Energy Vector and Time Vector in the Dirac Theory

Christian RAKOTONIRINA[1]

Abstract

A sign operator of energy, analogous to the helicity operator, but in the direction of what we call energy vector has been introduced. It is possible that there may be physical phenomena where energy vector should be considered. However, to write a wave function this energy vector needs a time vector. But, unlike the energy vector the time vector has no physical meaning yet. To make physical senses of the components of the time vector, the time dilation in special relativity has been studied and also the components of the time vector have been related to the tunneling times when an electron crosses a potential barrier. Physical results for quantum tunneling time will not be limited to this study.

Keywords: Tunneling time; helicity; time dilation; Dirac equation; superluminal velocity

1 Introduction

The mysterie of time increases as physics penetrates deeper and deeper into the secrets of the Universe. Muga et al. [1] emphasized that the treatment of time in quantum mechanics is one of the most important and challenging open questions in the foundations of quantum theory.

The title of the paper is reminiscent of multidimensional time. Three dimensional time theories are not anything new. Numerous authors [2–6] have already discussed this. From our studies, we found the topic equally interesting when we encountered the energy vector. However, given the popular time quantities such as: tunneling times, decay time, dwell time, delay time, arrival time, or jump time in quantum mechanics and proper time, time dilation in special relativity we are motivated to propose the concept of time vector in the Dirac theory which puts time and space on an equal footing.

The energy quantity $\sqrt{c^2p^2 + m^2c^4}$ of a free particle in special relativity is a combination of an energy due to the momentum and an energy due to the mass. We think that it is more fundamental to take such energy as the magnitude of a vector $\vec{E} = (cp; mc^2)$, whose components are an energy due to the momentum and an energy due to the mass. We call this vector "energy vector".

The resolution of the Dirac equation by using the tensor product or Kronecker product of matrices gives rise to an operator [7] whose eigenvalues are negative energy and positive energy. We called this operator the "sign operator of energy". Both this operator and the helicity operator are vectors in the Pauli algebra. Their components with respect to the Pauli basis $(\sigma_1; \sigma_2; \sigma_3)$ are, respectively the components of the energy vector and the momentum vector.

It is known that the phase $-\frac{i}{\hbar}(Et - \vec{p} \cdot \vec{x})$ of a wave function solution of the Dirac equation is a combination of the components of the momentum vector coupled with the components of the position vector, i.e. the scalar product $\vec{p} \cdot \vec{x}$, and the energy coupled with the classical time, i.e. Et. Thus, regarding the energy vector, a time vector would be needed in the phase of the wave function, in order that we have as phase of the wave function $-\frac{i}{\hbar}(\vec{E} \cdot \vec{t} - \vec{p} \cdot \vec{x})$, where $\vec{t}$ is the time vector. However, the components of this time vector should be given some senses, in order to know in what situations they should be considered.

We shall first study the time vector for a free electron and then try to explain the time dilation in special relativity.

The components of a time vector and any combinations of these components would evolve simultaneously from the beginning to the ending of a phenomenon like the passage time and the dwell time in quantum tunneling, from the entrance to the exit of a potential barrier. So, it is normal to think that it is possible to give senses to the components of the time vector by using the tunneling times in quantum tunneling.

The method proposed in this study consists of putting forward some hypotheses about the couplings of energies with different combinations of the components of the time vector, for example the magnitude of the energy vector couples with the magnitude of the time vector, $\|\vec{E}\| \|\vec{t}\| = \sqrt{c^2p^2 + m^2c^4}$

$\sqrt{t_0^2 + t_{00}^2}$, $mc^2 t_0$, etc..., and trying to find out what combination of the components of the time vector couples with the same energy as the energy coupled with such and such tunneling time. That will lead us to which combination of components of the time vector is equal to the tunneling time.

