Modification on Restrictive Taylor and Padé approximations

Abstract:

We review~~In the following paper we first provide an over view of~~ Taylor approximation (TA), Padé approximation (PA), Restrictive Taylor approximation (RTA) and Restrictive Padé approximation (RPA). After~~Then we~~ comparinged~~ed~~ these four approximation methods with~~ other~~ two other modified approximation methods: Modified Restrictive Taylor approximation(MRTA)) and Modified Restrictive Padé approximation (MRPA), we give test examples to illustrate how the modified approximations could be used. The mathematical principles behind all these approximations could be applied for the development of new computing methods~~.~~

Keyword**:** Taylor polynomial, Padé approximation, Restrictive Taylor approximation, Restrictive Padé approximation, Modified Restrictive Taylor approximation and Modified Restrictive Padé approximation.

1. Introduction

The relationship between the coefficients of a~~the Taylor~~ series expansion and of~~ a function and~~ the value~~s~~ of the function is both important~~a profound~~ mathematically~~question~~ and ~~an important~~significant practically~~ one~~. It is basic to the study of mathematical analysis, and to the practical calculation of mathematical models of nature throughout much of applied sciences and engineering problems, for example, electrical and dynamic System's transfer function, biology, physics, chemistry, medicine. Padé approximant[1,2,~~ and~~ 3] is ~~a~~ typical~~e~~ rational function of a given order.~~,~~ U~~u~~nder this technique, the approximant's power series agrees with the power series of the function it is approximating. The technique was first developed around 1890 by Henri ~~Padé, but~~Padé but goes back to Georg Frobenius who introduced the idea and investigated the features of rational approximations of power series. This technique is better than rational function [4].

Padé approximant was further developed~~introduced~~ by Brezinski[5]~~]~~;~~,~~Khan[6] used Padé –Hermite approximant;~~,~~ the convergence of multipoint Padé -type approximants ~~was~~were shown by Ysern and Lagomasino[7].

Restrictive Taylor approximation introduced by Ismail and Elbarbary to solve Parabolic Partial Differential Equations [8],~~,~~ Ismail et al. approximated a solution for

Convection –Diffusion equation and KdV-Burgers equation [9,10], and Rageh et al. [11] used RTA to solve Gardner and KdV equations.

Ismail et al. [13,14,15] applied Restrictive Padé approximation to solve Schrodinger equation and Generalized Burger's equation. Ismail [16] study the convergence of (RPA)to the exact solutions of IBVP of parabolic and hyperbolic types. Ismail [17] introduced solvability and uniqueness for both (RTA) and (RPA).

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2. Classsicaland Restrictiveand Restrictive Approximations

2.1 Taylor and Padé Approximations

Taylor series is a polynomial of infinite degree that can be used to represent many different functions, particular functions that are not polynomial. This means that Taylor expansion is a method through which we can convert a non-polynomial function into a polynomial function. Taylor approximation is useful and preferably to be useddesirable thing to do, since polynomials are easier to be evaluated at some particularvalues.particular valuesIt is alsoand easier to differentiate and integrate. Then the Taylor series of a real function $f(\varepsilon)$ about a point a is given by

$$f(\xi)=f(\mathrm{a})+\frac{(\xi-\mathrm{a})}{1!}f'(\mathrm{a})+\frac{(\xi-\mathrm{a})^2}{2!}f''(\mathrm{a})+\cdots \tag{1}$$

wWhere;

$a\in R$and$f(\xi)$ is infinity differentiable function on the interval I that containsing the pPointa.

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For evaluating finite number of terms, the Taylor polynomial of degree n is defined by

$$f(\xi)=\sum_{k=0}^{n}\frac{(f(\xi-a)^k f^{(k)}(a)}{k!}+R_n \tag{2}$$

Note 1: remove ‘ (f ’ from the above equation.

wWhere R_n is the e Lagrange remainder term are given by

$$R_n(\xi)=\frac{(\xi-a)^{n+1}}{(n+1)!}f^{(n+1)}(\varepsilon),\qquad a\le\varepsilon\le\xi \tag{3}$$

So that the maximum error after n terms of Taylor expansion is the maximum value obtained from equation (3) running through all $\xi\in[a,\varepsilon]$ $\varepsilon\in[a,\zeta]$

