Energy Vector and Time Vector in the Dirac Theory

Original research paper

Abstract

A sign operator of energy, analogous to the helicity operator, but in the direction of what we call energy vector has been introduced. It is possible that there will be physical phenomena where energy vector should be considered. However, to write a wave function this energy vector needs a time vector. But, unlike the energy vector the time vector has not physical meaning yet. To give physical senses to the components of the time vector, the time dilation in special relativity has been studied and also the components of the time vector have been related to the tunneling times when an electron crosses a potential barrier. Physical results for quantum tunneling time will not be limited in this study.

Keywords: Tunneling time; helicity; time dilation; Dirac equation; superluminal velocity

1 Introduction

The mysteries of time increase as physics penetrate deeper and deeper into the secrets of the Universe. [1] said "The treatment of time in quantum mechanics is one of the important and challenging open questions in the foundations of quantum theory".

The title of the paper makes us think immediately about multidimensional time. Three dimensional time theories are not something new. Many authors have already mentioned them, including [2–6]. From our side, we have by chance fallen to this question when we encountered the energy vector. But, many different time quantities: tunneling times, decay time, dwell time, delay time, arrival time, or jump time in quantum mechanics and proper time, time dilation in special relativity make us to dare to introduce time vector in the Dirac theory, a quantum relativistic theory which puts time and space on an equal footing.

The energy $\sqrt{c^2p^2 + m^2c^4}$ of a free particle in special relativity is a combination of an energy due to

the momentum and an energy due to the mass. We think that it is more fundamental to take such energy as the magnitude of a vector $\boldsymbol{E}$, whose components are an energy due to the momentum and an energy due to the mass. We call this vector "energy vector".
The resolution of the Dirac equation by using the tensor product or Kronecker product of matrices gives rise to an operator [7] whose eigenvalues are negative energy and positive energy. We called this operator the "sign operator of energy". Both this operator and the helicity operator are vectors in the Pauli algebra. Their components with respect to the Pauli basis $\left(\sigma^1, \sigma^2, \sigma^3\right)$ are, respectively the components of the energy vector and the momentum vector.
It is known that the phase $-\frac{i}{\hbar}\left(Et - \boldsymbol{p}\cdot\boldsymbol{x}\right)$ of a wave function solution of the Dirac equation is a combination of the components of the momentum vector coupled with the components of the position vector, *i.e.* the scalar product $\boldsymbol{p}\cdot\boldsymbol{x}$, and the energy coupled with the classical time, *i.e.* Et. Thus, regarding the energy vector, a time vector should be needed in the phase of the wave function, in order that we have as phase of the wave function $-\frac{i}{\hbar}\left(\boldsymbol{E}\cdot\boldsymbol{t} - \boldsymbol{p}\cdot\boldsymbol{x}\right)$, where $\boldsymbol{t}$ is the time vector. However, the components of this time vector should be given some senses, in order to know in what situations they should be considered.
We shall study at first the time vector for a free electron and shall try to explain the time dilation in special relativity.
The components of a time vector and any combinations of these components would evolve simultaneously from the beginning to the ending of a phenomenon like the passage time and the dwell time in quantum tunneling, from the entrance to the outrance of a potential barrier. So, it is normal to think that it is possible to give senses to the components of the time vector by using the tunneling times in quantum tunneling.
The method proposed in this study consists of putting forward some hypotheses about the couplings of energies with different combinations of the components of the time vector, for example the magnitude of the energy vector couples with the magnitude of the time vector, $E\cdot t = \sqrt{c^2p^2 + m^2c^4}\sqrt{t'^2 + t''^2}$, mc^2t', etc..., and trying to find out what combination of the components of the time vector couples with the same energy as the energy coupled with such and such tunneling time. That will lead us to which combination of components of the time vector is equal to the tunneling time.
The paper is organized as follows: in the first section we show the road which has led us to an energy vector; in the second section we introduce the time vectors for a free electron and for an electron crossing through a potential; in the last section we try to give senses to the components of the time vector compared with the quantum tunneling times when the electron crosses a potential barrier.

