Modification on Restrictive Taylor and Padé approximations

Abstract:

In the following paper we first provide an over view of Taylor approximation (TA), Padé approximation (PA), Restrictive Taylor approximation (RTA) and Restrictive Padé approximation (RPA). Then we compared these four approximation methods with other two modified approximation methods: Modified Restrictive Taylor approximation(MRTA)) and Modified Restrictive Padé approximation (MRPA).

***Keyword*:** Taylor polynomial, Padé approximation, Restrictive Taylor approximation, Restrictive Padé approximation, Modified Restrictive Taylor approximation and Modified Restrictive Padé approximation.

1. Introduction

The relation between the coefficients of the Taylor series expansion of a function and the values of the function is both a profound mathematical question and an important practical one. It is basic to the study of mathematical analysis, and to the practical calculation of mathematical models of nature throughout much of applied sciences and engineering problems, for example, electrical and dynamic System's transfer function, biology, physics, chemistry, medicine. Padé approximant[1,2 and 3] is a type rational function of given order, under this technique, the approximant's power series agrees with the power series of the function it is approximating. The technique was developed around 1890 by Henri Padé, but goes back to Georg Frobenius who introduced the idea and investigated the features of rational approximations of power series. This technique better than rational function [4].

Padé approximant was introduced by Brezinski[5] ,Khan[6] used Padé –Hermite approximant, the convergence of multipoint Padé -type approximants was shown by Ysern and Lagomasino[7].

Restrictive Taylor approximation introduced by Ismail and Elbarbary to solve Parabolic Partial Differential Equations [8], Ismail et al. approximated a solution for Convection –Diffusion equation and KdV-Burgers equation [9,10] and Rageh et al. [11] used RTA to solve Gardner and KdV equations.

Ismail et al. [13,14,15] applied Restrictive Padé approximation to solve Schrodinger equation and Generalized Burger's equation. Ismail [16] study the convergence of (RPA) to the exact solutions of IBVP of parabolic and hyperbolic types. Ismail [17] introduced solvability and uniqueness for both (RTA) and (RPA).

2. Clasic and Restrictive Approximation

2.1 Taylor and Padé Approximation

Taylor series is a polynomial of infinite degree that can be used to represent many different functions, particular functions that are not polynomial. This means that Taylor expansion is a method through which we can convert a non-polynomial function into polynomial function. Taylor approximation is useful and desirable thing to do, since polynomials are easier to evaluate at particular values and easier to differentiate and integrate. Then the Taylor series of a real function $f(\varepsilon)$ about a point a is given by

$$f(\xi) = f(\mathrm{a}) + \frac{(\xi - \mathrm{a})}{1!} f'(\mathrm{a}) + \frac{(\xi - \mathrm{a})^2}{2!} f''(\mathrm{a}) + \cdots \tag{1}$$

Where;

$a \in R$ and $f(\xi)$ is infinity differentiable function on the interval I containing the pointa.

For evaluating finite number of terms, the Taylor polynomial of degree n is defined by

$$f(\xi) = \sum_{k=0}^{n} \frac{(f(\xi - a)^k f^{(k)}(a)}{k!} + R_n \tag{2}$$

Where R_n is the Lagrange remainder term are given by

$$R_n(\xi) = \frac{(\xi - a)^{n+1}}{(n+1)!} f^{(n+1)}(\varepsilon), \qquad a \leq \varepsilon \leq \xi \tag{3}$$

So that the maximum error after n terms of Taylor expansion is the maximum value obtained from equation (3) running through all $\xi \in [a, \varepsilon]$.

Padé Approximation is an approximation of a function using the rational polynomials. Since the Taylor series use repeated differentiation to produce a polynomial approximation of a function then, Taylor series often can't extrapolate the function for very long before rapidly diverging. Padé approximants often follow the function more closely for longer. This technique was developed around 1890 by Henri Padé $(1863 - 1953)$, he introduce what is known as Padé approximant.

