Hyer-Ulam-Stability of Generalized Quadratic functional equation in Random Norm spaces

Abstact: The purpose of this paper to propose a finite variable generalized quadratic functional equation with solution. Also investigate Hyers Ulam stability in Random normed space by direct method.

Keywords: Quadratic functional equation, Hyers- Ulam stability, Random normed space.

Mathematics subject classification: 54E40, 39B82, 46S50.

1. Introduction: Functional equations play an important part in the study of stability. In 1940 , the stability problems of functional equations about group homomorphisms was introduced by Ulam [16]. In1941, Hyers [1] gave a affirmative answer to Ulam's quaestion for additive groups (under the assumption that groups are Banach spaces). Hyers theorem was generalized by Aoki [19] for additive mappings and by Rassias [23] for linear mappings by considering an unbounded Cauchy difference $\|f(x+y)-f(x)-f(y)\| \leq \varepsilon(\|x\|^p+\|y\|^p)$ for all $\varepsilon > 0$ and $p \in [0,1)$. Also Rassias[22] generalization theorem was delivered by Gavruta [13] whoreplaced $\varepsilon(\|x\|^p + \|y\|^p)$by a control function $\varphi(x,y)$.The paper of Rassias has significantly influenced the development of what we now call the Hyers-Ulam-Rassias stability of functional equations. In1982, J.M. Rassias [5] followed the modern approach of the Th.M.Rassias theorem in which he replaced the factor product of norms instead of sum of norms.

Recently the stability of many functional equations in various spaces like Banach spaces, modular spaces, fuzzy normed spaces and Random normed spaces etc. have been established by researchers[9,10,18-20]. Now we introduce new quadratic functional equation and obtain the Hyers- Ulam stability of quadratic function equation

$$\sum_{i=1}^{n} \emptyset\left(-2v_i + \sum_{j=1,i\neq j}^{n} v_j\right) = (n-6)\sum_{1\leq i<j\leq n} \emptyset\left(v_i+v_j\right) - (n^2-8n+3)\sum_{i=1}^{n} \emptyset(v_i) \tag{1.1}$$

in Random normed spaces.

We adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [11].

Throughout, Δ^+ denotes the distribution functions spaces , i.e., the space of all mapping f:R∪ $(-\infty,\infty)$ →[0,1] such that f is left continuous and increasing on R, f (0)=0 and f (+∞)=1. D^+

subset of Δ^+ consisting of all functions V of $\Delta^+$for which $\ell^- V(+\infty)=1$ where $\ell^- f(s)$ denotes $\ell^- f(s)=\lim_{t\to s^-} f(t)$. The space Δ^+ is partially order by usual wise ordering of functions, i.e.,

$f \leq g \Leftrightarrow f(s) = g(s), \forall s \in R$. The maximal element for Δ^+ in this order is the distribution function $\epsilon_0(s) = \begin{cases} 0, & if\ s \leq 0 \\ 1 & if\ s > 0 \end{cases}$

Definition 1.1: A Random Normed space (RN-space) is a triple (V,μ, T), where V is a vector space, T is a continuous t-norm and μ:V→ D^+ satisfying the following conditions:

(R_1). $\mu_v(t)= \epsilon_0(t)$ for all t>0 if and only if v=0,

(R_2). $\mu_{av}(t)= \mu_v(\frac{t}{|a|})$ for all v ϵV, t ≥ 0 and a ϵ R with a ≠ 0,

(R_3). $\mu_{v+u}(t+s) \geq T(\mu_v(t), \mu_u(s))$ for all v, u ϵ V and t, u ≥0.

Definition 1.2. Let (V,μ, T) be a RN- space.

(i) A sequence $\{v_n\}$ in V is said to be convergent to v ϵV if $\lim_{n\to\infty} \mu_{v_n - v}(t) = 1$, t > 0.

(ii) A sequence $\{v_n\}$ in V is said to be Cauchy sequence if $\lim_{n\to\infty} \mu_{v_n - v_m}(t) = 1$, t > 0.

(iii) A RN-space (V,μ, T) is complete if every Cauchy sequence is convergent in V.

2. General solution

Theorem 2.1. If any even function ∅: V→W satisfies the functional equation (1.1) for all v_1, v_2,………,v_n ϵ V, then the function ∅ is quadratic.