Padé Approximation~~is an~~ approximates~~ion of~~ a function using the rational polynomials. Since the Taylor series use repeated differentiation to produce a polynomial approximation of a function then, Taylor series often ~~can't~~cannotbe used to extrapolate the function for very long before becoming rapidly diverging. Padé approximants often follow the function more closely for longer. This technique was developed around 1890 by Henri Padé(1863-1953) ~~$(1863-1953)$, he introduce what is known as Padé approximant.~~

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For a given function $f(\xi)$ and two integers $m \geq 0$ and $n \geq 1$ the Padé approximant (PA) of order $[m/n]$ is

$$f_{[m/n]}(\xi) = \quad = \frac{a_{0+}a_1\xi + a_2\xi^2 + \cdots + a_M\xi^m}{b_{0+}b_1\xi + b_2\xi^2 + \cdots + b_N\xi^n} \tag{5}$$

Since PA are rational functions, with a denominator that ~~doesn't~~does not vanish at zero, and whose series expansion matches a given series as far as possible, then its coefficients are determined from the condition

$$(b_{0+}b_1\xi + b_2\xi^2 + \cdots + b_n\xi^n)(c_0 + c_1\xi + c_2\xi^2 + \cdots)$$
$$= a_0 + a_1\xi + a_2\xi^2 + \cdots + a_m\xi^m + O(\xi^{m+n+1})$$

Withcoefficient condition$b_0 = 1$, then

$$a_0 = c_0$$
$$a_1 = c_1 + b_1 c_0$$

$$a_m = c_m + \sum_{k=1}^{p} b_i\, c_{m-k}$$

$$a_m = c_m + \sum_{k=1}^{p} b_i\, c_{m-k}$$

Where$p = \min(m, n)$, and the coefficients c_i are obtained from the expansion of the function $f(\xi)$in Maclaurin series.

2.2 Restrictive Type Approximations

~~2.2~~

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In the following pages, we describe two forms for "***Restrictive***" type approximations in which we find the restrictive parameter that gives the exact solution at a given number of points in some type of problems.

Restrictive Taylor Approximation (RTA)

Consider a function $f(x)$ defined in a neighborhood of $x = a$ and it has derivatives up to order $(n+1)$(n + 1).

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Constructing a function

$$RT_{n,f(x)}(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{\varepsilon(x-a)^n}{n!} f^{(n)}(a) \quad (14)$$

Note 2: All equation numbers need to be changed from (14) onward, e.g. (14) change to (6) …etc…

wWhere ε is a parameter to be determined by adding the following condition[9,10 ,11]:

$$RT_{n,f}(x_a) = f(x_a)$$

Some points x_a Let x_a be some points in the domain of the function f. The function $RT_{n,f}(x)$ is called restrictive Taylor approximation of order n of the function $f(x)$ at the point $x = a$. The following theorem gives the value of the remainder term of this approximation.

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Assume the function $f(x)$ and its derivatives up to an order $n+1$ are continuous in a certain neighborhood of a point. Suppose, furthermore, that x is any value of the argument from the indicated neighborhood and ε is the restrictive Taylor parameter, then there is a point ξ which lies between the points a and x such that the formula:

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$$f(x) = RT_{n,f}(x) + \mathcal{R}_{n+1}(x, \varepsilon(x)) \quad (15)$$

is true, for which

$$\mathcal{R}_{n+1}(x, \varepsilon(x)) = \frac{\varepsilon(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) - \frac{n(\varepsilon-1)^{n+1}(x-a)^{n+1}}{(n+1)!\,(x-\xi)} f^{(n)}(\xi) \quad (16)$$

$,\xi \in [a, x]$

where $\mathcal{R}_{n+1}(x, 1)$ is the Taylor remaindear term.

Restrictive Padé Approximation (RPA)

We construct a restrictive type of Padé approximation [13, 14] of the function $f(x)$ with a parameter yet to be determined~~, which i~~.If it reduces the remainder term to zero, we will get the classical Padé approximation. The restrictive Padé approximation is a rational function in the form [12]:

$$RPA[M+\alpha/N]_{f(x)}(x)=\frac{\sum_{i=0}^{M}a_i x^i+\sum_{i=1}^{\alpha}\varepsilon_i x^{M+i}}{1+\sum_{i=1}^{N}b_i x^i} \tag{17}$$

where the positive integer α does not exceed the degree of the numerator,

$\alpha=0(1)n$

Such that

$$f(x)=RPA[M+\alpha/N]_{f(x)}(x)+O(X^{M+N+1}) \tag{18}$$

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and let f(x) has a Maclaurin series expansion