The paper is organized as follows: in the first section we show the road which has led us to an energy vector; in the second section we introduce the time vectors for a free electron and for an electron crossing through a potential; in the last section we try to give senses to the components of the time vector compared with the quantum tunneling times when the electron crosses a potential barrier.

2 Sign Operator of energy in the Dirac Theory

The Dirac equation [8]

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0 \quad (2.1)$$

is the quantum relativistic equation for a free spin-$\frac{1}{2}$ fermion, where the γ's are the gamma matrices.

In this equation $\hbar$ is the Planck constant, c the speed of light, m the mass of the spin-$\frac{1}{2}$ fermion and ψ is its wave function.

Throughout this paper we use the Dirac representation, where the gamma matrices are

$$\gamma^0 = \sigma_3 \otimes \sigma_0;\ \gamma^1 = i\sigma_2 \otimes \sigma_1;\ \gamma^2 = i\sigma_2 \otimes \sigma_2;\ \gamma^3 = i\sigma_2 \otimes \sigma_3$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices and $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the 2×2-unit matrix.

The wave function solution of the Dirac equation may be written as a Kronecker product or tensor product (See, for instance [9])

$$\psi(t; x) = \chi \otimes s\, e^{-\frac{i}{\hbar}(\epsilon Et - p\cdot x)} \quad (2.2)$$

of the energy state $\chi e^{-\frac{i}{\hbar}(\epsilon Et - p\cdot x)}$ and helicity state s, where $\chi = |\chi(E; p)\rangle = \sqrt{\frac{E+mc^2}{2E}} \begin{pmatrix} 1 \\ \frac{\epsilon cp}{E+mc^2} \end{pmatrix}$ is the eigenvector associated to the positive energy $E = +\sqrt{c^2p^2 + m^2c^4}$ or $\chi = \overline{\chi(E; p)} =$

$\sqrt{\frac{E+mc^2}{2E}} \begin{pmatrix} \square \frac{\_cp}{E+mc^2} \\ 1 \end{pmatrix}$ _, eigenvector associated to the negative energy $\square E = \square \sqrt{c^2p^2 + m^2c^4}$ of the hamiltonian operator $h_D = \_cp\_{}_1 + mc^2\_{}_3$, and s is the eigenvector of the helicity operator $\frac{\hbar}{2}\_.n$, i.e the spin operator in the direction of the momentum vector $p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$, with $n = \frac{p}{\|p\|} = \frac{p}{p} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$.

In all of that _ is the sign of the helicity or the handedness.

We call the operator $h_D = \_cp\_{}_1 + mc^2\_{}_3$ "sign operator of energy" [7, 10].

Let us introduce the "energy vector" $E = \begin{pmatrix} \_cp \\ 0 \\ mc^2 \end{pmatrix}$. Therefore, the operator $\frac{\hbar}{2}\frac{h_D}{E} = \frac{\hbar}{2}\frac{\_.E}{E}$ is the projection of the spin operator in the direction of the energy vector E. Let us call the eigenvalues of this operator "enginity" and this operator the "enginity operator". Therefore there is the probabilities of the particle of having the positive enginity $+\frac{\hbar}{2}$ or the negative enginity $\square\frac{\hbar}{2}$. So, a spin-$\frac{1}{2}$ particle can be in a superposition of a state of positive and a state of negative energy. For seeing that more clearly let us compare the enginity operator with the helicity operator.

However, as mentioned in the introduction, the energy vector $E = \begin{pmatrix} \_cp \\ 0 \\ mc^2 \end{pmatrix}$ requires the time vector

$t =$

0

@
t_1
t_2
t_3
1
A.

