Original Research Article

Null Controllability of Nonlinear Mixed Integrodifferential Systems

Abstract

The present paper investigates the existence of mixed solutions and provides sufficient conditions for null controllability of nonlinear mixed integrodifferential systems with unbounded linear operators in Banach space. The results are obtained using semi group of linear operators, fractional powers of operators and the schauder fixed point theorem.

Keywords: Controllability, Nonlinear systems, Mixed integrodifferential systems.

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1. Introduction

Controllability of linear and nonlinear systems of ordinary differential space has been exclusively studied. Several authors have extended the concept to infinite dimensional systems represented by the evolution equations with bounded operators in Banach spaces [1-3]. Recently Alhazmi and Abdou [5] established sufficient conditions for the local null controllability of nonlinear functional differential systems using the Schauder fixed point theorem with unbounded operators in Banach space. The essence of this work is to extend the use of fixed point theorems to integrodifferential systems with unbounded operators and infinite delay in Banach space [4,6,7]. In particular, results are obtained for the null controllability of nonlinear mixed integrodifferential systems with unbounded linear operators in Banach space.

2. Preliminaries

Consider the following nonlinear mixed Voltera – Fredholm integrodifferential equation of the form

$$X(t) = A(t)\, x(\phi) + \int_{-\infty}^{t} f(t,s)x(s)ds + (Bu)(t) + \int_{-\infty}^{t} k(t,s)x(s)ds; t \in J = [0,T] \qquad (1)$$

$$x(\phi) = \phi(t) \qquad t \in (-\infty, 0]$$

Here $\{A(t): t \geq 0\}$ is a family of unbounded linear operators mapping a Banach space X_c to itself. The state $x(t)$ takes the values in the Banach space X and the control function u is given in $L^2\ (J.U)$, a Banach space of admissible control with U as a Banach space. B is a bounded linear operator from U into X. Let X_α denotes the interpolation space defined in the α power of $A(0)$, that is

$$X_\alpha = \{x: x \in D(A^\alpha(0))\},$$

With $\|x\|_\alpha = \|A^\alpha(0)x\|$. The space C_α is the space of bounded uniformly continuous functions from $(-\infty, 0]$ to X_p endowed with the supremum norm.

$\|\phi\| C_\alpha = \sup\{\|\phi(0)\|_\alpha : \theta \in (-\infty, 0]\}$

Further let $\phi \in C_\alpha$ for some $\propto, 0 < \alpha < 1$ and f be a continuous nonlinear operator on $J \times J \times X_\alpha$ into X.

For the existence of a solution of (1), we need the following assumptions (see [6]).

(i) The domain $D(A)$ of $A(t), t \in]$ is dense in the Banach space X and independent of t.

(ii) For each $t \in (-\infty, 0]$, the resolvent $R(\lambda, A(t))$ exists for all λ such that $Re\lambda \geq 0$ and there exists $c > 0$ such that $\|R(\lambda, A(t)\| \leq \frac{c}{(|\lambda|+1)}$

(iii) for any $t, s, \tau \in J$, there exists a $0 < \delta < 1$ and $k > 0$ such that

$$\|(A(t) - A(\tau))A^{-1}(s)\| \leq K|t - \tau|^\delta$$

Conditions (i) and (ii) imply that for each $r \geq 0, -A(r)$ is the infinitesimal generator of an analytic semi group $\{e^{-tA(r)} : t \geq 0\}$.

The fact that $0 \in p\big(A(r)\big)$ and that $-A(r)$ generates an analytic semigroup implies that fractional powers of $A(r)$ can be defined for $0 < \alpha < 1$. We set

$$A^{-\alpha}(r) = \frac{1}{\Gamma(\alpha)} \int_0^\infty S^{\propto -1} e^{-SA(r)} ds$$

Where $\Gamma(.)$ denotes the Euclidean gamma function. The operator $A^{-\alpha}(r)$ can be shown to be a bounded linear operator with a well-defined inverse [7]. Let $A^\alpha(r) = A^{-\alpha}(r)^{-1}$. If conditions (i) – (iii) are satisfied, then there exists an operator valued function $X(t, \tau)$ which is defined on the triangle $0 \leq \tau \leq t < \infty$ and has values in $B(u), [8, P_{100}]$. $X(t, \tau)$ is strongly continuous jointly in t and in Y and maps X into $D(A)$ if $t > \tau$. The family $\{X(t, \tau) : 0 \leq \tau \leq t < \infty\}$ satisfy the identity