2 Sign Operator of energy in the Dirac Theory

The Dirac equation [8]

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0 \tag{2.1}$$

is the quantum relativistic equation for a free spin-$\frac{1}{2}$ fermion, where the γ^μ's are the gamma matrices. In this equation $\hbar$ is the Planck constant, c the speed of light, m the mass of the spin-$\frac{1}{2}$ fermion and ψ is its wave function.
Throughout this paper we use the Dirac representation, where the gamma matrices are

$$\gamma^0 = \sigma^3\otimes\sigma^0,\;\; \gamma^1 = i\sigma^2\otimes\sigma^1,\;\; \gamma^2 = i\sigma^2\otimes\sigma^2\;\; \gamma^3 = i\sigma^2\otimes\sigma^3$$

with

$$\sigma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\;\; \sigma^2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\;\; \sigma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

are the Pauli matrices and $\sigma^0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ the 2×2-unit matrix.
The wave function solution of the Dirac equation may be written as a Kronecker product or tensor

product (See, for instance [9])

$$\psi(t, \boldsymbol{x}) = \xi \otimes s e^{-\frac{i}{\hbar}(\pm Et - \boldsymbol{p}.\boldsymbol{x})} \tag{2.2}$$

of the energy state $\xi e^{-\frac{i}{\hbar}(\pm Et-\boldsymbol{p}.\boldsymbol{x})}$ and helicity state s, where $\xi = |\xi(E,p)\rangle = \sqrt{\frac{E+mc^2}{2E}}\begin{pmatrix}1\\ \frac{\epsilon cp}{E+mc^2}\end{pmatrix}$ is the eigenvector associated to the positive energy $E = +\sqrt{c^2p^2+m^2c^4}$ or $\xi = |\bar{\xi}(E,p)\rangle = \sqrt{\frac{E+mc^2}{2E}}\begin{pmatrix}-\frac{\epsilon cp}{E+mc^2}\\ 1\end{pmatrix}$, eigenvector associated to the negative energy $-E = -\sqrt{c^2p^2+m^2c^4}$ of the hamiltonian operator $h_D = \epsilon cp\sigma^1 + mc^2\sigma^3$, and s is the eigenvector of the helicity operator $\frac{\hbar}{2}\boldsymbol{\sigma}.\boldsymbol{n}$, *i.e* the spin operator in the direction of the momentum vector $\boldsymbol{p} = \begin{pmatrix}p^1\\ p^2\\ p^3\end{pmatrix}$, with $\boldsymbol{n} = \frac{\boldsymbol{p}}{\|\boldsymbol{p}\|} = \frac{\boldsymbol{p}}{p} = \begin{pmatrix}n^1\\ n^2\\ n^3\end{pmatrix}$.
In all of that ϵ is the sign of the helicity or the handedness.
We call the operator $h_D = \epsilon cp\sigma^1 + mc^2\sigma^3$ "sign operator of energy" [7, 10].

Let us introduce the "energy vector" $\boldsymbol{E} = \begin{pmatrix}\epsilon cp\\ 0\\ mc^2\end{pmatrix}$. Therefore, the operator $\frac{\hbar}{2}\frac{h_D}{E} = \frac{\hbar}{2}\frac{\boldsymbol{\sigma}.\boldsymbol{E}}{E}$ is the projection of the spin operator in the direction of the energy vector $\boldsymbol{E}$. Let us call the eigenvalues of this operator "enginity" and this operator the "enginity operator". Therefore there is the probabilities of the particle of having the positive enginity $+\frac{\hbar}{2}$ or the negative enginity $-\frac{\hbar}{2}$. So, a spin-$\frac{1}{2}$ particle can be in a superposition of a state of positive and a state of negative energy. For seeing that more clearly let us compare the enginity operator with the helicity operator.

Table 1: Comparison between helicity operator and enginity operator

$\boldsymbol{\sigma}.\boldsymbol{E} = \epsilon cp\sigma^1 + mc^2\sigma^3$ with $\boldsymbol{E} = \begin{pmatrix}\epsilon cp\\ 0\\ mc^2\end{pmatrix}$ energy vector	$\boldsymbol{\sigma}.\boldsymbol{p} = p_1\sigma^1 + p_2\sigma^2 + p_3\sigma^3$ with $\boldsymbol{p} = \begin{pmatrix}p_1\\ p_2\\ p_3\end{pmatrix}$ momentum vector
$E = \|\boldsymbol{E}\| = \sqrt{m^2c^4 + c^2p^2}$ the energy	$p = \|\boldsymbol{p}\| = \sqrt{p_1^2+p_2^2+p_3^2}$ the impulsion
$\frac{\hbar}{2E}\boldsymbol{\sigma}.\boldsymbol{E} = \frac{\hbar}{2E}\epsilon cp\sigma^1 + \frac{\hbar}{2E}mc^2\sigma^3$ enginity operator or spin operator in the direction of $\boldsymbol{E}$	$\frac{\hbar}{2p}\boldsymbol{\sigma}.\boldsymbol{p} = \frac{\hbar}{2p}p_1\sigma^1 + \frac{\hbar}{2p}p_2\sigma^2 + \frac{\hbar}{2p}p_3\sigma^3$ helicity operator or spin operator in the direction of $\boldsymbol{p}$
$\frac{\boldsymbol{\sigma}.\boldsymbol{E}}{E} = \frac{\epsilon cp}{E}\sigma^1 + \frac{mc^2}{E}\sigma^3$ enginity sign operator	$\frac{\boldsymbol{\sigma}.\boldsymbol{p}}{p} = \frac{p_1}{p}\sigma^1 + \frac{p_2}{p}\sigma^2 + \frac{p_3}{p}\sigma^3$ helicity sign operator
Probability for having positive or negative enginity (energy)	Probability for having positive or negative helicity