$[\alpha/0]$	$[\alpha/1]$	$[\alpha/2]$	...	$[\alpha/N]$	...
$[\alpha+1/0]$	$[\alpha+1/1]$	$[\alpha+1/2]$	...	$[\alpha+1/N]$	...
$[\alpha+2/0]$	$[\alpha+2/1]$	$[\alpha+2/2]$	...	$[\alpha+2/N]$	...
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	...
$[\alpha+M/0]$	$[\alpha+M/1]$	$[\alpha+M/2]$	...	$[\alpha+M/N]$	...
$\vdots$	$\vdots$	$\vdots$		$\vdots$	

Table 1 Restrictive Padé table

~~In t~~Table1~~ 2 we represent the Restrictive Padé, It~~ is ~~called~~ the restrictive Padé table for $f(x)$.~~,t~~The first jth columns disappear when $\alpha>j-1$.~~t~~The case $\alpha=0$ gives the ~~c~~ classical Padé table 1.

The case $\alpha=1$ gives the following selected elements of RPA table:

$$RPA[1/1]_{f(x)}(x)=\frac{a_0+a_1 x}{1+b_1 x}$$

where

$$a_0 = c_0$$
$$a_1 = \varepsilon_1$$
$$b_1 = \frac{\varepsilon_1 - c_1}{c_0}$$

$$RPA[2/1]_{f(x)}(x) = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x}$$

where

$$a_0 = c_0$$
$$a_1 = c_1 + \frac{c_0 + (\varepsilon_1 - c_2)}{c_1}$$
$$a_2 = \varepsilon_1$$
$$b_1 = \frac{\varepsilon_1 - c_2}{c_1},$$

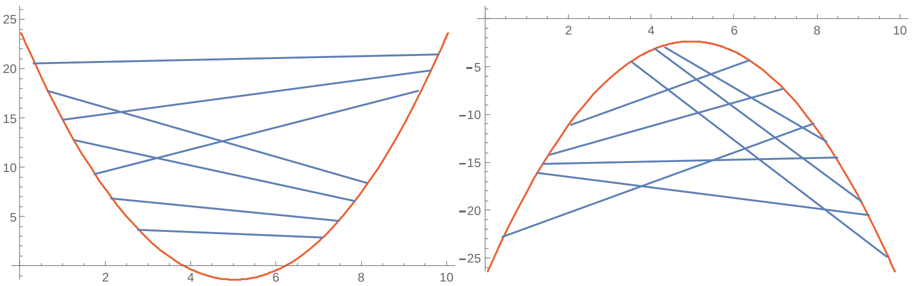
$$RPA[3/1]_{f(x)}(x) = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x}$$

where

$$a_0 = c_0$$
$$a_1 = c_1 + \frac{c_0 + (\varepsilon_1 - c_2)}{c_1}$$
$$a_2 = \varepsilon_2 + \frac{c_1(\varepsilon_1 - c_3)}{c_2}$$
$$a_3 = \varepsilon_1$$
$$b_1 = \frac{\varepsilon_1 - c_3}{c_2}$$

3. Concave and Convex functions

We use ~~C~~concave and convex ~~are words that we use~~ to describe the shape or the curvature of the curve. The ~~twin~~ definitions concavity and convexity of a single variable function are widely used in economic ~~theory, and~~theory and are also central to optimization theory. The function $f(x)$ is said to be a concave function if every line segment joining any two points on the curve of this function lie below the graph at any point. On the other hand, a function $f(x)$ is said to be convex if every line segment joining two points on the curve of the function lie above the graph at any point. S~~s~~ee Fig. 1. below:



~~A convex function: no line segment joining A concave function: no line segment two points on the graph lies below the joining two points on the graph lies above graph. the graph.~~

Fig. 1. Convex (on left) and Concave (on right) functions.