3 The components of the Time Vector

Consider an electron with a mass m, moving freely along an x axis, from a point O to a point A of this axis. An observer observes the motion of the electron in a frame where the electron is at rest. So, this observer can measure the time, the proper time _ = t_3 that the electron takes to move from O to A. To calculate the energy of the electron the observer uses the formula E = mc_2.
3
Now, another observer in a frame fixed at the point O measures the time that the electron takes to move from O to A with velocity v. For this observer, A is at a distance L from O. The electron takes the impulsion p = p mv
1□(v=c)2 , and the observer measures the time _ 0 for the passage of the
electron from O to A and uses the formula E =
p
m_2c_4 + c_2p_2 to calculate the energy of the electron.
Then, the energy is the magnitude of the energy vector E =
0
@
_cp;
0
mc_2
1
A which needs the time vector
t_0 =
0
@
t_0
1
0
t_0
3
1
A, where _ is the sign of the helicity.
mc_2t_3 = mc_2t_0
3 + _cpt0
1 □ px (3.1)
c_2t_{23}
= c_2t_{02}
3 + c_2t_{02}
1 □ x_2 (3.2)
From these equations, t_0
3 = t_3 if, and only if, for helicity positive, t_0
1 = x
c and for helicity negative,
t_0
1 = □x
c .
Otherwise, given t_0
3 6= t_3, solving this system of two equations there are two time vectors. But,
according to (3.2), these two time vectors have the same euclidian norm, and according to the special relativity of Einstein
_ 0 =
q
t_{02}
3 + t_{02}
1 =
1 q
1 □ (v=c)2

_ (3.3)
t_0
3 _ t_3 if, and only if for helicity positive, t_0
1 _ x
c and for helicity negative, t_0
1 _ □x
c , that is t_0
1 is the
time of a subluminal velocity for moving from O to A. Then, according to the formula (3.3) the time t_0
1
which appears when the electron takes the impulsion p is responsible for the time dilation in special relativity.
But otherwise, t_0
3 > t_3, where t_0
1 < x
c for helicity positive or t_0
1 > □x
c for helicity negative. In this case,
we cannot say what the contribution of t_0
1 to the dilation of time is.
The classical time _ 0 =
p
t_{02}
3 + t_{02}
1 and the component times t_0
3, t_0
1 evolve from O to A, but only the
classical time can be observed. Then, the wave function is of the form (2.2).
Now, suppose that from O to A the electron moves in a uniform potential U. For the observer at the frame where the electron is fixed the energy vector is E_0 =
0
@
0;
U
mc_2
1
A and the time vector is
T =
0
@
0
T_2
T_3
1
A. Whereas for the observer in the second frame the energy vector is E =
0
@
_cp;
U
mc_2
1
A whose
components are respectively the energy due to the impulsion, the energy due to the mass and the potential energy, i.e the energy due to the space, which makes the second component of the time vector appear, T 0 =
0
@
T 0
1
T 0
2
T 0
3

1
A. It follows
_ = $T_3 mc^2$ + $T_2 U$ = $mc^2 T_0$
$_3$ + _cpT_0
$_1$ + UT_0
$_2$ px (3.4)
$c^2 T_2$
$_3$ + $c^2 T_2$
$_2$ = $c^2 T_0^2$
$_3$ + $c^2 T_0^2$
$_1$ + $c^2 T_0^2$
$_2$ x^2 (3.5)
and q
T_0^2
$_3$ + T_0^2
$_1$ + T_0^2
$_2$ =
1 p
1 $(v=c)^2$
q
T_2
$_3$ + T_2
$_2$
The total energy of the electron is the magnitude
E =
p
$m^2c^4 + c^2p^2 + U^2$ (3.6)
4
of the energy vector, which is like the one in [11] for the extension to the Klein-Gordon equation, and we suppose that the magnitude
T_0 =
q
T_0^2
$_3$ + T_0^2
$_1$ + T_0^2
$_2$
of the time vector T_0 is the classical time.
The following hypotheses are put forward for possible couplings of energy with time in the phase of the wave function:
ET_0 =
p
$m^2c^4 + c^2p^2 + U^2$
q
T_0^2
$_3$ + T_0^2
$_1$ + T_0^2
$_2$ (3.7)
ET =
p
$m^2c^4 + c^2p^2$
q
T_0^2
$_3$ + T_0^2
$_1$ (3.8)
$mc^2 T_0$
$_3$ (3.9)
UT_0
$_2$ (3.10)
p
$c^2p^2 + U^2$
q
T_0^2
$_2$ + T_0^2