But, as mentioned in the introduction, the energy vector $\boldsymbol{E} = \begin{pmatrix}\epsilon cp\\ 0\\ mc^2\end{pmatrix}$ need time vector $\boldsymbol{t} = \begin{pmatrix}t_1\\ t_2\\ t_3\end{pmatrix}$.

3 The components of the Time Vector

Consider an electron with a mass m, moving freely along an x axis, from a point O to a point A of this axis. An observer observes the motion of the electron in a frame where the electron is at rest. So, this observer can measure the time, the proper time $\tau = t_3$ that the electron takes to move from O to A. To calculate the energy of the electron the observer uses the formula $E = mc^2$.

Now, another observer in a frame fixed at the point O measures the time that the electron takes to move from O to A with velocity v. For this observer, A is at a distance L from O. The electron takes the impulsion $p = \frac{mv}{\sqrt{1-(v/c)^2}}$, and the observer measures the time τ' for the passage of the electron from O to A and uses the formula $E = \sqrt{m^2c^4 + c^2p^2}$ to calculate the energy of the electron. Then, the energy is the magnitude of the energy vector $\boldsymbol{E} = \begin{pmatrix} \epsilon cp, \\ 0 \\ mc^2 \end{pmatrix}$ which needs the time vector $\boldsymbol{t'} = \begin{pmatrix} t'_1 \\ 0 \\ t'_3 \end{pmatrix}$, where ϵ is the sign of the helicity.

$$mc^2 t_3 = mc^2 t'_3 + \epsilon cpt'_1 - px \tag{3.1}$$

$$c^2 t_3^2 = c^2 t_3'^2 + c^2 t_1'^2 - x^2 \tag{3.2}$$

From these equations, $t'_3 = t_3$ if, and only if, for helicity positive, $t'_1 = \frac{x}{c}$ and for helicity negative, $t'_1 = -\frac{x}{c}$.

Otherwise, where $t'_3 \neq t_3$, solving this system of two equations there are two time vectors. But, according to (3.2), these two time vectors have the same euclidian norm, and according to the special relativity of Einstein

$$\tau' = \sqrt{t_3'^2 + t_1'^2} = \frac{1}{\sqrt{1-(v/c)^2}}\tau \tag{3.3}$$

$t'_3 \leq t_3$ if, and only if for helicity positive, $t'_1 \geq \frac{x}{c}$ and for helicity negative, $t'_1 \leq -\frac{x}{c}$, that is t'_1 is the time of a subluminal velocity for moving from O to A. Then, according to the formula (3.3) the time t'_1 which appear when the electron takes the impulsion p is responsible of the time dilation in special relativity.

But otherwise, $t'_3 > t_3$, where $t'_1 < \frac{x}{c}$ for helicity positive or $t'_1 > -\frac{x}{c}$ for helicity negative, we cannot say what the contribution of t'_1 to the dilation of time is.

The classical time $\tau' = \sqrt{t_3'^2 + t_1'^2}$ and the component times t'_3, t'_1 evolve from O to A, but only the classical time can be observed. Then, the wave function is of the form (2.2).

Now, suppose that from O to A the electron moves in a uniform potential U. For the observer at the frame where the electron is fixed the energy vector is $\boldsymbol{E'} = \begin{pmatrix} 0, \\ U \\ mc^2 \end{pmatrix}$ and the time vector is $\boldsymbol{\mathcal{T}} = \begin{pmatrix} 0 \\ \mathcal{T}_2 \\ \mathcal{T}_3 \end{pmatrix}$. Whereas for the observer at the second frame the energy vector is $\boldsymbol{\mathcal{E}} = \begin{pmatrix} \epsilon cp, \\ U \\ mc^2 \end{pmatrix}$ whose components are respectively the energy due to the impulsion, the energy due to the mass and the potential energy, *i.e* the energy due to the space, which makes the second component of the time vector appear, $\boldsymbol{\mathcal{T}'} = \begin{pmatrix} \mathcal{T}'_1 \\ \mathcal{T}'_2 \\ \mathcal{T}'_3 \end{pmatrix}$. It follows

