{t_j^{-\beta}}{\gamma(-\beta)} \int_0^1 F(w) w^{-\beta-1} dw, \quad (1.6)$$

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130 where $F(w) = u(t_j - t_j w) - u(0)$.

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132 Using the quadrature formula [24], the integral is replaced with $t_j = n/j, n = 0$, for
133 each j , to obtain

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$$135 \quad {}^R_0D_t^\beta u(x, t) = \frac{1}{\gamma(-\beta)} \left[\sum_{r=0}^j \beta_{rj} u(t_j - t_r) + G_j(g) \right],$$

136 where,

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$$138 \quad \|G_j(g)\| \leq k_j^{\beta-2} \sup_{0 \leq t \leq T} \|u''(t_j - t_j t)\|.$$

139 Thus,

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$$141 \quad {}^R_0D_t^\beta u(x, t_j) = \frac{\Delta t^{-\beta}}{r(2-\beta)} \sum_{r=0}^j (-\beta)(1-\beta) j^{-\beta} \beta_{rj} u(t_j - t_r) + \frac{t_j^{-\beta}}{\gamma(-\beta)} G_j(g)$$

142

$$143 \quad = \Delta t^{-\beta} \sum_{r=0}^j \frac{(-\beta)(1-\beta) j^{-\beta} \beta_{rj}}{r(2-\beta)} u(t_j - t_r) + \frac{t_j^{-\beta}}{\gamma(-\beta)} G_j(g)$$

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$$145 \quad = \Delta t^{-\beta} \sum_{r=0}^j w_{rj} u(t_j - t_r) + \frac{t_j^{-\beta}}{\gamma(-\beta)} G_j(g),$$

146 where

$$\gamma(2 - \beta)w_{rj} = (-\beta + \beta^2)j^{-\beta}\beta_{rj},$$

148

149 such that w_{rj} and β_{rj} satisfies

150

$$w_{rj} = \frac{1}{\gamma(2-\beta)} \begin{cases} 1, & r = 0 \\ -2r^{1-\beta} + (r-1)^{1-\beta} + (r+1)^{1-\beta}, & r = 1, 2, \dots, j \\ -(\beta-1)r^{-\beta} + (r-1)^{1-\beta} - r^{1-\beta}, & r = j \end{cases}$$

152

$$\beta_{rj} = \frac{1}{\beta(1-\beta)j^{-\beta}} \begin{cases} -1, & r = 0 \\ -2r^{1-\beta} - (r-1)^{1-\beta} - (r+1)^{1-\beta}, & r = 1, 2, \dots, j \\ (\beta-1)r^{-\beta} - (r-1)^{1-\beta} + r^{1-\beta}, & r = j \end{cases}$$

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156 2.1 Discretization in Time

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158 Let consider the finite difference method [25] of (1.4) at $t = t_j$, we get

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$${}^R D_t^\beta [u(x, t) - u_0(x, t)]|_{t=t_j} + u_t(x, t_j) = g(x, t_j) + u_{xx}(x, t_j) \quad (1.7)$$

160

161

$$\Rightarrow \Delta t^{-\beta} \sum_{r=0}^j w_{rj} [u(t_j - t_r) - u(0)] + \frac{t_j^{-\beta}}{\Gamma(-\beta)} G_j(g) + u_t(x, t_j) = g(x, t_j) + u_{xx}(x, t_j),$$

162

163 or

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$${}^R D_t^\beta [u(x, t) - u_0(x, t)]|_{t=t_j} = \Delta t^{-\beta} \sum_{r=0}^j w_{rj} [u(t_j - t_r) - u(0)] + \frac{t_j^{-\beta}}{\Gamma(-\beta)} G_j(g). \quad (1.8)$$

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167 Denote $u_j \approx u(x, t_j)$, we obtain,

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$$\Delta t^{-\beta} \sum_{r=0}^j w_{rj} [u_{j-r} - u_0] + \frac{u_j^{-u_{j-1}}}{\Delta t} = G(x, t_j) + u_{xx}(x, t_j).$$

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171 Let $r = 0$, we obtain,

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178 But from, we find [30]

$$\sum_{r=0}^j w_{rj} = \frac{-\beta(1-\beta)j^{-\beta}}{(2-\beta)} \left(\frac{-1}{\beta}\right) = j^{-\beta} h_\beta,$$

182 where $h_\beta = \frac{1}{\Gamma(1-\beta)}$.