$_1$ (3.11)
_cpT $_0$
$_1$ (3.12)

4 Components of the Time Vector and the Tunneling Times of an Electron

To provide physical meaning to the components of the time vector, we think that it is normal to try to find their possible relations with the tunneling times. Firstly, the Dirac type equation for the electron in a potential less than the kinetic energy of the electron has to be determined.

4.1 A Dirac equation with parity violation

The Dirac equation to be determined is a Dirac equation which has the energy vector E =
0
@
_cp;
U
mc_2
1
A,
i.e whose operator enginity is
H = _cp_$_1$ + U_$_2$ + mc_2_$_3$ (4.1)
with U <
p
$c_2p_2 + m_2c_4$.
The search for a solution of the form = A(p)e□ $_i$
$_{\sim (Et□p\_x)}$ for the Dirac-Sidharth equation [12]
i~_@_ □ mc □ i
p
_I~$_5$_ = 0
employing the kronecker product leads to the operator enginity
H_0 = _cp_$_1$ □ c
p
_p_2 I
~
_$_2$ + mc_2_$_3$
with $_5$ = i_{0123} = _$_1$ _$_0$.
Then, following the backward way, from the operator enginity (4.1) we obtain as a Dirac equation for discribing electron in a potential U the following equation
i~_@_ □ mc + i
U
c
$_5$ = 0 (4.2)
Because of the presence of $_5$, the parity is violated [13]. Looking for a wave function of the form
= A(p)e□ $_i$
$_{\sim (Et□p\_x)}$
5
i.e. of the form of (2.2), by using the kronecker product of matrices, the following
=
r
E + mc_2
2E
1 p
2 (1 + n_3)
_
1
□cp+iU
E+mc_2
_
_
□n_1 + in_2
1 + n_3
_
_ e□ $_i$

~ (Et□p_x) (4.3)
is obtained as solution with positive enginity and negative helicity. The resolution of this equation using the kroneker product is given in the appendix.

4.2 The Components of the Time Vector and Tunneling Times

The components of the time vector make us think to one of the controversial issues of modern quantum theory, the question of tunneling time, i.e. the time a particle takes to move from one side of a barrier of potential to the other side [14]. Some experimental investigations have supported a nonzero tunneling time, while others supported a zero tunneling time, [15, 16]. However, to make sense to the components of the time vector we have to choose the nonzero tunneling time and consider the case of one dimensional tunneling of an electron through a potential barrier.
Quantum tunneling is a phenomenon in which particles penetrate a potential energy barrier with a height greater than the total energy of the particles. The phenomenon is interesting and important because it violates the principles of classical mechanics. Suppose that an uniform and time-independent beam of electrons with an energy E traveling along the x-axis (in the positive direction to the right) encounters a potential barrier (Fig. 1) described by (See, for instance [17])
U(x) =
8><
>:
0 when x < 0
U when 0 _ x _ L
0 when x > L
When both the width L and the height U are finite, a part of the incident quantum wave packet on one side of the barrier can penetrate the barrier boundary and continue its motion inside the barrier, where it is gradually attenuated on its way to the other side. A part of the incident quantum wave packet eventually emerges on the other side of the barrier in the form of the transmitted wave packet that tunneled through the barrier. How much of the incident waves can tunnel through a barrier depends on the barrier's width L and its height U , and on the energy E of the incident quantum particle. For such transmitted waves, there are four widely used tunneling times calculated by finding the transmission amplitude given by: T = jTj ei_ [18], where _ and jTj are the phase and the magnitude of the transmission amplitude, respectively. The two of them are : Larmor time [20,21], _LM and Eisenbud-Wigner times [22], _EW. The first has been called resident or dwell time:
_LM = □~ @_
@U
(4.4)
The second has been called the passage time,
_EW = ~ @_
@E
+
L
k
(4.5)
The phase of the transmitted wave behind the barrier (region III) is given by [19]
_III = _ +
p
~
x (4.6)
An additional term, L=k is present in _EW, where L and k are the barrier width and the electron velocity, respectively. This additional term would correspond to the propagation of the electron in the barrier region if that barrier were absent, and has to be added to get the total time [23], since the first term only gives a relative time shift [22].
However, the quantum tunneling phenomena and the consideration of the time vector lead to think