$$\phi = \mathcal{T}_3 mc^2 + \mathcal{T}_2 U = mc^2 \mathcal{T}'_3 + \epsilon cp \mathcal{T}'_1 + U \mathcal{T}'_2 - px \tag{3.4}$$

$$c^2 \mathcal{T}_3^2 + c^2 \mathcal{T}_2^2 = c^2 \mathcal{T}_3'^2 + c^2 \mathcal{T}_1'^2 + c^2 \mathcal{T}_2'^2 - x^2 \tag{3.5}$$

and

$$\sqrt{\mathcal{T}_3'^2 + \mathcal{T}_1'^2 + \mathcal{T}_2'^2} = \frac{1}{\sqrt{1-(v/c)^2}}\sqrt{\mathcal{T}_3^2 + \mathcal{T}_2^2}$$

The total energy of the electron is the magnitude

$$\mathcal{E} = \sqrt{m^2c^4 + c^2p^2 + U^2} \tag{3.6}$$

of the energy vector, which is like the one in [11] for the extension to the Klein-Gordon equation, and we suppose that the magnitude

$$\mathcal{T}' = \sqrt{\mathcal{T}_3'^2 + \mathcal{T}_1'^2 + \mathcal{T}_2'^2}$$

of the time vector $\boldsymbol{\mathcal{T}'}$ is the classical time.
The following hypotheses are put forward for possible couplings of energy with time in the phase of the wave function:

$$\mathcal{E}\mathcal{T}' = \sqrt{m^2c^4 + c^2p^2 + U^2}\sqrt{\mathcal{T}_3'^2 + \mathcal{T}_1'^2 + \mathcal{T}_2'^2} \tag{3.7}$$

$$E\mathcal{T} = \sqrt{m^2c^4 + c^2p^2}\sqrt{\mathcal{T}_3'^2 + \mathcal{T}_1'^2} \tag{3.8}$$

$$mc^2\mathcal{T}_3' \tag{3.9}$$

$$U\mathcal{T}_2' \tag{3.10}$$

$$\sqrt{c^2p^2 + U^2}\sqrt{\mathcal{T}_2'^2 + \mathcal{T}_1'^2} \tag{3.11}$$

$$\epsilon cp\mathcal{T}_1' \tag{3.12}$$

4 Components of the Time Vector and the Tunneling Times of an Electron

To give physical senses to the components of the time vector, we think that it is normal to try to find their possible relations with the tunneling times. But firstly, the Dirac type equation for the electron in a potential less than the kinetic energy of the electron has to be determined.

4.1 A Dirac equation with parity violation

The Dirac equation to be determined is a Dirac equation which has the energy vector $\boldsymbol{\mathcal{E}} = \begin{pmatrix} \epsilon cp, \\ U \\ mc^2 \end{pmatrix}$, *i.e* whose operator enginity is

$$H = \epsilon cp\sigma^1 + U\sigma^2 + mc^2\sigma^3 \tag{4.1}$$

with $U < \sqrt{c^2p^2 + m^2c^4}$.
The search for a solution of the form $\psi = A(p)e^{-\frac{i}{\hbar}(\mathcal{E}t - \boldsymbol{p}\cdot\boldsymbol{x})}$ of the Dirac-Sidharth equation [12]

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi - i\sqrt{\alpha}l\hbar\gamma^5\Delta\psi = 0$$

by using the kronecker product leads to the operator enginity

$$H' = \epsilon cp\sigma^1 - c\sqrt{\alpha}p^2\frac{l}{\hbar}\sigma^2 + mc^2\sigma^3$$

with $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \sigma^1 \otimes \sigma^0$.

Then, following the backward way, from the operator enginity (4.1) we obtain as a Dirac equation for discribing electron in a potential U the following equation

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi + i\frac{U}{c}\gamma^5\psi = 0 \tag{4.2}$$

Because of the presence of γ^5, the parity is violated [13]. Looking for a wave function of the form

$$\psi = A(\boldsymbol{p})e^{-\frac{i}{\hbar}(\mathcal{E}t-\boldsymbol{p}\cdot\boldsymbol{x})}$$

i.e. of the form of (2.2), by using the kronecker product of matrices, the following

$$\psi = \sqrt{\frac{\mathcal{E}+mc^2}{2\mathcal{E}}}\frac{1}{\sqrt{2\left(1+n^3\right)}}\begin{pmatrix}1\\ \frac{-cp+iU}{\mathcal{E}+mc^2}\end{pmatrix}\otimes\begin{pmatrix}-n^1+in^2\\ 1+n^3\end{pmatrix} \times e^{-\frac{i}{\hbar}(\mathcal{E}t-\boldsymbol{p}\cdot\boldsymbol{x})} \quad (4.3)$$

is obtained as solution with positive enginity and negative helicity. The resolution of this equation by using the kroneker product has been given in appendix.