$X(t, \tau) = X(t, s) \times (s, \tau)$, for $0 \leq \tau \leq S \leq t < \infty$

The derivative $\frac{\delta \times (t, T)}{\delta t}$ exists in the strong operator topology and belongs to X whenever$0 \leq \tau < t$, $[9, P_{129}]$. Finally $X(t, \tau)$ satisfies the following initial value problem.

$$\frac{\delta X(t, \tau)}{\delta t} = A(t) \times (t, \tau) \; for \; t > r$$

$X(t, \tau) = I$ (I = identity operator)

Further, if $0 \leq v \leq 1$, $0 \leq \beta \leq \delta < 1 + \mu$, then for any $0 \leq \tau \leq t < t + \Delta t < T$ and $S \in \tau$, there exists a $K(B, v, \delta)$ such that;

$$\left\|A^r(s)[X(t + \Delta t, \tau) - X(t, \tau)]A^{-\beta}(\tau)\right\| \leq K(\beta, v, S)(\Delta t)^{\delta - r}|t - \tau|^{\beta - \delta} \quad (2)$$

(iv) There exist a $\beta_0 > \alpha$ and $\omega > 0$ such that for all$0 \leq \beta < \beta_0$, there exists $K\beta > 0$ satisfying for all $0\left\|A^\beta(0) \times (t, s)\right\| \leq K\beta(t - s)^{-\beta} e^{-\omega(t-s)}$ (3)

(v) The function $f : J \times J \times X_\alpha \to X$ is continuous, $f(t, s, 0) = 0$ for all $s \leq t$ and there exists an $L > 0$ and $v = 0$ such that

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$$\|f(t,s,x) - f(t,s,y)\| \leq e^{-v(t-s)}L\|x-y\|_\alpha \quad (4)$$

(vi) The bounded linear operator $A^{-\alpha}(t)$ is compact for all $\alpha \in (0,1]$

Note that (v) is satisfied by the function;

$$f(t,s,x) = \begin{cases} 0, & for\ s \leq t \\ ce^{-\alpha(t-s\_} & \end{cases} arctanx, \alpha > 0\ and\ c > 0$$

Suppose that (i) – (iii) and (v) are satisfied, if $\phi \in c_\alpha\ and\ \ \phi(0) \in D(A^\beta(0))$for some $\beta > 0$ and (iv) is satisfied for some $\beta_0 > \beta$, then there exists a unique function $x(\phi)(.) : J \to X$ such that

$$x(\phi) = X(t,0)\phi(0) + \int_0^t X\,(t,s) \int_{-\infty}^s f\big(s,\tau,\delta_{\tau(\phi)}\big) d\tau ds + \int_0^t X(t,s)Bu(s)ds + \int_0^t X(t,s)K\big(t,s,x(s)\big)ds \quad \to \quad (5a)$$

$$x(\phi) = \phi(t), t \in (-\infty, v] \quad \to \quad (5b)$$

Moreover, $x(\phi)$ is continuously differentiable for $t > 0$ and satisfies (1)

Definition: The system (1) is said to be NULL controllable on the interval J if for every continuous initial function $\phi \in c_\alpha$, there is a control $u \in L^2(J,u)$ such that the solution $x(\phi)$ of (λ) satisfies $x(\phi) = 0$

3. Main Results

Theorem 3.1 : Suppose conditions $(i) - (vi)$ hold and the linear operator W from u to X defined by $Wu = \int_0^T X(T,s)Bu(s)\,ds$ has an invertible operator W^{-1} defined on $L^2(J,u)/K_{\alpha N}$ if there exists positive constants N_1, N_2 such that $\|B\| \leq N_1$ and $\|W^{-1}\| \leq N_2$. Then the system (1) is Null controllable on J.