Figure 1: An electron e with kinetic energy E **moves along the** x **-axis and interacts with a rectangular barrier with height** U**,** U > E**, and width** L**.**

that the Dirac equation inside the potential barrier (E < U) is not of the form (4.2) [25]. So, let us construct the wave function of the electron inside the barrier in terms of the components of the time vector. The energy vector E =
0
@
_cp

0
mc^2
1
A before the barrier region becomes E =
0
@
_cp
U
mc^2
1
A when the
particle is inside the barrier region. The time T =
p
T 02
3 + T 02
1 and T 0
2 can be qualified respectively
as passage time T_p =
p
T 02
3 + T 02
1 and resident time T_r = T 0
2 . These three types of time, the
classical time T 0 =
p
T 02
3 + T 02
1 + T 02
2 , the passage time T_p and the resident time T_r evolve from the
entrance to the outrance of the barrier region. But according to the quantum tunneling phenomena, the classical time can not be observed, whereas at least one of the passage time and the resident time can be. Actually,
T 0 > T_p T 0 > T_r
All these times evolve from zero to positive values.
Let us search for _ in (4.4) and (4.5) in terms of $\_{LM}$ and $\_{EW}$. From (4.4)
_ = □
1
~
$_{LM}$U + K(E)
where K(E) is a function of E. Then,
@_
@E
= K_0(E)
in substituting in (4.5)
K_0(E) =
1
~
$\_{EW}$ □
1
~
L
v
7
Using the relations p = mv
1□v2
c2
and E = mc^2
r
1□v2
c2
(See for instance, [24]), we have
K_0(E) =
1
~

_EW
1
c~
E
p
E2 m2c4
L
K(E) =
1
~
E_EW
1
c~
p
E2 m2c4L + _(L)
and then,
_ =
1
~
(E_EW U_LM pL) + _(L) (4.7)
with _(L) independant of E and U, such that U > E.
It follows that according to the couplings (3.8) and (3.10), the times evolve from 0 to _EW and _LM
are respectively
p
T 02
3 + T 02
1 and T 0
2 . Then, to give senses to the components of the time vector, the
phase which evolves from the phase at x = 0 to x = L, inside the potential barrier, should be defined
as
_II =
1
~
_
E
q
T 02
3 + T 02
1 + UT 0
2 px
_
(4.8)
with at x = L,
p
T 02
3 + T 02
1 = _EW and T 0
2 = _LM. The equation (4.6) yields that the phase for the
trasmitted wave in region III is
_III =
1
~
(E_EW + U_LM + pL px) + _(L)
At x = L, _II = _III . Then, we have _(L) = p
~L and for the case of positive enginity and negative
helicity: incident, reflected and transmitted wave functions are
I (x) =
_
1
cp
E+mc2
_

_
□1
1
_
e□ i
~
_
E
p
t02
3 +t02
1 □px
_
+ A
_
1
cp
E+mc2
_

_
□1
1
_
e□ i
~
_
E
p
t02
3 +t02
1 +px
_
(x < 0)
II (x) = B
_
1
□cp+iU
E+mc2
_

_
□1
1
_
e□ i
~
_
□E
p
T 02
3 +T 02
1 +UT 0
2□px
_
+ C
_
1
cp+iU
E+mc2
_

_
□1
1
_
_ e□ i
~
_
□E
p
T 02
3 +T 02
1 +UT 0

2+px

(0 < x < L) (4.9)
III (x) = D

1
cp
E+mc2

1
1

e i
~ (E_EW+U_LMpx) (L < x) (4.10)

The form of each term of the wave function (4.9) inside the barrier is not like the one that has been thought in [25]. It is a wave function solution, not of (1+1) spacetime Dirac equation, like a particular case of (4.3), but a (1 + 2) spacetime Dirac equation.