We can easily conclude that the graph of any differentiable concave function lies below the tangent at point x_0, which its slope is $f'(x_0)$. ~~Similarly~~Similarly, the graph of any differentiable convex function lies above the tangent at point x_0, which its slope is $f'(x_0)$. ~~Thus~~Thus, the definition of the concave and convex function may be written as follows:

Definition:

Let f is a single variable function defined on the interval I. Then the function f is

Concave function if for all $x \in I, all\ y \in I$ and $\lambda \in [0,1]$ we have $f($ $)$ λy λf $x \in I, all\ y \in I$ $\lambda \in [0,1]$

$($ λ $)$ $($ $\lambda)$ $($ $)$

$($ λ $f($ $)$ λy f $\lambda f(y)$

Convex function if for $x \in I, all\ y \in I$ all and we have

$1 +$ $x +$ $\geq$ $1 +$ x $+$

Both convex and concave and convex function if for ~~all and~~all and $\lambda \in [0,1]$ we have

$$f((1+\lambda)x+\lambda y)=(1+\lambda)f(x)+\lambda f(y)$$

~~Thus~~Thus, the function is both convex and concave if and only if it is linear function.

In this paper our approximated functions (RTA& RPA) will be classified into either concave or convex function. We are looking forward to ~~reduce~~reducing the maximum norm error L_∞ resulting from our approximated functions (RTA& RPA) my using this classification as we will show in the next part.

4. Modification in Restrictive Taylor and Padé Approximation

After we use~~approximate~~ the restrictive approximation we find the more~~ of~~ its convergesn~~ce~~, ~~we calculate~~ the larger the maximum L_∞ error norm we obtain.~~which means the maximum error,The~~the new modification is to eliminate this max error and to force the absolute error to be near zero around the region that has the maximum error.

From the analysis of error and plotting of the absolute error between the function needed to be approximated and the restrictive approximation, we can classify the modification intofor two types according to whether if the function is be convex or concave.

Case 1: Convex Functions

From Fig. 2.b, the RA lies aboveup the function so the error will be positivenegative; so we subtract the modification term.

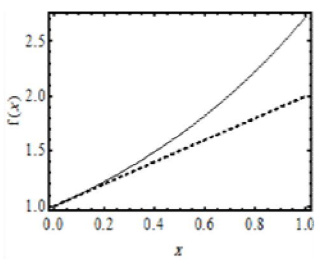


Fig. 2.a

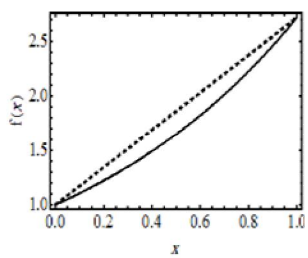


Fig. 2.b

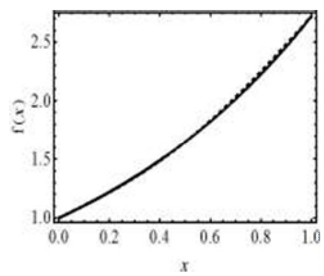


Fig. 2.c

Case 2: Concave Functions

From Fig. 3.b the RA lies below the function so the error will be negative; so we will adding the modification term.

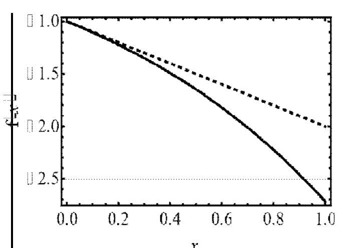


Fig. 3.a

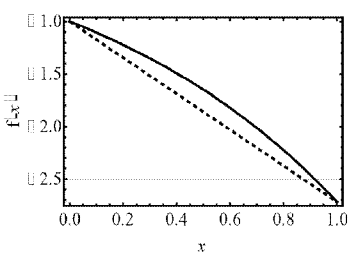


Fig. 3.b

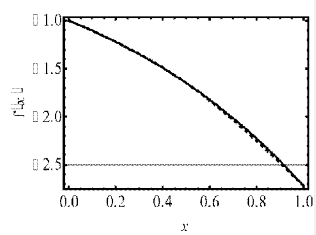


Fig. 3.c

We choose the modification term to satisfy three conditions:

1) At $x = 0$, the modification term vanishes (equal zero). See point A in Fig. 4
2) At the point B, have the maximum error (L_∞ error norm) the modification term has maximum value and equal to the L_∞ error norm;thus forced the new approximations is used to force the error to be near zero around the point having e the L_∞ error norm.
3) At x_a where we calculate the restrictive term the error equal to zerozero, so we does not need to the modification term at point C in Fig. 4