184 Thus, the implicit difference formula for (1.4) is given as

185 $(u_j - u_0)w_{0,j} + \sum_{r=1}^j w_{rj}u_{j-r} + j^{-\beta}h_{\beta}u_0 + \Delta t^{\beta} \left(\frac{u_j - u_{j-1}}{\Delta t} - \frac{u_{j+1} + 2u_j - u_{j-1}}{\Delta x^2} \right) = g(x, t_j). \quad (1.9)$

186 **3.1 Discretization in Time with Mamadu-Njoseh Polynomials**

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188 Suppose that an approximation to (1.4) is declined by

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$$u_j(x, t) \cong u(x, t) = \sum_{r=0}^j a_r(t)\varphi_r(x), \quad (1.10)$$

191 where,

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193 $a_r(t), r = 0(1)j$, are unknown constants to be estimated, and $\varphi_r(x), r = 0(1)j$, are Mamadu –

194 Njoreh basis functions.

195 Using (1.10) on (1.9), we obtain a new **implicit scheme** given by

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$$\sum_{r=0}^j a_r(t)\varphi_r(x)w_{0,j} + \frac{j^{-\beta}}{\Gamma(1-\beta)}a_0(t) + \sum_{r=1}^j w_{rj} \left(\sum_{r=0}^{j-r} a_r(t)\varphi_r(x) \right) +$$

198
$$\frac{\Delta t^{\beta}}{\Delta t} \sum_{r=0}^j a_r(t)\varphi_r(x) - \frac{\Delta t^{\beta}}{\Delta t} \sum_{r=0}^{j-1} a_r(t)\varphi_r(x) - \frac{\Delta t^{\beta}}{\Delta t} \sum_{r=0}^j a_r(t)\varphi_r(x) + \frac{2\Delta t^{\beta}}{\Delta x^2} \sum_{r=0}^j a_r(t)\varphi_r(x) +$$

199
$$\frac{\Delta t^{\beta}}{\Delta x^2} \sum_{r=0}^{j-1} a_r(t)\varphi_r(x) = g(x, t) \quad (1.11)$$

200

201 Equation (1.11) is collocated orthogonally ([26]-[28]) for any $j > r$ to obtain a system of

202 $(j + 1)$ linear equations which can be solve for the unknowns $a_r(t), r = 0(1)j$, via a suitable

203 mathematical software with estimate $\beta, \Delta t, \Delta t^2, w_{rj}$ clearly given. The approximate solution to

204 (1.4) is obtain by substituting the known estimate into (1.10).

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207 **3. MAIN RESULTS**

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209 In this section, we carry out a precise analysis of error estimate on the derived implicit

210 formula (1.11).

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212 Define $\beta = u_t(x, t) - u_{xx}(x, t), \Delta(\beta) = \{u: u(0) = u(T) = 0, u', u'' \in L_2(0, T)\}$, then the

213 equation (1.4) can be reformulated in abstract sense as

214

215
$${}_0^R D_t^{\beta} [u(t) - u_0(t)]|_{t=t_j} + Bu(t) = g(t), 0 \leq x \leq T, t > 0, \quad (1.12)$$

216

217 Using compound quadratic formula on (1.12), we have

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219
$$\frac{t^{-\beta}}{\Gamma(-\beta)} \sum_{r=0}^j \beta_{rj} [u(t_j - t_r) - u_0] + G_j(g) + Bu(t_j) = g(t_j) \quad (1.13)$$

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221 Evaluating (1.13) at $r = 0$ yields

222
$$[\beta_{0j} + t_j^{\beta} \Gamma(-\beta)\beta]u(t_j) = t_j^{\beta} \Gamma(-\beta)g_j - \sum_{r=1}^j \beta_{rj}(t_j - t_r) + \sum_{r=0}^j \beta_{rj}u_0 - G_j(g) \quad (1.14)$$

223

224
$$\Rightarrow [\beta_{0j} + t_j^{\beta} \Gamma(-\beta)\beta]u(t_j) = t_j^{\beta} \Gamma(-\beta) - \sum_{r=0}^j \beta_{rj}u_{jor} + \sum_{r=0}^j \beta_{rj}u_0, \quad (1.15)$$