Proof. Let function ∅: V→W satisfies (1.1). Taking v_1= v_2=………=v_n=0 in (1.1), we have $\emptyset(0) = 0$. Replacing (v_1, v_2,………,v_n) by (v,0,……..,0) in (1.1), we get

$$\emptyset(-2v)+ (n-1)\emptyset(v) = (n-6)(n-1)\emptyset(v) - (n^2 - 8n + 3)\emptyset(v) \tag{2.1}$$

$$\emptyset(-2v)=(n^2 - 7n+6 - n^2 +8n - 3 - n +1)\emptyset(v) \tag{2.2}$$

$$\emptyset(-2v)=4\emptyset(v) \tag{2.3}$$

Now taking v_1=v_2=v and v_3=……=v_n=0 in (1.1), we have

$$2\emptyset(-v)+ (n-2)\emptyset(2v) = (n-6)\emptyset(2v) + 2(n-6)(n-2)\emptyset(v) - 2(n^2 - 8n + 3)\emptyset(v)$$

$$2\emptyset(-v)+4\emptyset(2v) = 18\emptyset(v) \tag{2.4}$$

Since ∅ is an even mapping, hence by (2.4) we have

$\emptyset(2v)=2^2\emptyset(v)$, for all vϵV.

Now, replace v by 2v, we have

$\emptyset(2^2 v) = 2^4\emptyset(v)$

Continue like this, we generalize

$\emptyset(2^n v) = 2^{2n}\emptyset(v)$ (2.5)

For all vϵV and for any n≥0.

Similarly, we have

$\emptyset(2^{-n} v) = 2^{-2n}\emptyset(v)$ (2.6)

Remark 2.2: Let V be a linear space and a mapping $\emptyset: V \rightarrow W$ satisfies the functional equation (1.1), then

(i) $\emptyset(r^n v) = r^n\emptyset(v)$ for all vϵV,rϵQ, tϵZ.
(ii) $\emptyset(v)$=v$\emptyset(1)$ for all vϵV if $\emptyset$ is continuous.

3. Hyers-Ulam Stability : Direct method

For notational handiness, we denote (V,Ψ, T), and (W,Ψ, T) are Complete RN spaces and define a mapping $\emptyset: V \rightarrow W$ by

$$D\emptyset(v_1, v_2, \dots\dots\dots\dots\dots, v_n) = \sum_{i=1}^{n} \emptyset\left(-2v_i + \sum_{j=1, i\neq j}^{n} v_j\right) - (n-6)\sum_{1\leq i<j\leq n} \emptyset\left(v_i + v_j\right) + (n^2 - 8n + 3)\sum_{i=1}^{n} \emptyset(v_i) \quad (3.1)$$

for all $v_1, v_2, \dots\dots\dots, v_n \epsilon V$.

Theorem 3.1. If an even mapping $\emptyset$: V→W with $\emptyset(0) = 0$ for which there exists a mapping $\Phi: V^n \rightarrow D^+$ for some 0<α<4,

$$\Phi_{2v_1, 2v_2, \dots\dots\dots, 2v_n}(\varepsilon) \geq \Phi_{v_1, v_2, \dots\dots\dots, v_n}\left(\frac{\varepsilon}{\alpha}\right) \quad (3.2)$$

And $\lim_{t\to\infty} \Phi_{2^t v_1, 2^t v_2, \dots\dots\dots, 2^t v_n}(2^{2t}\varepsilon) = 1$ (3.3)

for all $v_1, v_2, \dots\dots\dots, v_n \epsilon$ V and all ε > 0 such that

$$\mu_{D\emptyset(v_1, v_2, \dots\dots\dots, v_n)}(\varepsilon) \geq \Phi_{v_1, v_2, \dots\dots\dots, v_n}(\varepsilon). \quad (3.4)$$

Then there exists a unique quadratic mapping Q_2: V→W satisfying the functional equation (1.1) with $\mu_{Q_2(v) - \emptyset(v)}(\varepsilon) \geq \Phi_{v, 0, \dots\dots\dots, 0}((2^2 - \alpha)2\varepsilon)$ (3.5)

for all v ϵ V and all ε > 0. The mapping Q_2: V→W is defined by

$$\mu_{Q_2(v)}(\varepsilon) = \lim_{t\to\infty} \mu_{\frac{\emptyset(2^t v)}{2^{2t}}}(\varepsilon) \qquad (3.6)$$

for all v ϵ V and all ε > 0.

Proof: Replace $(v_1, v_2, \dots\dots\dots, v_n)$ by (v,0,………,0) in (3.4), we have

$$\mu_{\emptyset(2v)-2^2\phi(v)}(\varepsilon) \geq \Phi_{v,0,\dots\dots\dots,0}(\varepsilon).$$

$$\mu_{\frac{\emptyset(2v)}{2^2}-\phi(v)}(\varepsilon) \geq \Phi_{v,0,\dots\dots\dots,0}(2^2\varepsilon). \qquad (3.7)$$