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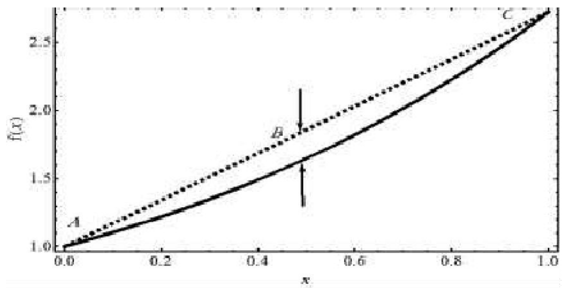

Fig. 4

5. Numerical Examples

In this section we introduce two three examples solved by Taylor approximation (TA), Padé approximation (PA), Restrictive Taylor approximation (RTA) and Restrictive Padé approximation (RPA). Then comparing this result by the two Modified restrictive Taylor approximation (MRTA) and Restrictive Padé approximation (MRPA).,finally graph bothFig. 5 – 6 show the function used together with the expansions and errors.

Example 1.

$$f(x) = Exp(cx) \tag{20}$$

For $c = 1$ the restrictive terms are calculated at For $x = 1$ for both RTA and RPA. We limit expand the expanstion domain to [0, 1] and plot TA, PA, RTA,RPA, MRTA and MRPAand in MRPA in Fig. 5. Also, the errors are in Fig. 6.

$$_1 = 1 + x$$

$$R\ _1 = 1 + \varepsilon x$$

wWhere the ε is the restrictive term to be determine by

$$RTA_1 = 1 + 1.718281828459045x$$

	TA_1	RTA_1	$MRTA_1$
L_2	1.0915436502789486	0.488301679070951	0.039990197968343945
L_∞	0.7182818284590451	0.21041964352939435	0.02540297101555078
	Table 2 error norm for Ttaylor expanstions for ex 1		

$$PA_1 = \frac{1+\frac{x}{2}}{1-\frac{x}{2}} RPA_1 = \frac{1+\varepsilon x}{1-b_1 x}$$

wWhere the ε is the restrictive term to be determine by

$$RPA_1 = \frac{1 + 0.5819767068693265x}{1 - 0.41802329313067355x}$$

$$MRPA_1 = RTA_1 + L_\infty \operatorname{Sin}(x)$$

$$MRPA_1 = \frac{1 + 0.5819767068693265x}{1 - 0.41802329313067355x} + 0.02419074111857622 \operatorname{Sin}(x)$$

	PA1	RPA1	MPTA1
L_2	0.35755397907149267	0.0483757253399784844	0.028051263981545418
L_∞	0.2817181715409549	0.02419074111857622	0.01522650542301296

Table 3 Eerror norm for Pade expanstions for x 1

x	TA_1	TA_2	RTA_1	RTA_2	$MRTA_1$	$MRTA_1$
0	0	0	0	0	0	0
0.1	0.005170918075	0.00017091807564	0.066657264770	0.00201190020894	0.00163401896935	0.009794214106360721
0.2	0.021402758160	0.00140275816016	0.122253607531	0.00732851497819	0.00142795572757	0.015128048925197923
0.5	0.148721270700	0.02372127070012	0.210419643529	0.03084918641463	0.	0.007356202059822792
0.7	0.313752707470	0.06875270747047	0.189044572450	0.03820538847445	0.01881150488525	0.007296579921924451
0.9	0.559603111156	0.15460311115695	0.086850534456	0.02220516989487	0.02182728865528	0.010399055579572991
1	0.718281828459	0.21828182845904	0	0	0	0