225 where

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Let $e_j = u - u_i$ be the error function, where u is the analytic solution and u_j , then (1.15) can also reformulated in reference to (1.11) as

$$[\beta_{0j} + t_j^{\beta} \Gamma(-\beta) \left(\frac{u_j - u_{j-1}}{\Delta t} - \frac{u_{j+1} + 2u_j - u_{j-1}}{\Delta x^2} \right)]u_j = t_j^{\beta} \Gamma(-\beta) - \sum_{r=0}^j \beta_{rj}u_{j-r} + \sum_{r=0}^j \beta_{rj}u_0. \quad (1.16)$$

232

233 **Lemma 1 [24]:** For $\beta \in (1,2)$, let $\{h_j\}_{j=1}^{\infty}$ with $h_1 = 1$, then

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$$235 \quad 1 \leq h_j \leq \frac{\sin \pi \beta}{\pi \beta (1-\beta)} j^\beta, j = 1(2)\infty.$$

236

237 **Lemma 2 :** Suppose $p_0 = 1, p_j = \beta(1-\beta)j^{-\beta} \sum_{r=1}^j \beta_{rj} p_{j-r}, j = 1(2)\infty$, then, $p_j \leq 1$.

238

239 **Theorem 1:** Let U_j and u_j denotes the solution of (1.4) with the prescribed conditions, then

$$e_j \leq k \Delta t^{2-\beta} + e_0,$$

where $e_0 = \|u_0 - u\|$.

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242 **Proof:** Let subtract (1.15) from (1.13) to obtain the error equation as

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$$244 \quad (\beta_{0j} + t_j^\beta \Gamma(-\beta)B) e_j = -(\sum_{r=0}^j \beta_{rj} e_{j-r} + G_j(g)) \quad (1.17)$$

245

$$246 \quad \Rightarrow e_j = (-\beta_{0j} - t_j^\beta \Gamma(-\beta)B)^{-1} (\sum_{r=0}^j \beta_{rj} e_{j-r} + G_j(g)).$$

247

248 By definition of L_∞ norm, we obtain

$$249 \quad \|e_j\| \leq \|(-\beta_{0j} - t_j^\beta \Gamma(-\beta)B)^{-1}\| (\sum_{r=0}^j \beta_{rj} \|e_{j-r}\| + \|G_j(g)\|), \quad (1.18)$$

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251 where B is symmetric and positive definite operator [29]. Now for any function $f(x)$, we have

252 via spectral method that $\|f(B)\| = \sup_{\tau>0} |f(\tau)|$.

253

254 Hence, using the definition of β_{rj} and (1.18), we have

$$255 \quad \|(-\beta_{0j} - t_j^\beta \Gamma(-\beta)B)^{-1}\| = \left\| \left(\frac{1}{\beta(1-\beta)j^{-\beta}} - t_j^\beta \Gamma(-\beta)B \right)^{-1} \right\| \quad (1.19)$$

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$$257 \quad = \left\| \beta(1-\beta)j^{-\beta} (1 - \beta(1-\beta)j^{-\beta} t_j^\beta \Gamma(-\beta)B)^{-1} \right\|$$

258

$$259 \quad = \beta(1-\beta)j^{-\beta} \sup_{\tau>0} (1 - \beta(1-\beta)j^{-\beta} t_j^\beta \Gamma(-\beta)\tau)^{-1}.$$

260

261 By definition of gamma function, it is obvious $\Gamma(-\beta) > 0$. This implies

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$$263 \quad \sup_{\tau>0} (1 - \beta(1-\beta)j^{-\beta} t_j^\beta \Gamma(-\beta)\tau)^{-1} \leq 1.$$

264

$$265 \quad \text{Therefore,} \quad \left\| (-\beta_{0j} - t_j^\beta \Gamma(-\beta)B)^{-1} \right\| \leq \beta(1-\beta)j^{-\beta}.$$

266

267 Thus, (1.18) becomes

$$268 \quad \|e_j\| \leq \beta(1-\beta)j^{-\beta} [\sum_{r=0}^j \beta_{rj} \|e_{j-r}\| + K j^\beta m^{-2} \sup_{0 \leq t \leq T} \|u_{tt}''(t_j - t_r)\|], \quad (1.20)$$

269 $\Delta t = 1, t_m = m$.