Proof: Define the control

$$u(t) = -W^{-1}[X(T,0)\phi(0) + \int_0^T X(T,s)) \int_{-\infty}^0 f\big(s,\tau,x(\phi)\big)d\tau ds](t) + \int_{-\infty}^t \overset{K}{(t,s)}x(s)ds \quad \to \quad (6)$$

Now it is shown that, when using this control, the operator defined by

$\Phi x(\phi) = \phi(t)$ for $t \in (-\infty, 0]$

$$\Phi x(\phi) = X(t,0)\phi(0) - \int_0^t X(t,u)B\,W^{-1}[X(T,0)\phi(0) + \int_0^T X)T,s) \int_{-\infty}^0 f\big(s,\tau,x(\phi)\big)d\tau ds](\mu)d\mu + \int_0^t X(t,s) \int_{-\infty}^0 f(s,\tau,x(\phi)d\tau ds + \int_{-\infty}^t K(t,s)x(s)ds \quad \to \quad (7)$$

has a fixed point.

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This fixed point is a solution of equation (1) clearly $\Phi x(\emptyset) = 0$ which means that the control u steers the nonlinear mixed integrodifferential system from the initial function $\emptyset$ to 0 in time T provided that the nonlinear operator Φ has a fixed point.

Define,

$S = \{x \in c((-\infty, T]: X_0(0): \|x(0)\| = \|\emptyset(t)\|$ on $t \in (-\infty, o]$ and $\|x(\emptyset)\| \leq N, t \in J\}$

Consider the transformation

$\Phi: s \rightarrow c(-\infty, T]: X_{\alpha(0)})$ defined by (7)

It is claimed that $\Phi: s \rightarrow s$ for that

$$\|\Phi x(0)\|_\infty \leq \|\Phi(0)\| K_\alpha t^{-\infty} e^{-\omega t} + K_\alpha L \int_0^t (t-s)^{-\alpha} e^{-\omega(t-s)} \int_{-\infty}^0 e^{-\infty(s-\tau} \|x(\phi)\|_\alpha d\tau ds + K_\alpha \int_0^t (t-\mu)^{-\alpha} e^{-\omega(t-\mu)} N_1 N_2 \{\|\phi(0)\| K_\alpha t^{-\alpha} e^{-\omega t} + K_\alpha L \int_0^T (T-s)^{-\alpha} e^{-\omega(T-s)} \int_{-\infty}^0 e^{-v(s-\tau)} \|x(\emptyset)\|_\alpha d\tau ds\}(\mu) d\mu$$

We choose $\delta > 0$ and observe that

$$e^{-\delta t}\|\Phi x(0)\|_\infty \leq K_1 K_2 L \int_0^t (t-s) e^{-(\omega+\delta)(t-s)} \int_{-\infty}^0 e^{(v+\delta)(s-\tau)} e^{-\delta\tau} \|x(\phi)\|_\alpha \delta t ds + K_\alpha N_1 N_2 \int_0^t (t-\mu)^{-\alpha} e^{-\omega(t-\mu)} \Big\{K_1 + K_2 L \int_0^T (T-s)^{-\alpha} e^{-(\omega+\delta)(T-s)} \int_{-\infty}^0 e^{-(v+\delta)(s-\tau)} e^{-\delta t} \|x(\emptyset)\|_\alpha d\tau ds\Big\}(\mu) d\mu$$

for $K_2 > \sup\{K_1, \|\emptyset\| c_\alpha\}$ and for $y(t) = e^{-\delta t}\|x(0)\|_\alpha \|\Phi y(t)\|_\alpha \leq K_2 + K_\alpha L \int_0^t (t-s)^{-\alpha} e^{(\omega+\delta)(t-s)} \int_{-\infty}^0 e^{-(v+\delta)(s-\tau)} y(\tau) d\tau ds\} (\mu) d\mu$

if $K_3 > 0$, introduce a function $z(.)$by $z(s) = \begin{cases} K_{3,} & if\ s \in [0, T_0] \\ K_3\ e^{-\delta s} & s < 0 \end{cases}$

and note that for $t \geq 0$

$$K_2 + K_\alpha L \int_0^t (t-s)^{-\alpha} e^{-(\omega+\delta)(t-s)} \int_{-\infty}^0 e^{-(\omega+\delta)(s-\tau)} y(\tau) d\tau ds + N_1 N_2 K_\alpha \int_0^T (t-\mu)^{-\alpha} e^{-\omega(t-\mu)} \{K_2 + K_0\ L \int_0^T (T-s)^{-\alpha} e^{-(\omega+\delta)(T-s)} \int_{-\infty}^0 e^{-(v+\delta)(s-\tau)} y(\tau)\, d\tau ds\}(\mu) d\mu \leq K_2 + K_\alpha L \Gamma(1-\alpha) \left(\frac{1}{v}\right) (\omega+\delta)^{\alpha-1} K_3 + N_1 N_2 K_\alpha \int_0^T (t-\mu)^{\alpha} e^{-\omega(t-\mu)} \{K_2 + K_\alpha L L \Gamma(1-\alpha) \left(\frac{1}{v}\right) (\omega+\delta)^{\alpha-1} (\mu) d\mu$$