Table 4. Eerrorinfor example 1

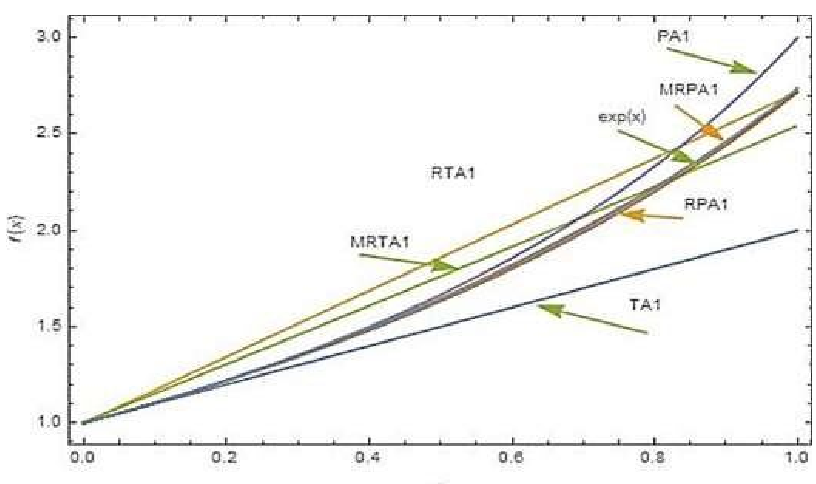

Fig. 5. Expansion of example 1

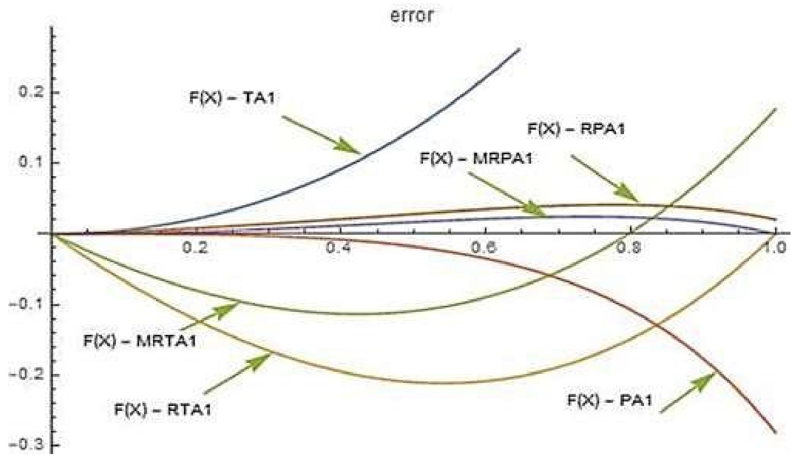

Fig. 6. The errors inof example 1

Example 2.

$$f(x) = \sqrt{\frac{1 + \frac{1}{2}x}{1 + 2x}} \tag{21}$$

Baker and Morris [1] have shown the accurate of PA via TA in this example. Wwe introduce a beatter methods (RTA and RPA) with results as shown in Fig. 7 and the error in Fig. 8.

$$TA_1 = 1 - 0.75\,x RTA_1 = 1 + \varepsilon x$$

Where the ε is the restrictive term to be determine by

$$RTA_1 = 1 - 0.2928932188x MRTA_1 = RTA_1 - L_\infty \operatorname{Sin}(x)$$

$$MRTA_1 = 1 - 0.2928932188x - 0.06634613154689295 \operatorname{Sin}(x)$$

	TA1	RTA1	MRTA1
L_2	0.8045311072982965	0.1522780428123673	0.02842734952458836
L_∞	0.4571067811865476	0.06634613154689295	0.016018818139471036
	Table 5 error norm for Taylor expanstions for example 2		

	PA1	RPA1	MRPA1
L_2	0.012674300329633452	0.002665666175542447	0.0001662632781343035
L_∞	0.007178933099166618	0.001204511854803969	0.00008268011675605091

Table 6.E~~e~~rror norm for Pade expans~~t~~ions in~~for~~ example 2

$$PA_1 = \frac{1 + 0.875x}{1 + 1.625x}$$

$$RPA_1 = \frac{1 + \varepsilon x}{1 - b_1 x}$$

Where the ε is the restrictive term to be determine by

$$RPA_1 = \frac{1 + 0.8106601718x}{1 + 1.560660172x}$$

$$MRPA_1 = RTA_1 + L_\infty \, \mathrm{Sin}(x)$$

$$MRPA_1 = \frac{1 + 0.8106601718x}{1 + 1.560660172x} + 0.001204511854803969 \, \mathrm{Sin}(x)$$

X	TA1	TA2	RTA1	RTA2	MRTA1	MRTA1
0	0	0	0	0	0	0
0.1	0.0104143466	0.001773153306	0.03529633142	0.005843278881	0.01479424926	0.01088927245
0.2	0.0364052604	0.01234473957	0.05501609580	0.01812098918	0.01601881813	0.01370621478
0.5	0.1655694150	0.13911808495	0.06298397555	0.05129271974	0.00336215599	0.00285495381
0.7	0.275	0.32218750000	0.04497474683	0.05101767721	0.008700401101	0.00721128910
0.9	0.3946229171	0.59256458287	0.01677318593	0.02436642436	0.00372889622	0.00763387303
1	0.4571067811	0.76164321881	$1.11022302462 \times 10^{-16}$	$1.11022302462 \times 10^{-16}$	$1.11022302462 \times 10^{-16}$	$1.11022302462 \times 10^{-16}$