270

271 Equation (1.20) can be reformulated into the form

$$272 \quad \|e_j\| \leq b + \beta(1-\beta)j^{-\beta} \sum_{r=0}^j \beta_{rj} \|e_{j-r}\|, \quad (1.21)$$

273 where, $b = \beta(1-\beta)j^{-\beta} K m^{-2} \|u''\|_{L_\infty}$.

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275 Let $j = n$, then (1.21) becomes

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$$\begin{aligned} \|e_n\| &\leq b + \beta(1 - \beta)n^{-\beta} \left[\sum_{r=1}^{n-1} \beta_{rn} \|e_{n-r}\| + \beta_{nn} \|e_0\| \right] \\ &\leq b + \beta(1 - \beta)n^{-\beta} \left[\sum_{r=1}^{n-1} \beta_{rn} (bh_{n-r} + p_{n-r} \|e_0\|) + \beta_{nn} \|e_0\| \right] \\ &= ah_n + p_n \|e_0\| \end{aligned}$$

where

$$\begin{aligned} h_n &= 1 + \beta(1 - \beta)n^{-\beta} \sum_{r=1}^{n-1} \beta_{rn} h_{n-r}, n = 2, 3, \dots, \\ p_n &= \beta(1 - \beta)n^{-\beta} \sum_{r=1}^n \beta_{rn} p_{n-r}, n = 1(2)\beta, p_0 = 1. \end{aligned}$$

280 Now using lemma 1 and lemma 2, we have from (1.21),

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$$\begin{aligned} \|e_j\| &\leq bh_j + p_j \|e_0\| \leq \beta(1 - \beta)Km^{-2} \|u''\|_{L_\infty} \cdot h_j + p_j \|e_0\| \\ &\leq \beta(1 - \beta)Km^{-2} \|u''\|_{L_\infty} \frac{\sin\pi\beta}{\pi\beta(1-\beta)} j^\beta + \|e_0\| \\ &\leq 1 \\ &\leq k\Delta t^{2-\beta} + \|e_0\|. \end{aligned}$$

5. Numerical Illustration

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To test the reliability and accuracy of the proposed method, we consider the example below:

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295 ${}^C_0 D_t^\beta u(x, t) + u_t(x, t) - u_{xx}(x, t) = 2(x^2 - x)t \left(\frac{\Gamma(3-\beta) + t^{1-\beta}}{\Gamma(3-\beta)} \right) - 2t^2, x \in [0, 1], t \in (0, 1],$ (1.22)

296 with the initial conditions

297
$$\left. \begin{aligned} u(0, 0) &= 0 \\ u_t(x, 0) &= 0 \end{aligned} \right\}$$

298 and boundary conditions

299
$$\left. \begin{aligned} u(0, t) &= 0 \\ u(1, t) &= 0 \end{aligned} \right\}, t > 0.$$

300

301 Applying the scheme (1.11) on (1.22) at $j = 3$ with parameters

302 $\beta = 1.5, \Delta x = 1/64, \Delta t = 1/1000$ at $t = 0$, and values of $w_{rj}, r = 0(1)3$, estimated as

303 $w_{0,3} = 1, w_{1,3} = -0.7294368868, w_{2,3} = 0.09204003089$ and $w_{3,3} = 0.1817856084$. Results

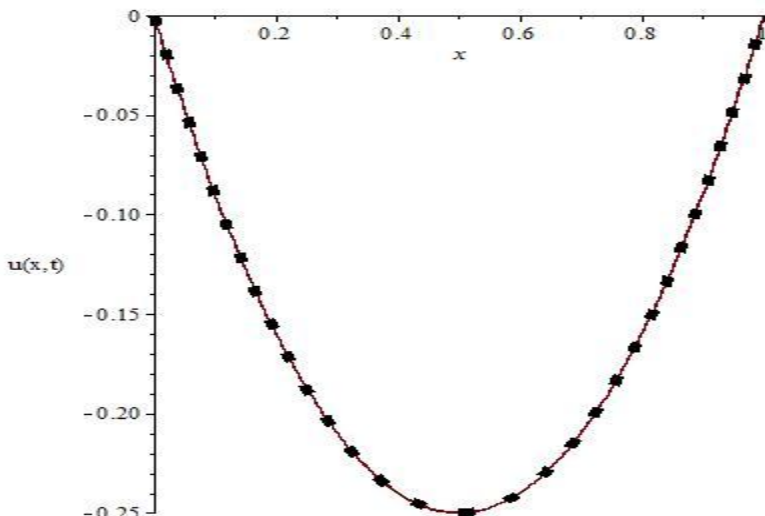
304 are presented **Table 1** and **Figure 1** using MAPLE 18.