In the case where the energy of the electron is higher than the value of the potential (E > U), the wave function inside the potential will be of the form

8
II (x) = A0

1
cpiU
E+mc2

1
1

e i
~
E
p
T 02
3 +T 02
2 +T 02
1 px

= A0

1
cpiU
E+mc2

1
1

e i
~
E
p
_2E
W+_2
LM
px

because according to (3.7) the classical time t in the wave function (4.3) is t =
p
T 02
3 + T 02
2 + T 02
1 .

Since the Eisenbud-Wigner time τ_{EW} is positive and $\sqrt{T_3'^2 + T_1'^2}$ should evolve from 0 to τ_{EW}, then there is negative energy $-E$ in the phase inside the barrier.

5 Results and Discussion

The first component t'_1 of the time vector which occurs when the electron takes an impulsion is responsible for the time dilation in special relativity, if it is the time of a subluminal velocity. Otherwise, we cannot say what the contribution of t'_1 to the dilation of time is.

Under the hypotheses that the classical time is the magnitude of the time vector and that the energy couples with time under the form of the expressions (3.7) to (3.12), the following results have been obtained, in the Dirac representation, during the tunneling of the electron through a potential barrier. Only for the second component a physical meaning can be given. It can be defined as the Larmor time τ_{LM} or the dwell time, $T'_2 = \tau_{LM}$. It is not possible to give physical senses to the first and the third components. But, the magnitude of the projection of the time vector into the plan of the fiprst and third components can be defined as the Eisenbud-Wigner time τ_{EW} or the passage time, $\sqrt{T_3'^2 + T_1'^2} = \tau_{EW}$. Then it follows the following relation

$$T' = \sqrt{\tau_{EW}^2 + \tau_{LM}^2}$$

between the classical time, the Eisenbud-Wigner time and the Larmor time. Then, we can see that the classical time is higher than the Eisenbud-Wigner time and the Larmor time. Thus,

$$v = \frac{L}{\|T'\|} < \frac{L}{\tau_{EW}} \quad \text{and} \quad v = \frac{L}{\|T'\|} < \frac{L}{\tau_{LM}}$$

This show the possibility of supeluminality.

Due to the minus sign in the equation (4.4), a negative energy ($-E$) couples with $\sqrt{T_3'^2 + T_1'^2}$ in the phase (4.8) inside the barrier. According to the Dirac interpretation of negative energy (See, for example, [26]), can we interpret it that it is not the electron which spends the passage time but its antiparticle a positron?

Finally, for a free electron, it is not possible to make sense of t'_3 or t'_1 separately. We think that a possible observability of these two components of the time vector would be in a phenomenon of free superluminal spin-$\frac{1}{2}$ particle, whose wave function would be a solution of a (3 + 2) spacetime Dirac equation. Thus, this study agrees with the authors of [27,28] who said :"The problem of representation and localizations of superluminal particles has been solved only by the use of higher dimensional space and it has been claimed that the localization space for tachyons is T_4- space with one space

and three times". As a consequence, the wave function of a spin- 1
2 particle inside a potential barrier
would be of the form _II = Ae_PT 0
1□UT 0
2+mc2T 0
3 , if the particle was a free superluminal spin- 1
2 particle
before the potentiel barrier U, U >
p
$P_2 + m_2c_4$ [25].