4.2 The Components of the Time Vector and Tunneling Times

The components of the time vector make us think to this one of the controversial issues of modern quantum theory, the question of tunneling time, *i.e.* the time a particle takes to move from one side of a barrier of potential to the other side [14]. Some experimental investigations have supported a nonzero tunneling time, while others supported a zero tunneling time, [15, 16]. However, to give senses to the components of the time vector we have to opt to the nonzero tunneling time and let us consider the case of one dimensional tunneling of an electron through a potential barrier.

Quantum tunneling is a phenomenon in which particles penetrate a potential energy barrier with a height greater than the total energy of the particles. The phenomenon is interesting and important because it violates the principles of classical mechanics. Suppose that an uniform and time-independent beam of electrons with an energy E traveling along the x-axis (in the positive direction to the right) encounters a potential barrier (Figure4.2) described by (See, for instance [17])

$$U(x) = \begin{cases} 0 & \text{when } x < 0 \\ U & \text{when } 0 \leq x \leq L \\ 0 & \text{when } x > L \end{cases}$$

When both the width L and the height U are finite, a part of the incident quantum wave packet on one side of the barrier can penetrate the barrier boundary and continue its motion inside the barrier, where it is gradually attenuated on its way to the other side. A part of the incident quantum wave packet eventually emerges on the other side of the barrier in the form of the transmitted wave packet that tunneled through the barrier. How much of the incident waves can tunnel through a barrier depends on the barrier's width L and its height U , and on the energy E of the incident quantum particle. For such transmitted waves, there are four widely used tunneling times calculated by finding the transmission amplitude given by: $T = |T|\, e^{i\theta}$ [18].The two of them are : Larmor time [19,20], τ_{LM} and Eisenbud-Wigner times [21], τ_{EW}. The first has been called resident or dwell time:

$$\tau_{LM} = -\hbar\frac{\partial\theta}{\partial U} \quad (4.4)$$

The second has been called the passage time,

$$\tau_{EW} = \hbar\frac{\partial\theta}{\partial E} + \frac{L}{k} \quad (4.5)$$

An additional term, L/k is present in τ_{EW}, where L and k are the barrier width and the electron velocity, respectively. This additional term would correspond to the propagation of the electron in the barrier region if that barrier were absent, and has to be added to get the total time [22], since the first term only gives a relative time shift [21].

However, the quantum tunneling phenomena and the consideration of the time vector lead to think

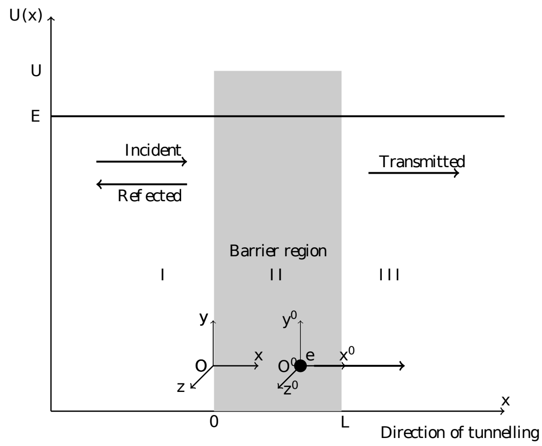

Figure 1: An electron e with kinetic energy E moves along the x -axis and interacts with a rectangular barrier with height U, $U > E$, and width L.