Thus if $\delta > 0$ is chosen large enough to ensure that $K_\alpha L \Gamma(1-\alpha) \left(\frac{1}{v}\right) (\omega+\delta)^{\alpha-1} < 1$ and

K_3 is such $K_3 > K_2 [1 - 3K_0 L\, \Gamma(1-\alpha) \left(\frac{1}{v}\right) (\omega+\delta)^{\alpha-1}]^{-1}$

Then for suitable N_1, N_2, it follows that

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$Sup\|\Phi x(0)\| \leq K_3 e^{\delta t} = N, s \in [0, T_\alpha],$

Hence the result follows by applying Schauder's fixed point theorem to the mapping.

4. Application

Consider the partial nonlinear integrodifferential equations

$$\frac{\partial u}{\partial t} + A(t,u,D)_x = \int_{-\infty}^{t} g(t,s) f(\nabla u) ds + B(v)(t)$$

$u(x,t) = 0 \ (x,t) \in \partial\Omega \times \mathrm{R} \quad \rightarrow$ (8)

Where $A(t,u,D) = \sum_{|x| \leq 2} a_n(x,t) D^n$

The operators $A(t,x,D)$ are assumed to be uniformly elliptic. Thus there exists a constant $C_0 > 0$ such that $-Re \sum an(x,t)\zeta^n \geq C_0|\zeta|^2$

Where Ω is a bounded domain in R^3 with smooth boundary. The coefficients $an(x.t)$ are smooth functions of $(x.t) \in \Omega \times R^T$ and there exists constants $C_3 > 0$ and $\mu \in (0,1)$ such that

$|an(x,t) - an(x,\tau| \leq C_2|t-\tau|^\mu$ for $t_1\tau > 0, x \in \Omega$

Finally, stipulate that the coefficients $an(.): \overline{\Omega} \rightarrow R$ are such that

$\lim_{C \rightarrow 0} sup\ |an(x,t) - an(x)| = 0$

The nonlinearity $f(.)$: $R^3 \rightarrow R$ vanishes at zero and has the property that there exists a $C_3 > 0$ satisfying

$$|f(u) - f(v)| \leq C_3 \sum_{i=1}^{3} |u_i - v_i|$$

The function $g(.)$:$R^+ \rightarrow R$ is Lipschitz continuous. Moreover, there exists $C_4 e^{-vs}$,

for $s \in (-\infty, 0]$

Let $X = L^2(\Omega)$. A family of operators $\{A(t): t \geq 0\}$ is introduced on $L^2(\Omega)$ by specifying

$D = D(A(t) = H^2(\Omega) \cap H_0^1(\Omega)$ and letting $A(t)u = A(t,x,D)\ u$ for $u \in D$. If $\alpha > 3/4$ define

g: $R^+ \times R \times X_\alpha \rightarrow X$ by setting $g(t,s,u) = g(t,-s)\ f(\nabla \mathrm{u})$

X_α will denote the interpolation space obtained from $A^\alpha(0)$. Suppose $\emptyset \in C_2$ and identity $u(x,t) = x(\emptyset)(x)$, then (8) assumes the form of the function space integrodifferential equation

$x(\emptyset) + Ax(\emptyset) = \int_{-\infty}^{t} f(t,s,x(\emptyset)ds + Bu(t) \quad \rightarrow$ (9)

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If $\emptyset \in C(\alpha > {}^{3}/_{4})$then from [6] there exists a unique function$x(\emptyset): B \rightarrow X$ which satisfies equation (9) and $\|x(\emptyset) - x(\Psi)\|_{0,\infty} \leq C\|x(\emptyset) - x(\Psi)\|_{\alpha}$ for some $C > 0$

This yields the solution and the system is controllable in J.

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