Replace v by 2^tv in (3.7) we have

$$\mu_{\frac{\emptyset(2^{t+1}v)}{2^2}-\phi(2^t v)}(\varepsilon) \geq \Phi_{2^t v,0,\dots\dots\dots,0}(2^2\varepsilon)$$

$$\mu_{\frac{\emptyset(2^{t+1}v)}{2^{2(t+1)}}-\frac{\phi(2^t v)}{2^{2t}}}(\varepsilon) \geq \Phi_{3^t v,0,\dots\dots\dots,0}(2^{2(t+1)}\varepsilon)$$

$$\geq \Phi_{v,0,\dots\dots\dots,0}\left(\frac{2^{2(t+1)}\varepsilon}{\alpha^t}\right). \qquad (3.8)$$

for all v ϵ V and all ε > 0. Since

$$\frac{\phi(2^m v)}{2^{2m}} - \phi(v) = \sum_{i=0}^{m-1} \frac{\phi(2^{i+1}v)}{2^{2(i+1)}} - \frac{\phi(2^i v)}{2^{2i}} \qquad (3.9)$$

From (3.8) and (3.9), we have

$$\mu_{\frac{\emptyset(2^m v)}{2^{2m}}-\phi(v)}\left(\sum_{i=0}^{m-1} \frac{\alpha^i}{2^{2(i+1)}}\frac{\varepsilon}{2}\right) \geq \Phi_{v,0,\dots\dots\dots,0}(\varepsilon).$$

$$\mu_{\frac{\emptyset(2^m v)}{2^{2m}}-\phi(v)}(\varepsilon) \geq \Phi_{v,0,\dots\dots\dots,0}\left(\frac{2\varepsilon}{\sum_{i=0}^{m-1}\frac{\alpha^i}{2^{2(i+1)}}}\right) \qquad (3.9)$$

Replace v by 2^2v , we got

$$\mu_{\frac{\emptyset(2^{m+n}v)}{2^{2(m+n)}}-\frac{\phi(2^n v)}{2^{2n}}}(\varepsilon) \geq \Phi_{v,0,\dots\dots\dots,0}\left(\frac{2\varepsilon}{\sum_{i=n}^{m+n-1}\frac{\alpha^i}{2^{2(i+1)}}}\right)$$

For all vϵV and all ε > 0. As $\Phi_{v,0,\dots\dots\dots,0}\left(\frac{2\varepsilon}{\sum_{i=n}^{m+n-1}\frac{\alpha^i}{2^{2(i+1)}}}\right) \to 1$ as $m, n \to \infty$ then $\left\{\frac{\phi(2^n v)}{2^{2n}}\right\}$ is a Cauchy sequence in (W,μ, T), Since (W,μ, T) is complete RN- space, thus sequence $\left\{\frac{\phi(2^n v)}{2^{2n}}\right\}$ converges to some Q_2(v) ϵW. fix v ϵV and put n=0 , we obtain

$$\mu_{\frac{\phi(2^m v)}{2^{2m}}-\phi(v)}(\varepsilon) \geq \Phi_{v,0,\ldots\ldots,0}\left(\frac{2\varepsilon}{\sum_{i=0}^{m-1}\frac{\alpha^i}{2^{2(i+1)}}}\right) \tag{3.10}$$

And so , for every δ>0, we get

$$\mu_{Q_2(v)-\phi(v)}(\varepsilon+\delta) \geq T\left(\mu_{Q_2(v)-\frac{\phi(2^m v)}{2^{2m}}}(\delta), \mu_{\frac{\phi(2^m v)}{2^{2m}}-\phi(v)}(\varepsilon)\right)$$

$$\geq T\left(\mu_{Q_2(v)-\frac{\phi(2^m v)}{2^{2m}}}(\delta), \Phi_{v,0,\ldots\ldots,0}\left(\frac{2\varepsilon}{\sum_{i=0}^{m-1}\frac{\alpha^i}{2^{2(i+1)}}}\right)\right)$$

For all vϵV and all ε, δ >0.Taking the limit m→∞,

$$\mu_{Q_2(v)-\phi(v)}(\varepsilon+\delta) \geq \Phi_{v,0,\ldots\ldots,0}(2(2^2-\alpha)\varepsilon) \tag{3.11}$$

For all vϵV and all ε, δ >0. Since δ was arbitrary, taking δ→0,

$$\mu_{Q_2(v)-\phi(v)}(\varepsilon) \geq \Phi_{v,0,\ldots\ldots,0}(2(2^2-\alpha)\varepsilon) \tag{3.12}$$

For all vϵV and all ε >0. Replacing $(v_1, v_2, \ldots\ldots\ldots, v_n)$ by $(2^t v_1, 2^t v_2, \ldots\ldots\ldots, 2^t v_n)$ in (3.4),

$$\mu_{DQ_2(2^t v_1, 2^t v_2, \ldots\ldots, 2^t v_n)}(\varepsilon) \geq \Phi_{2^t v_1, 2^t v_2, \ldots\ldots, 2^t v_n}(2^{2t}\varepsilon). \tag{3.13}$$