that the Dirac equation inside the potential barrier ($E < U$) is not of the form (4.2) [24]. So, let us construct the wave function of the electron inside the barrier in terms of the components of the time vector. The energy vector $\boldsymbol{E} = \begin{pmatrix} \epsilon cp \\ 0 \\ mc^2 \end{pmatrix}$ before the barrier region becomes $\boldsymbol{\mathcal{E}} = \begin{pmatrix} \epsilon cp \\ U \\ mc^2 \end{pmatrix}$ when the particle is inside the barrier region. The time $\mathcal{T} = \sqrt{\mathcal{T}_3'^2 + \mathcal{T}_1'^2}$ and $\mathcal{T}_2'$ can be qualified respectively as passage time $\mathcal{T}_p = \sqrt{\mathcal{T}_3'^2 + \mathcal{T}_1'^2}$ and resident time $\mathcal{T}_r = \mathcal{T}_2'$. These three types of time, the classical time $\mathcal{T}' = \sqrt{\mathcal{T}_3'^2 + \mathcal{T}_1'^2 + \mathcal{T}_2'^2}$, the passage time $\mathcal{T}_p$ and the resident time $\mathcal{T}_r$ evolve from the entrance to the outrance of the barrier region. But according to the quantum tunneling phenomena, the classical time can not be observed, whereas at least one of the passage time and the resident time can be. Actually,

$$\mathcal{T}' > \mathcal{T}_p \qquad \mathcal{T}' > \mathcal{T}_r$$

All these times evolve from zero to positive values.
Let us search for θ in (4.4) and (4.5) in terms of τ_{LM} and τ_{EW}. From (4.4)

$$\theta = -\frac{1}{\hbar}\tau_{LM}U + K(E)$$

where $K(E)$ is a function of E. Then,

$$\frac{\partial\theta}{\partial E} = K'(E)$$

in substituting in (4.5)

$$K'(E) = \frac{1}{\hbar}\tau_{EW} - \frac{1}{\hbar}\frac{L}{v}$$

Using the relations $p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$ and $E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$ (See for instance, [23]), we have

$$K'(E) = \frac{1}{\hbar}\tau_{EW} - \frac{1}{c\hbar}\frac{E}{\sqrt{E^2 - m^2c^4}}L$$

$$K(E) = \frac{1}{\hbar}\tau_{EW} - \frac{1}{c\hbar}\sqrt{E^2 - m^2c^4}L + \lambda(L)$$

and then,

$$\theta = \frac{1}{\hbar}\left(E\tau_{EW} - U\tau_{LM} - pL\right) + \lambda(L) \tag{4.6}$$

with $\lambda(L)$ independant of E and U, such that $U > E$.
It will be possible to have the value of the constant $\lambda(L)$ if boundary conditions on the phase difference θ are determined. But, according to the couplings (3.8) and (3.10), to give senses to the components of the time vector, the phase which evolves from the phase at $x = 0$ to $x = L$, inside the potential barrier, should be defined as

$$\theta_{II} = -\frac{1}{\hbar}\left(E\sqrt{\mathcal{T}_3'^2 + \mathcal{T}_1'^2} - U\mathcal{T}_2' - px\right) \tag{4.7}$$

with at $x = L$, $\sqrt{\mathcal{T}_3'^2 + \mathcal{T}_1'^2} = \tau_{EW}$ and $\mathcal{T}_2' = -\tau_{LM}$. Then, we have $\lambda(L) = 0$ and for the case of positive enginity and negative helicity: incident, reflected and transmitted wave functions are

$$\psi_I(x) = \begin{pmatrix} 1 \\ \frac{-cp}{E+mc^2} \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-\frac{i}{\hbar}\left(E\sqrt{t_3'^2+t_1'^2}-px\right)} + A\begin{pmatrix} 1 \\ \frac{cp}{E+mc^2} \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-\frac{i}{\hbar}\left(E\sqrt{t_3'^2+t_1'^2}+px\right)} \quad (x<0)$$

$$\psi_{II}(x) = B\begin{pmatrix} 1 \\ \frac{-cp+iU}{\mathcal{E}+mc^2} \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-\frac{i}{\hbar}\left(E\sqrt{\mathcal{T}_3'^2+\mathcal{T}_1'^2}-U\mathcal{T}_2'-px\right)} + C\begin{pmatrix} 1 \\ \frac{cp+iU}{\mathcal{E}+mc^2} \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} \times e^{-\frac{i}{\hbar}\left(E\sqrt{\mathcal{T}_3'^2+\mathcal{T}_1'^2}-U\mathcal{T}_2'+px\right)} \quad (0<x<L) \tag{4.8}$$

$$\psi_{III}(x) = D\begin{pmatrix} 1 \\ \frac{-cp}{E+mc^2} \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} \times e^{-\frac{i}{\hbar}\left(E\sqrt{t_3'^2+t_1'^2}-px-E\tau_{EW}+U\tau_{LM}+pL\right)} \quad (L < x) \tag{4.9}$$