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Table 1: Maximum error at $\beta = 1.5$

j	L_∞ Error (Proposed method)	L_∞ Error [11]
20	5.6445E-008	9.877022E-003
40	3.5198E-006	3.477002E-003
80	1.5441E-005	1.232302E-003

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Figure 1: Comparison of computed solutions and exact solutions.

Similarly at $j = 3$ with parameters $\beta = 1.8$, $\Delta x = 1/32$, $\Delta t = 1/1000$ at $t = 0$, and values of $w_{rj}, r = 0(1)3$, estimated as $w_{0,3} = 1$, $w_{1,3} = -0.3105422252$, $w_{2,3} = 0.05806019726$ and $w_{3,3} = 0.06480727693$. Results are presented in **Table 2** and **Figure 2** using MAPLE 18.

Table 2: Maximum error at $\beta = 1.5$

j	L_∞ Error (Proposed method)	L_∞ Error [11]
20	6.6376E-004	1.1484477E-002
40	1.4188E-006	5.148878E-003
80	2.4280E-007	1.998394E-003
160	3.9604E-008	8.980387E-004

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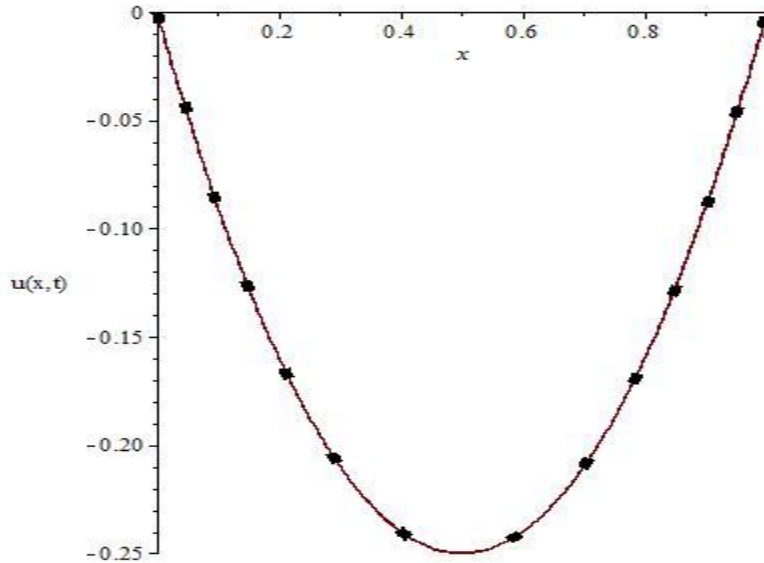


Figure 2: Comparison of computed solutions and exact solutions.

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6. Discussion of Results

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Numerical evidences to test the reliability and accuracy of the proposed method are presented in tables and figures. **Tables 1** and **2** shows the maximum error of the proposed method at $j = 20, 40, 80,$ and 160 . In comparison with the finite difference method in [30-32] show the superiority of the proposed method with maximum errors of order 10^{-8} and 10^{-8} , respectively. Results are also presented in graphs showing the comparison of results as shown in the **figures 1** and **2**.

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7. Conclusion

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In this paper, we have successively proposed an implicit finite difference method with Mamadu-Njoseh polynomials as basis functions for the time fractional telegraph equation. Numerical illustration of the proposed method showed convergence and accuracy than the standard finite difference method. The optimal error analysis of the proposed method was investigated in L_∞ norm for two dimensional case. The result showed that the method is of order $(2 - \beta)$.

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ACKNOWLEDGEMENTS

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Not Applicable.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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