Conclusion

The energy vector in the Dirac theory came out when the purpose was to show the analogy between helicity and enginity. This energy vector needs a time vector. A component of the time vector appears when the electron takes an impulsion. This component time vector is responsible for time dilation in special relativity.

Under some hypotheses, during the tunneling of the electron through a potential barrier, one component can be defined as the Larmor time, whereas the Eisenbud-Wigner time is a combination of the two other components. Then, a relation between the classical time, the Larmor time and the Eisenbud-Wigner time has been obtained. When the electron is inside the barrier, in the phase of the wave function a negative energy couple with the Eisenbud-Wigner time. So, what actually about the times spend by the electron and its antiparticle inside the barrier?

References

[1] Muga, J-G, Sala Mayato, R. and I L Egusquiza, I-L. (Eds.) Time in Quantum Mechanics. Lect. Notes Phys. 734. 2nd edn. (Berlin Heidelberg : Springer) Vol.1. Preface to the First Edition p VII. 2008. DOI 10.1007/978-3-540-73473-4.

[2] Bonacci, E. Hypothetical six-dimensional reference frames. Int Jr. of Mathematical Sciences & Applications. 2015;Vol 5 No 2 493.

[3] Dartora, C-A. and Cabrera, G-G. The Dirac Equation in Six-dimensional SO(3,3) Symmetry Group and a Non-chiral Electroweak Theory. Int. J. Theor. Phys. 2010;49 (51). https://doi.org/10.1007/s10773-009-0177-9.

[4] Cole, E-A-B. Emission and absorption of tachyons in six-dimensional relativity. Phys. Let. 1979; A75 (1-2) 29.

[5] Lunsford, D-R. Gravitation and Electrodynamics over SO(3; 3). Int. J. Theor. Phys. 2004;43 (1) 161.

[6] Ziino, G. On a straightforward experimental test for a three-dimensional time. Lettere Al Nuovo Cimento. 1980;28 (16) 551.

[7] Raoelina Andriambololona and Rakotonirina, C. A Study of the Dirac-Sidharth Equation. EJTP 2011;8 No.25 177.

[8] Dirac, P-A-M. Proc. R. Soc. 1928;A117 (778) 610. Proc. R. Soc. 1930;A126 (801), 360.

[9] Schweber, S-S. An Introduction to Relativistic Quantum Field Theory. (New York : Harper and Row) 1961. Part 1, Chap 4, Sec d, p 84.

[10] Rakotonirina, C. Produit Tensoriel de Matrices en Th´eorie de Dirac. Th`ese de Doctorat de Troisi `eme Cycle (Supervised by Raoelina Andriambololona) (University of Antananarivo, Madagascar). 2003. p 61. viXra : 1608.0232, https://vixra.org/abs/1608.0232. French.

[11] Xu, D., Wang, T. and Xue, X. Quantum tunneling Time: Relativistic Extensions. Found. Phys. 2013;43 (11) 1257. https://doi.org/10.1007/s10701-013-9744-2

[12] Sidharth, B-G. The Modified Dirac Equation. EJTP. 2010;Vol 7 No 24 211.

[13] Bjorken, J-D. and Drell, S-D. Relativistic Quantum Mechanics (New York : Mc-Graw Hill). 39. 1964.

[14] Razavy, M. Quantum Theory of tunneling. (Singapore : World Scientific) 2nd Edition xix. 2014.

[15] Shu, Z., Hao, X-L., Li, W-D. and Chen, J. General way to define tunneling time. Chin. Phys. B. 2019;Vol 28 No 5 050301.

[16] Hofmann, C., Landsman, A-S. and Keller, U. Attoclock revisited on electron tunneling time. J. Mod. Opt. 2019;66 (10) 1052. DOI: 10.1080/09500340.2019.1596325.

[17] Hooge, C. "BCIT Physics 8400 Textbook." 2018. https://opentextbc.ca/postsecondary/chapter/bcit/.