For a given function $f(\xi)$ and two integers $m \geq 0$ and $n \geq 1$ the Padé approximant of order $[m/n]$ is

$$f_{[m/n]}(\xi) = \quad = \frac{a_{0+}a_1\xi + a_2\xi^2 + \cdots + a_M\xi^m}{b_{0+}b_1\xi + b_2\xi^2 + \cdots + b_N\xi^n} \tag{5}$$

Since PA are rational functions, with a denominator that doesn't vanish at zero, and whose series expansion matches a given series as far as possible, then its coefficients are determined from the condition

$$(b_{0+}b_1\xi + b_2\xi^2 + \cdots + b_n\xi^n)(c_0 + c_1\xi + c_2\xi^2 + \cdots)$$
$$= a_0 + a_1\xi + a_2\xi^2 + \cdots + a_m\xi^m + O(\xi^{m+n+1})$$

With coefficient condition $b_0 = 1$, then

$$a_0 = c_0$$
$$a_1 = c_1 + b_1c_0$$

$$a_m = c_m + \sum_{k=1}^{p} b_i\, c_{m-k}$$

Where $p = \min(m, n)$, and the coefficients c_i are obtained from the expansion of the function in Maclaurin series.

2.2 Restrictive Type Approximations

In the following pages, we describe two forms for "***Restrictive***" type approximations in which we find the restrictive parameter that gives the exact solution in some type of problems.

Restrictive Taylor Approximation (RTA)

Consider a function $f(x)$ defined in a neighborhood of $x = a$ and it has derivatives up to order $(n + 1)$

Constructing a function

$$RT_{n,f(x)}(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{\varepsilon(x-a)^n}{n!}f^{(n)}(a) \tag{14}$$

Where ε is a parameter to be determined by adding the following condition[9,10 , 11]:

$$RT_{n,f}(x_a) = f(x_a)$$

Some points x_a in the domain of the function f. The function $RT_{n,f}(x)$ is called restrictive Taylor approximation of order n of the function $f(x)$ at the point $x = a$. The following theorem gives the value of the reminder term of this approximation.

Assume the function $f(x)$ and its derivatives up to an order $n + 1$ are continuous in a certain neighborhood of a point. Suppose, furthermore, that x is any value of the argument from the indicated neighborhood and ε is the restrictive Taylor parameter, then there is a point ξ which lies between the points a and x such that the formula:

$$f(x) = RT_{n,f}(x) + \mathcal{R}_{n+1}(x, \varepsilon(x)) \quad (15)$$

is true, for which

$$\mathcal{R}_{n+1}(x, \varepsilon(x)) = \frac{\varepsilon(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) - \frac{n(\varepsilon-1)^{n+1}(x-a)^{n+1}}{(n+1)!\,(x-\xi)} f^{(n)}(\xi) \quad (16)$$

$$, \xi \in [a, x]$$

where $\mathcal{R}_{n+1}(x, 1)$ is the Taylor remindar term.

Restrictive Padé Approximation (RPA)

We construct a restrictive type of Padé approximation [13, 14] of the function $f(x)$ with parameter to be determined, which if it reduces to zero, we will get the classical Padé approximation. The restrictive Padé approximation is a rational function in the form [12]:

$$RPA[M + \alpha/N]_{f(x)}(x) = \frac{\sum_{i=0}^{M} a_i x^i + \sum_{i=1}^{\alpha} \varepsilon_i x^{M+i}}{1 + \sum_{i=1}^{N} b_i x^i} \quad (17)$$

where the positive integer α does not exceed the degree of the numerator,

$\alpha = 0(1)n$

Such that

$$f(x) = RPA[M + \alpha/N]_{f(x)}(x) + O(X^{M+N+1}) \quad (18)$$

and let f(x) has a Maclaurin series expansion

$[\alpha/0]$	$[\alpha/1]$	$[\alpha/2]$	…	$[\alpha/N]$	…
$[\alpha+1/0]$	$[\alpha+1/1]$	$[\alpha+1/2]$	…	$[\alpha+1/N]$	…
$[\alpha+2/0]$	$[\alpha+2/1]$	$[\alpha+2/2]$	…	$[\alpha+2/N]$	…
⋮	⋮	⋮	⋱	⋮	…
$[\alpha+M/0]$	$[\alpha+M/1]$	$[\alpha+M/2]$	…	$[\alpha+M/N]$]	…
⋮	⋮	⋮		⋮	

Table 1 Restrictive Padé table

In table 2 we represent the Restrictive Padé, It is called the restrictive Padé table for $f(x)$, the first j<u>th</u> columns disappear when $\alpha > j-1$. the case $\alpha = 0$ gives the classical Padé table 1.

