SOME NEW SERIES OF INEQUALITIES PREMISED ON VERMA MEASURES OF INFORMATION

Abstract

In this communication we derive some inequalities on the basis of Verma [8, 9] measures of information. These inequalities are useful in the study of channel capacity in wired and wireless communication system in the presence of noise and solving many problems related to information sciences.

Key words: Inequality, Measures of Information, Beta Distribution, Binomial Distribution, Truncated Binomial Distribution, Truncated Normal Distribution, Geometric Distribution, Exponential Distribution.

1. INTRODUCTION:

In 1972, J. P. Burg [1] gave his well-known measure as

$$V_0(P) = \sum_{i=1}^{n} \ln p_i \tag{1.1}$$

to measure its uncertainty or entropy for any given discrete-variate probability distribution $P = (p_1, p_2, \dots \dots \dots, p_n)$ and later in 2012, Verma [8, 9] measure of uncertainty or entropy for the same probability distribution P is given by

$$V_a(P) = -\sum_{i=1}^{n} \ln(1 + ap_i) + \sum_{i=1}^{n} \ln p_i + \ln(1 + a), \quad a > 0. \tag{1.2}$$

It can be easily verified that $V_a(P) \to V_0(P)$. Using this concept we can develop a number of inequalities for this measure [8, 9] by the help of various theoretical probability distributions in both cases discrete as well as continuous.

Since 1997, Kapur [4] gave a number of inequalities for various measures of entropy in the case of discrete as well as continuous. He used the concept that if $P = (p_1, p_2, \dots \dots \dots, p_n)$ be a probability distribution and $H_n(p_1, p_2, \dots \dots \dots, p_n)$ be any measure of entropy, then since entropy is always maximum for the uniform distribution, then we get the inequality

$$H_n(p_1, p_2, \dots \dots \dots, p_n) \le H_n\left(\frac{1}{n}, \frac{1}{n}, \dots \dots \dots, \frac{1}{n}\right). \tag{1.3}$$

In particular by using measures of entropy due to Shannon [7], Havrda and Charvat [2], Renyi [6], Kapur [3] we get the inequalities

$$-\sum_{i=1}^{n} p_i \ln p_i \le \ln n \tag{1.4}$$

$$\frac{1}{1-\alpha}\left(\sum_{i=1}^{n} p_i^{\alpha} - 1\right) \le \frac{1}{1-\alpha}\left(n^{1-\alpha} - 1\right), \qquad \alpha \ne 1, \ \ \alpha > 0 \tag{1.5}$$

$$\frac{1}{1-\alpha}\left(\ln \sum_{i=1}^{n} p_i^{\alpha}\right) \le \ln n, \qquad \alpha \ne 1, \ \ \alpha > 0 \tag{1.6}$$

$$-\sum_{i=1}^{n} p_i \ln p_i + \frac{1}{a}(1 + ap_i)\ln(1 + ap_i) \le \ln n + \frac{1}{a}(n + a)\ln\left(1 + \frac{a}{n}\right), \qquad a \ge -1 \tag{1.7}$$

Here $P = (p_1, p_2, \dots \dots \dots, p_n)$ may be any positive number whose sum is unity. Alternatively, we may take any n arbitrary positive numbers $a_1, a_2, \dots \dots \dots, a_n$ and replace in each of the inequalities (1.4) to (1.7) p_i by $\frac{a_i}{\sum_{i=1}^{n} a_i}$ to get inequalities holding between any n positive numbers.

Similarly, let $f(x)$ be the density function for a continuous random variable defined over the interval $[a, b]$ and let a measure of entropy for the distribution be given by

$$H(f) = \int_a^b \emptyset\big(f(x)\big)\, dx \tag{1.8}$$

then since the maximum entropy occurs when the distribution is uniform, we get the inequality

$$\int_a^b \emptyset\big(f(x)\big)\, dx \le (b - a)\emptyset\left(\frac{1}{b-a}\right) \tag{1.9}$$

In particular, by using the continuous variate versions of the entropy measures used in previous case, we get the inequalities

$$-\int_a^b f(x) \ln f(x) dx \le \ln(b - a) \tag{1.10}$$

$$\frac{1}{1-\alpha}\int_a^b f^{\alpha}(x)\, dx \le \frac{1}{1-\alpha}(b - a)^{1-\alpha}, \qquad \alpha \ne 1, \ \ \alpha > 0 \tag{1.11}$$

$$\frac{1}{1-\alpha}\ln \int_a^b f^{\alpha}(x) dx \le \ln(b - a), \qquad \alpha \ne 1, \ \ \alpha > 0 \tag{1.12}$$

$$-\int_a^b f(x)\ln f(x)dx + \frac{1}{c}\int_a^b\big(1+cf(x)\big)\ln\big(1+cf(x)\big)dx \le$$

$$\ln(b-a) + \frac{1}{c}\big((b-a)+c\big)\ln\frac{(b-a)+c}{b-a} \tag{1.13}$$

Since we can have an infinity of density functions $f(x)$, each of these gives us an infinity of inequalities. The similar argument we can use for directed-divergence.