Table 7.E~~e~~rrorin~~for~~ example 2

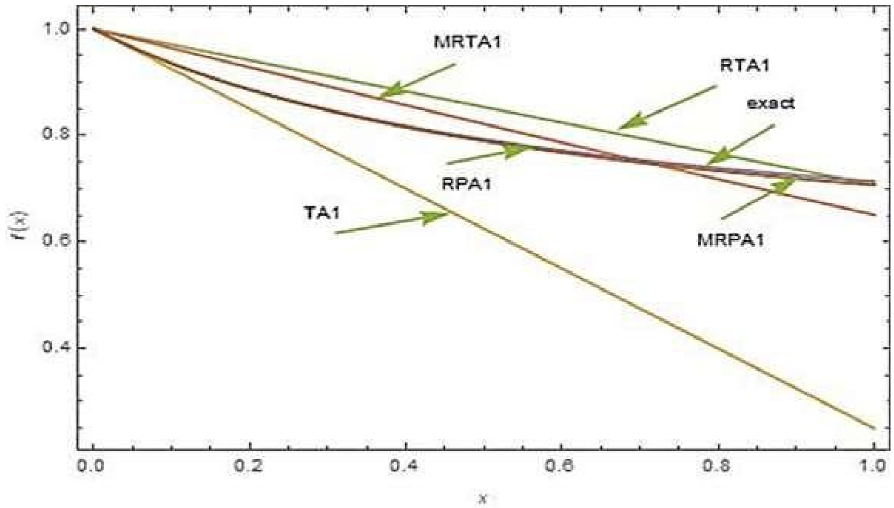

Fig.7. Expansion of example 2

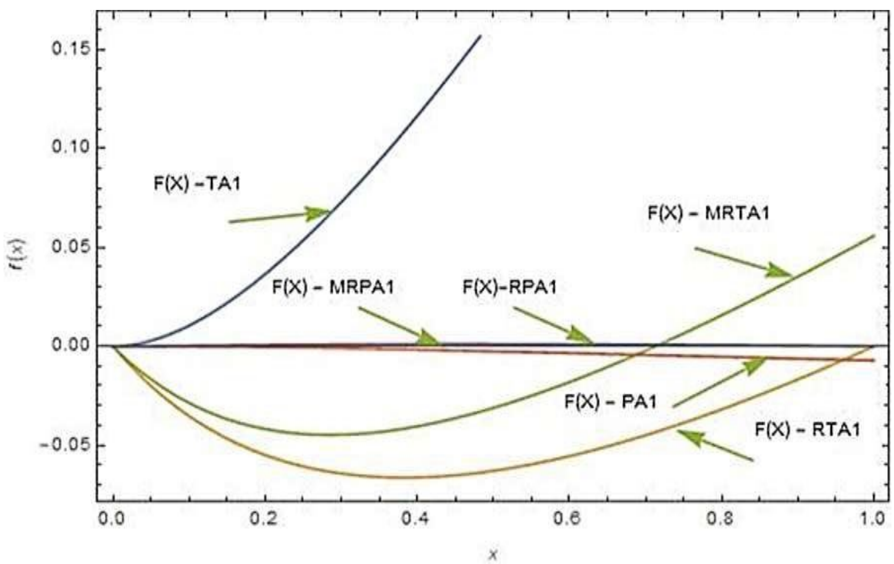

Fig.8. ~~the e~~Errors in ~~of~~ example 2

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6. ~~Results and~~ Conclusion

The numerical examples show better~~the high~~ accurate for~~of~~ Modified Restrictive Taylor approximation (MRTA) and Modified Restrictive Padé approximation (MRPA) than Restrictive Taylor approximation (RTA) and Restrictive Padé approximation (RPA~~).We~~). We know that TA and PA gives the exact solution at $x = 0$ also at the point used to ~~of~~ calculate the restrictive ~~term as~~term as shown in Fig 2.b or Fig 3.b. As we show in example 1 and ~~&~~2 (Fig 5,6,~~.~~7 and 8) we use~~force~~ RTA and RPA to force the approximation to be close to the exact curve by adding modified terms so that ~~to decrease~~ the error is close to zero.

The mathematical principles behind all these approximations and their modifications could be applied for the development of new computing methods.

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