The form of each term of the wave function (4.8) inside the barrier is not like the one that has been thought in [24]. It is a wave function solution, not of $(1+1)$ spacetime Dirac equation, like a particular case of (4.3), but a $(1+2)$ spacetime Dirac equation.
In the case where the energy of the electron is higher than the value of the potential $(E > U)$, the wave function inside the potential will be of the form

$$\psi_{II}(x) = A'\begin{pmatrix} 1 \\ \frac{-cp-iU}{\mathcal{E}+mc^2} \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-\frac{i}{\hbar}\left(\mathcal{E}\sqrt{\mathcal{T}_3'^2+\mathcal{T}_2'^2+\mathcal{T}_1'^2}-px\right)} = A'\begin{pmatrix} 1 \\ \frac{-cp-iU}{\mathcal{E}+mc^2} \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-\frac{i}{\hbar}\left(\mathcal{E}\sqrt{\tau_{EW}^2+\tau_{LM}^2}-px\right)}$$

because according to (3.7) the classical time t in the wave function (4.3) is $t = \sqrt{\mathcal{T}_3'^2 + \mathcal{T}_2'^2 + \mathcal{T}_1'^2}$.
Since the larmor time τ_{LM} is positive and the the second component of the time vector should evolve from 0 to the future, that is $\mathcal{T}_2'$ should be positive, then there is negative energy $-U$ in the phase inside the barrier.

5 Results and Discussion

The first component t'_1 of the time vector which occurs when the electron takes an impulsion is responsible for the time dilation in special relativity, if it is the time of a subluminal velocity. Otherwise, we cannot say what the contribution of t'_1 to the dilation of time is.
Under the hypotheses that the classical time is the magnitude of the time vector and that the energy couples with time under the form of the expressions (3.7) to (3.12), the following results have been obtained, in the Dirac representation, during the tunneling of the electron through a potential barrier. Only for the second component a physical meaning can be given. It can be defined as the Larmor time τ_{LM} or the dwell time, $\mathcal{T}'_2 = \tau_{LM}$. But, due to the minus sign in the equation (4.4), a negative energy $(-U)$ couples with $\mathcal{T}'_2$ in the phase (4.7) inside the barrier. According to the Dirac interpretation of negative energy (See, for example, [25]), can we interpret it that it is not the electron which spends the dwell time but its antiparticle a positron?
It is not possible to give physical senses to the first and the third components. But, the magnitude of the projection of the time vector into the plan of the first and third components can be defined as the Eisenbud-Wigner time τ_{EW} or the passage time, $\sqrt{\mathcal{T}'^2_3 + \mathcal{T}'^2_1} = \tau_{EW}$. Then it follows the following relation

$$\|\boldsymbol{\mathcal{T}}'\| = \sqrt{\tau^2_{EW} + \tau^2_{LM}}$$

between the classical time, the Eisenbud-Wigner time and the Larmor time. Then, we can see that the classical time is higher than the Eisenbud-Wigner time and the Larmor time. Thus,

$$v = \frac{L}{\|\boldsymbol{\mathcal{T}}'\|} < \frac{L}{\tau_{EW}} \quad \text{and} \quad v = \frac{L}{\|\boldsymbol{\mathcal{T}}'\|} < \frac{L}{\tau_{LM}}$$

This show the possibility of supeluminality.

Finally, for a free electron, it is not possible to give senses to t'_3 or t'_1 separately. We think that a possible observability of these two components of the time vector would be in a phenomenon of free superluminal spin-$\frac{1}{2}$ particle, whose wave function would be a solution of a $(3+2)$ spacetime Dirac equation. Thus, this study agrees with the authors of [26,27] who said :"The problem of representation and localizations of superluminal particles has been solved only by the use of higher dimensional space and it has been claimed that the localization space for tachyons is T^4- space with one space and three times". As a consequence, the wave function of a spin-$\frac{1}{2}$ particle inside a potential barrier would be of the form $\Phi_{II} = Ae^{\epsilon P\mathcal{T}'_1 - U\mathcal{T}'_2 + mc^2\mathcal{T}'_3}$, if the particle was a free superluminal spin-$\frac{1}{2}$ particle before the potentiel barrier U, $U > \sqrt{P^2 + m^2c^4}$ [24].

Conclusion

The energy vector in the Dirac theory came out when the purpose was to show the analogy between helicity and enginity. This energy vector needs a time vector. A component of the time vector appears when the electron takes an impulsion. This component time vector is responsible of time dilation in special relativity.
Under some hypotheses, during the tunneling of the electron through a potential barrier, one component can be defined as the Larmor time, whereas the Eisenbud-Wigner time is a combination of the two other components. Then, a relation between the classical time, the Larmor time and the Eisenbud-Wigner time has been obtained. When the electron is inside the barrier, in the phase of the wave function a negative energy couple with the Larmor time. We interpret it that it is the antiparticle which spends the Larmor time.