This paper highlights the inequalities on the basis of the new parametric measures of entropy $i.e.$ Verma [8, 9, 10] measures of entropy under the pre-described limiting condition $i.e.$

$$V_a(P) \to V_0(P) \tag{1.14}$$

It is obvious that we can write any number of inequalities. Some of these may not be interesting or useful, but some of these can be useful. Moreover it may be difficult to prove these without using the measure of entropy. These inequalities are illustrate in the following section 2 by taking various theoretical probability distributions.

2. NEW RESULTS

1. USE OF BETA DISTRIBUTION

Theorem: 1.1 If m and n are the positive integers, which are causally related to each other, then show that

$$\beta(m,n) \ge e^{-(m+n-2)}, \qquad m,n > 0. \tag{1.1.1}$$

Proof: The probability density function for the beta distribution is given by,

$$f(x) = \frac{1}{\beta(m,n)}x^{m-1}(1-x)^{n-1}. \tag{1.1.2}$$

Now, in continuous variate case

$$V_{a=0}(P) = \int_0^1 \ln f(x)dx \tag{1.1.3}$$

so that, $$V_{a=0}(P) = \int_0^1 \ln\left[\frac{1}{\beta(m,n)}x^{m-1}(1-x)^{n-1}\right]dx \tag{1.1.4}$$

$$= \int_0^1[-\ln\beta(m,n) + (m-1)\ln x + (n-1)\ln(1-x)]dx \tag{1.1.5}$$

$$= -\ln \beta(m,n) + (m-1)(-1) + (n-1)(-1) \leq 0 \tag{1.1.6}$$

$$= \ln \beta(m,n) + (m+n-2) \geq 0 \tag{1.1.7}$$

or, $\beta(m,n) \geq e^{-(m+n-2)}, \qquad m,n > 0.$

2. USE OF BINOMIAL DISTRIBUTION

Theorem: 2.1 If r and n are causally related to each other, then show that

$$\sum_{r=0}^{n} \ln p^r q^{n-r} \leq n(n+1) \ln \frac{1}{2}$$

and, in the case of uniform distribution

$$\sum_{r=0}^{n} \ln n_{c_r} \leq (n+1)(n \ln 2 - \ln(n+1)).$$

Proof: Since, for the binomial distribution

$$P_r = n_{c_r} p^r q^{n-r} . \tag{2.1.1}$$

So that, $$\sum_{r=0}^{n} \ln P_r = \sum_{r=0}^{n} \ln n_{c_r} p^r q^{n-r} \tag{2.1.2}$$

$$= \sum_{r=0}^{n} \ln n_{c_r} + \ln p \sum_{r=0}^{n} r + \ln q \sum_{r=0}^{n} (n-r) \tag{2.1.3}$$

$$= \sum_{r=0}^{n} \ln n_{c_r} + \ln p . \frac{n(n+1)}{2} + \ln q . \frac{n(n+1)}{2} \tag{2.1.4}$$

$$= \sum_{r=0}^{n} \ln n_{c_r} + \frac{n(n+1)}{2} \ln pq \tag{2.1.5}$$

This is maximum when $p = q = \frac{1}{2}$, so that

$$\sum_{r=0}^{n} \ln p^r q^{n-r} \leq n(n+1) \ln \frac{1}{2}.$$

which is otherwise obvious.

Again, in the case of uniform distribution, the maximum entropy of binomial distribution for this measure will be $\leq -(n+1)\ln(n+1)$, so that

$$\sum_{r=0}^{n} \ln n_{c_r} + \frac{n(n+1)}{2} \ln pq \leq -(n+1)\ln(n+1) \tag{2.1.6}$$

so that, $\sum_{r=0}^{n} \ln n_{c_r} \leq (n+1)(n \ln 2 - \ln(n+1))$

3. USE OF TRUNCATED BINOMIAL DISTRIBUTION

Theorem: 3.1 If r and n are causally related to each other, then prove that

$$\sum_{r=1}^{n-1} \ln \frac{n_{c_r} p^r q^{n-r}}{1-p^n-q^n} \leq \sum_{r=1}^{n-1} \ln \frac{n_{c_r}(1/2)^n}{1-2(1/2)^n}$$

and, $$\frac{(1-2(1/2)^n)^{2/n}}{[1-p^n-(1-p)^n]^{2/n}} \leq \frac{1/4}{p(1-p)}.$$