[18] Landsman, A-S. and Keller, U. Attosecond science and the tunneling time problem. Phys. Rep. 2015;547 1–24.

[19] Winful, H-G. Relation between quantum tunneling times for relativistic particles. Phys. Rev. A. 2004; 70, 052112. DOI: 10.1103/PhysRevA.70.052112.

[20] B¨ uttiker, M. Larmor precession and the traversal time for tunneling. Phys. Rev. B. 1983;27 (10)

6178.
[21] Rybachenko, V-F. Time of Penetration of a Particle through a Potential Barrier. Sov. J. Nucl. Phys. 1967;5 635.
[22] Wigner, E-P. Lower Limit for the Energy Derivative of the Scattering Phase Shift. Phys. Rev. 1955;98 (1) 145.
[23] Hauge, E-H. and Støvneng, J-A. Tunneling times: a critical review. Rev. Mod. Phys. 1989;61 (4) 917.
[24] Baym, G. Lectures on Quantum Mechanics. (W.A. Benjamin, Reading) Chap. 23. 1976.
[25] Rakotonirina, C. Operator of sign of energy : Toward a consequence. 11th Int. Conf. in High-Energy Physics, HEPMAD 19. Antananarivo, Madagascar, 2020;October 14-20, 2019, (eds) S Narison.
[26] Thomson, A. Modern Particle Physics. (Cambridge : University Press) 98. 2013.
[27] Bisht, P-S., Jivan Singh and Negi, O-P-S. Quaternionic Reformulation of Generalized Superluminal Electromagnetic Fields of Dyons. Communications in Physics. 2008;Vol 18 No 1 27.
[28] Pant, M-C., Bisht, P-S., Negi, O-P-S. and Rajput, B-S. Supersymmetric behaviour of tachyons. Ind. J. Pure and Appl. Phys. 2000;38 (06) 440.

Appendix A. Resolution of the Dirac equation i~_@_ □ mc + iU
c 5 = 0

To make appear the operator enginity we use, for solving the equation (4.2), the kronecker product. Put this solution of the form

_
= A(p)e□ i
~ (Et□p:x)

Then,

i~_3 _0 @
c@t
_
+ i~i_2 _j @
@xj
_
□ mc_
+ i
U
c
_1 _0_
= 0

i~_0 _0 @
@t
_
+ ic~_1 _j @
@xj
_
□ mc2_3 _0_
□ U_2 _0_
= 0

Let ^pj = □i~ @
@xj
the mometum operator in the direction of xj and ^ E = i~ @
@t the energy operator,
^pj_
= pj_
and ^ E_
= E_ . Then, we have

E_0 _0A(p) □ cpj_1 _jA(p) □ mc2_3 _0A(p) □ U_2 _0A(p) = 0

But let us search for a solution where A(p) is of the form A(p) = ' u with u is an eigenvector of the helicity operator

□ ~
2_ _ n
_
=
_
~

2
__p
p
_
, whose eigenvalues are the helicities +~
2 and □~
2 .
E_0 _0' u □
2
~
cp_1
_
~
2
_ _ n
_
' u □ mc2_3 _0' u □ U_2 _0' u = 0
11
E' u □ cp_1' _u □ mc2_3' u □ U_2' u = 0
or □
E' □ _cp_1' □ mc2_3' □ U_2'
_
u = 0
where _ is the sign of helicity. However, u 6= 0, thus
E' =
□
_cp_1 + mc2_3 + U_2_
' (5.1)
That is ' is the eigenvector of the operator
hD = _cp_1 + mc2_3 + U_2 = _ _ E
in the Dirac representation, whose eigenvalues are the negative energy □E and the positive energy +E. The enginity operator ~
2
hD
E = ~
2
__E
E in the direction of the energy vector E will be the analogous of the helicity operator.
The resolution will be finished in solving the equation (5.1) and the equation
□
_1n1 + _2n2 + _3n3_
u = _u