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Appendix A. Resolution of the Dirac equation $i\hbar\gamma^{\mu}\partial_{\mu}\psi - mc\psi + i\frac{U}{c}\gamma^5\psi = 0$

To make appear the operator enginity we use, for solving the equation (4.2), the kronecker product. Put this solution of the form

$$\breve{\Psi} = A(\boldsymbol{p})e^{-\frac{i}{\hbar}(\mathcal{E}t - \boldsymbol{p}.\boldsymbol{x})}$$

Then,

$$i\hbar\sigma^3 \otimes \sigma^0 \frac{\partial}{c\partial t}\breve{\Psi} + i\hbar i\sigma^2 \otimes \sigma^j \frac{\partial}{\partial x_j}\breve{\Psi} - mc\breve{\Psi} + i\frac{U}{c}\sigma^1 \otimes \sigma^0\breve{\Psi} = 0$$

$$i\hbar\sigma^0 \otimes \sigma^0 \frac{\partial}{\partial t}\breve{\Psi} + ic\hbar\sigma^1 \otimes \sigma^j \frac{\partial}{\partial x_j}\breve{\Psi} - mc^2\sigma^3 \otimes \sigma^0\breve{\Psi} - U\sigma^2 \otimes \sigma^0\breve{\Psi} = 0$$

Let $\hat{p}^j = -i\hbar\frac{\partial}{\partial x_j}$ the mometum operator in the direction of x_j and $\hat{\mathcal{E}} = i\hbar\frac{\partial}{\partial t}$ the energy operator, $\hat{p}^j\breve{\Psi} = p^j\breve{\Psi}$ and $\hat{\mathcal{E}}\breve{\Psi} = \mathcal{E}\breve{\Psi}$. Then, we have

$$\mathcal{E}\sigma^0 \otimes \sigma^0 A(\boldsymbol{p}) - cp_j\sigma^1 \otimes \sigma^j A(\boldsymbol{p}) - mc^2\sigma^3 \otimes \sigma^0 A(\boldsymbol{p}) - U\sigma^2 \otimes \sigma^0 A(\boldsymbol{p}) = 0$$

But let us search for a solution where $A(\boldsymbol{p})$ is of the form $A(\boldsymbol{p}) = \varphi \otimes u$ with u is an eigenvector of the helicity operator $\left(\frac{\hbar}{2}\boldsymbol{\sigma}\cdot\boldsymbol{n}\right) = \left(\frac{\hbar}{2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p}\right)$, whose eigenvalues are the helicities $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.

$$\mathcal{E}\sigma^0 \otimes \sigma^0\varphi \otimes u - \frac{2}{\hbar}cp\sigma^1 \otimes \left(\frac{\hbar}{2}\boldsymbol{\sigma}\cdot\boldsymbol{n}\right)\varphi \otimes u - mc^2\sigma^3 \otimes \sigma^0\varphi \otimes u - U\sigma^2 \otimes \sigma^0\varphi \otimes u = 0$$

$$\mathcal{E}\varphi \otimes u - cp\sigma^1\varphi \otimes \epsilon u - mc^2\sigma^3\varphi \otimes u - U\sigma^2\varphi \otimes u = 0$$

or

$$\left(\mathcal{E}\varphi - \epsilon cp\sigma^1\varphi - mc^2\sigma^3\varphi - U\sigma^2\varphi\right) \otimes u = 0$$

where ϵ is the sign of helicity. However, $u \neq 0$, thus

$$\mathcal{E}\varphi = \left(\epsilon cp\sigma^1 + mc^2\sigma^3 + U\sigma^2\right)\varphi \tag{5.1}$$

That is φ is the eigenvector of the operator

$$h_D = \epsilon cp\sigma^1 + mc^2\sigma^3 + U\sigma^2 = \boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{E}}$$

in the Dirac representation, whose eigenvalues are the negative energy $-\mathcal{E}$ and the positive energy $+\mathcal{E}$. The enginity operator $\frac{\hbar}{2}\frac{h_D}{\mathcal{E}} = \frac{\hbar}{2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{E}}}{\mathcal{E}}$ in the direction of the energy vector $\boldsymbol{\mathcal{E}}$ will be the analogous of the helicity operator.

The resolution will be finished in solving the equation (5.1) and the equation

$$\left(\sigma^1 n^1 + \sigma^2 n^2 + \sigma^3 n^3\right) u = \epsilon u$$