Proof: Now, consider the truncated binomial distribution

$$P_r = \frac{n_{c_r} p^r q^{n-r}}{1-p^n-q^n}, \qquad r = 1,2, \dots\dots\dots, n-1. \tag{3.1.1}$$

So that,

$$\sum_{r=1}^{n-1} \ln P_r = \sum_{r=1}^{n-1} \ln \frac{n_{c_r} p^r q^{n-r}}{1-p^n-q^n} \tag{3.1.2}$$

$$= \sum_{r=1}^{n-1} \ln n_{c_r} p^r q^{n-r} - \sum_{r=1}^{n-1} \ln(1-p^n-q^n) \tag{3.1.3}$$

$$= \sum_{r=1}^{n-1} \ln n_{c_r} + \sum_{r=1}^{n-1} \ln p^r + \sum_{r=1}^{n-1} \ln q^{n-r} -$$

$$\sum_{r=1}^{n-1} \ln(1-p^n-q^n). \tag{3.1.4}$$

Let, $$F(P) = \text{Constant} + \ln p \sum_{r=1}^{n-1} r + \ln q \sum_{r=1}^{n-1} (n-r) -$$

$$\sum_{r=1}^{n-1} \ln(1-p^n-q^n) \tag{3.1.5}$$

$$= \text{Constant} + \ln p . \frac{n(n-1)}{2} + \ln q . \frac{n(n-1)}{2} - \sum_{r=1}^{n-1} \ln(1-p^n-q^n) \tag{3.1.6}$$

Thus,

$$F(P) = \text{Constant} + \frac{n(n-1)}{2}[\ln p + \ln(1-p)] -$$

$$(n-1)\ln(1-p^n-(1-p)^n) \tag{3.1.7}$$

so that,

$$F'(P) = \frac{n(n-1)}{2}\left[\frac{1}{p} - \frac{1}{1-p}\right] -$$

$$(n-1).\frac{1}{1-p^n-(1-p)^n}\left[-np^{n-1} + n(1-p)^{n-1}\right] \quad (3.1.8)$$

$$= 0 \text{ when } p = q = \frac{1}{2} \quad (3.1.9)$$

and,

$$F''(P) = \frac{n(n-1)}{2}\left[-\frac{1}{p^2} - \frac{1}{(1-p)^2}\right] +$$

$$n(n-1)\left[\frac{(n-1)p^{n-2}+(n-1)(1-p)^{n-2}}{1-p^n-(1-p)^n} - \frac{p^{n-1}-(1-p)^{n-1}}{(1-p^n-(1-p)^n)^2}.\left(-np^{n-1} + n(1-p)^{n-1}\right)\right]$$

(3.10)

It is easily shown that $F''(P) < 0$ when $p = q = \frac{1}{2}$, then this entropy is maximum when $p = q = \frac{1}{2}$.

Hence, this entropy is maximum when $p = q = \frac{1}{2}\, i.e.$

$$\sum_{r=1}^{n-1} \ln \frac{n_{c_r} p^r q^{n-r}}{1-p^n-q^n} \leq \sum_{r=1}^{n-1} \ln \frac{n_{c_r}(1/2)^n}{1-2(1/2)^n}$$

and, $$\frac{n}{2}\ln p(1-p) - \ln(1-p^n-(1-p)^n) \leq \frac{n}{2}\ln\frac{1}{4} - \ln\left[1-2\left(\frac{1}{2}\right)^n\right]$$

or, $$\frac{(1-2(1/2)^n)^{2/n}}{[1-p^n-(1-p)^n]^{2/n}} \leq \frac{1/4}{p(1-p)}.$$

4. USE OF TRUNCATED NORMAL DISTRIBUTION

Theorem: 4.1 If a and x are causally related to each other, then show that

$$\int_0^a e^{-\frac{1}{2}x^2} dx \leq a e^{-\frac{a^2}{6}}.$$

Proof: The probability density function, for the truncated normal distribution, is given by

$$f(x) = \frac{e^{-\frac{1}{2}x^2}}{\int_0^a e^{-\frac{1}{2}x^2}dx}, \qquad 0 \le x \le a.$$

so that, $$\int_0^a \ln f(x)dx = \int_0^a \ln\left(\frac{e^{-\frac{1}{2}x^2}}{\int_0^a e^{-\frac{1}{2}x^2}dx}\right)dx \qquad (4.1.1)$$

$$= \int_0^a \ln e^{-\frac{1}{2}x^2}dx - \int_0^a \ln\left(\int_0^a e^{-\frac{1}{2}x^2}dx\right)dx \qquad (4.1.2)$$