Since $\lim_{t\to\infty}\Phi_{2^t v_1, 2^t v_2, \ldots\ldots, 2^t v_n}(2^{2t}\varepsilon)$=1, so Q_2 satisfies the functional equation (1.1). To prove the uniqueness of Quadratic mapping Q_2. Assume that there exists another Quadratic mapping Q'_2, which satisfies inequality (3.11). Fix v ϵV. Clearly, $Q_2(2^t v)=2^{2t}Q_2(v)$ and $Q'_2(2^t v)=2^{2t}Q'_2(v)$ for all vϵV. from (3.10), we have

$$\mu_{Q_2(v)-Q'_2(v)}(\varepsilon) = \mu_{\frac{Q_2(2^t v)}{2^{2t}}-\frac{Q'_2(2^t v)}{2^{2t}}}(\varepsilon)$$

$$\geq T\left(\mu_{\frac{Q_2(2^t v)}{2^{2t}}-\frac{\phi(2^t v)}{2^{2t}}}\left(\frac{\varepsilon}{2}\right), \mu_{\frac{Q'_2(2^t v)}{2^{2t}}-\frac{\phi(2^t v)}{2^{2t}}}\left(\frac{\varepsilon}{2}\right)\right)$$

$$\geq \Phi_{2^t v,0,\ldots\ldots,0}\left(\left(\frac{2^2}{\alpha}\right)^t (2^2-\alpha)4\varepsilon\right) \tag{3.14}$$

$\lim_{t\to\infty} 4(2^2-\alpha)\left(\frac{3}{\alpha}\right)^t \varepsilon = \infty$, we have $\mu_{Q(v)-Q'(v)}(\varepsilon) = 1$ for all $\varepsilon > 0$.Thus, $Q_2(v) = Q_2'(v)$, for all $v \in V$. Hence, the proof is complete.

Theorem 3.2. If an even mapping ϕ: V→W with $\phi(0)=0$ for which there exists a mapping $\Phi: V^n \to D^+$ for some $0<\alpha<4$,

$$\Phi_{\frac{v_1}{2},\frac{v_2}{2},\ldots\ldots\ldots,\frac{v_n}{2}}(\varepsilon) \geq \Phi_{v_1,v_2,\ldots\ldots\ldots,v_n}(\alpha\varepsilon) \quad (3.15)$$

And $\lim_{t\to\infty} \Phi_{\frac{v_1}{2^t},\frac{v_2}{2^t},\ldots\ldots\ldots,\frac{v_n}{2^t}}(\frac{\varepsilon}{2^{2t}})=1$ (3.16)

for all $v_1, v_2,\ldots\ldots\ldots,v_n \epsilon V$ and all $\varepsilon > 0$ such that

$$\mu_{D\phi(v_1,v_2,\ldots\ldots\ldots,v_n)}(\varepsilon) \geq \Phi_{v_1,v_2,\ldots\ldots\ldots,v_n}(\varepsilon). \quad (3.17)$$

Then there exists a unique quartic mapping Q_2: V→W satisfying the functional equation (1.1) with $\mu_{Q_2(v)-\phi(v)}(\varepsilon) \geq \Phi_{v,0,\ldots\ldots\ldots,0}(\frac{\varepsilon}{(2^2-\alpha)2})$ (3.18)

for all vϵV and all ε>0. The mapping Q_2: V→W is defined by

$$\mu_{Q_2(v)}(\varepsilon) = \lim_{t\to\infty} \mu_{2^{2t}\phi(\frac{v}{2^t})}(\varepsilon) \quad (3.19)$$

for all vϵV and all ε>0.

Corolarry 3.3. If an even mapping $\emptyset$: V→W with $\emptyset(0)=0$ for which there exists a mapping $\Phi: V \to D^+$ satisfying

$$\mu_{D\emptyset(v_1,v_2,\ldots\ldots\ldots,v_n)}(\varepsilon) \geq \Phi_{\sum_{i=1}^{n}||v_i||^{\theta}}(\varepsilon). \quad (3.20)$$

Then there exists a unique quadratic mapping Q_2: V→W satisfying the functional equation (1.1) with $\mu_{Q_2(v)-\emptyset(v)}(\varepsilon) \geq \Phi_{||v||^{\theta}}((2^2-2^p)2\varepsilon)$ (3.21)

for all vϵV, where p<2 and all ε>0

We take $\alpha=2^{p-2}$ and $v_1, v_2, \ldots\ldots\ldots, v_n=\sum_{i=1}^{n}||v_i||^{\theta}$ in the theorem 3.1.

4.Conclusion. In this paper a new type quadratic functional equation in introduced and proved its stability in random norm space.

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