On taking $\emptyset(a) = \int_0^a e^{-\frac{1}{2}x^2}dx$ we get,

$$\int_0^a \left(-\frac{1}{2}x^2 - \ln \emptyset(a)\right)dx \le -a \ln a \qquad (4.1.3)$$

or, $$\frac{a^3}{6} + a \ln \emptyset(a) \le a \ln a \qquad (4.1.4)$$

or, $$\int_0^a e^{-\frac{1}{2}x^2}dx \le ae^{-\frac{a^2}{6}}$$

5. USE OF GEOMETRIC DISTRIBUTION

Theorem: 5.1 If r and N are causally related to each other, then show that

$$\ln\frac{\rho^{N/2}}{1+\rho+\rho^2 \ldots +\rho^N} \le \ln\frac{1}{N+1}.$$

Proof: In Geometric distribution,

$$P_r = \frac{(1-\rho)\rho^r}{1-\rho^{N+1}}, \qquad r = 0,1,2,\ldots,N. \qquad (5.1.1)$$

So that, $$\sum_{r=0}^{N} \ln P_r = \sum_{r=0}^{N} \ln\left(\frac{(1-\rho)\rho^r}{1-\rho^{N+1}}\right) \qquad (5.1.2)$$

$$= \sum_{r=0}^{N} \ln(1-\rho) + \sum_{r=0}^{N} \ln \rho^r - \sum_{r=0}^{N} \ln(1-\rho^{N+1}) \qquad (5.1.3)$$

$$= (N+1)\ln(1-\rho) + \ln \rho.\frac{N(N+1)}{2} - (N+1)\ln(1-\rho^{N+1}) \qquad (5,1.4)$$

$$= (N+1).\left[\ln(1-\rho) + \frac{N}{2}\ln\rho - \ln(1-\rho^{N+1})\right] \quad (5.1.5)$$

or, $$(N+1)\ln\left(\frac{(1-\rho)\rho^{N/2}}{(1-\rho^{N+1})}\right) \leq -(N+1)\ln(N+1) \quad (5.1.6)$$

or, $$\ln\frac{\rho^{N/2}}{1+\rho+\rho^{2}\ldots+\rho^{N}} \leq \ln\frac{1}{N+1}.$$

The maximum value of L.H.S. occur when $\rho = 1$.

6. USE OF EXPONENTIAL DISTRIBUTION

Theorem: 6.1 If a and b are causally related to each other, then show that

$$ab \leq 2\sin h\, ab/2, \text{ when } ab > 0.$$

Proof: In an exponential distribution, the probability density function is given by

$$f(x) = \frac{ae^{-ax}}{1-e^{-ab}}, \qquad 0 \leq x \leq b. \quad (6.1.1)$$

So that,

$$\int_0^b \ln f(x)dx = \int_0^b \ln\left(\frac{ae^{-ax}}{1-e^{-ab}}\right)dx \quad (6.1.2)$$

$$= \int_0^b \ln a\, dx + \int_0^b \ln e^{-ax}dx - \int_0^b \ln(1-e^{-ab})dx \quad (6.1.3)$$

$$= b\ln a - \frac{ab^2}{2} - b\ln(1-e^{-ab}). \quad (6.1.4)$$

Now, $$b\ln a - \frac{ab^2}{2} - b\ln(1-e^{-ab}) \leq -b\ln b \quad (6.1.5)$$

or, $$\ln a - \frac{ab}{2} - \ln(1-e^{-ab}) \leq \ln\frac{1}{b} \quad (6.1.6)$$

or, $$\ln a - \ln\frac{1}{b} - \ln(1-e^{-ab}) \leq \frac{ab}{2} \quad (6.1.7)$$

or, $$\ln\left(\frac{ab}{1-e^{-ab}}\right) \leq \frac{ab}{2} \quad (6.1.8)$$

or, $$\frac{ab}{1-e^{-ab}} \leq e^{ab/2} \qquad (6.1.9)$$

If $ab = x$ then, $$\frac{x}{1-e^{-x}} \leq e^{x/2} \qquad (6.1.10)$$

or, $$x \leq e^{x/2} - e^{-x/2}$$

or, $x \leq 2\sin h\, x/2$, when $x > 0$.

This can be easily proved otherwise since

If $f(x) = x - 2\sin h\, x/2$, $f(0) = 0$

and, $f'(x) = 1 - \cos h\, x/2$, $f'(0) = 0$

also, $f'(0) < 0$ when $x < 0$ or $x > 0$.

3. CONCLUSION:

These inequalities are useful in the study of channel capacity in wired and wireless communication system in the presence of noise and solving many problems related to information sciences.

On the other hand these inequalities are useful in the domain of marketing management where a number of unbalance problems arises such as price and demand, demand and supply, transportation problems etc. and by the help of these inequalities we can make our strategy and plan. Such type of problems can be solved by optimization technique under subject to the some constraints.

Thus, these inequalities are useful for making some new ideas and thoughts in the domain of marketing management, science and technology.

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