The case $\alpha = 1$ gives the following selected elements of RPA table:

$$RPA[1/1]_{f(x)}(x) = \frac{a_0 + a_1 x}{1 + b_1 x}$$

where

$$a_0 = c_0$$
$$a_1 = \varepsilon_1$$
$$b_1 = \frac{\varepsilon_1 - c_1}{c_0}$$

$$RPA[2/1]_{f(x)}(x) = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x}$$

where

$$a_0 = c_0$$
$$a_1 = c_1 + \frac{c_0 + (\varepsilon_1 - c_2)}{c_1}$$
$$a_2 = \varepsilon_1$$
$$b_1 = \frac{\varepsilon_1 - c_2}{c_1},$$

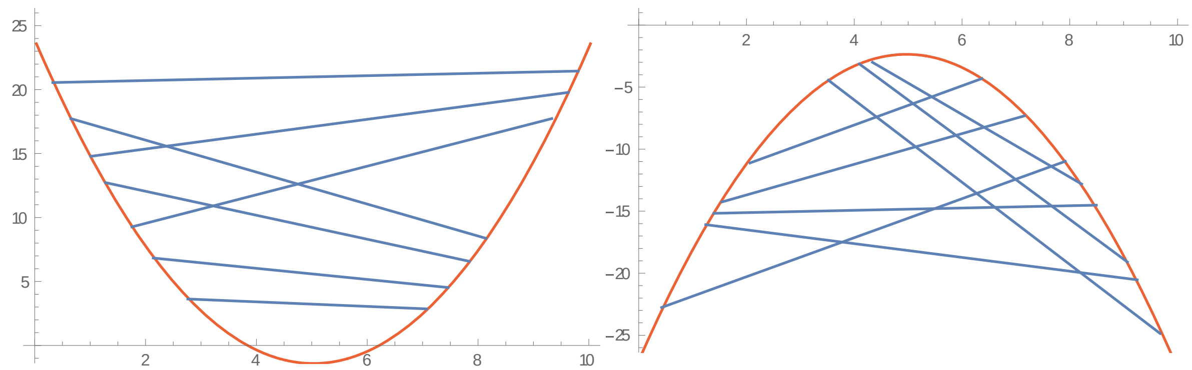
$$RPA[3/1]_{f(x)}(x) = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x}$$

where

$$a_0 = c_0$$
$$a_1 = c_1 + \frac{c_0 + (\varepsilon_1 - c_2)}{c_1}$$
$$a_2 = \varepsilon_2 + \frac{c_1(\varepsilon_1 - c_3)}{c_2}$$
$$a_3 = \varepsilon_1$$
$$b_1 = \frac{\varepsilon_1 - c_3}{c_2}$$

3. Concave and Convex functions

Concave and convex are words that we use to describe the shape or the curvature of the curve. The twin definitions concavity and convexity of a single variable function are widely used in economic theory, and are also central to optimization theory. The function $f(x)$ is said to be concave function if every line segment joining any two points on the curve of this function lie below the graph at any point. On the other hand a function $f(x)$ is said to be convex if every line segment joining two points on the curve of the function lie above the graph at any point see Fig. 1.



A convex function: no line segment joining two points on the graph lies below the graph.

A concave function: no line segment joining two points on the graph lies above the graph.

Fig. 1. Convex and Concave functions.

We can easily conclude that the graph of any differentiable concave function lies below the tangent at pointx_0, which its slope is $f'(x_0)$. Similarly the graph of any differentiable convex function lies above the tangent at point x_0, which its slope is $f'(x_0)$. Thus the definition of the concave and convex function may be written as follows:

Definition:

Let f is a single variable function defined on the interval I**.** Then the function f is

Concave function if for all $x \in I, all\ y \in I$ and $\lambda \in [0,1]$ we have

$$f((1+\lambda)x+\lambda y) \geq (1+\lambda)f(x)+\lambda f(y)$$

Convex function if for all $x \in I, all\ y \in I$ and $\lambda \in [0,1]$ we have

$$f((1+\lambda)x+\lambda y) \geq (1+\lambda)f(x)+\lambda f(y)$$

Both convex and concave and convex function if for all $x \in I, all\ y \in I$ and $\lambda \in [0,1]$ we have

$$f((1+\lambda)x+\lambda y) = (1+\lambda)f(x)+\lambda f(y)$$

Thus the function is both convex and concave if and only if it is linear function.

In this paper our approximated functions (RTA& RPA) will be classified into either concave or convex function. We are looking forward to reduce the maximum norm error L_∞ resulting from our approximated functions (RTA& RPA) my using this classification as we will show in the next part.

4. Modification in Restrictive Taylor and Padé Approximation

After we approximate the restrictive approximation we find the more of its convergence, we calculate the L_∞ error norm which means the maximum error, the new modification is to eliminate this max error and force the absolute error to be near zero around the region that has the maximum error.

From the analysis of error and plot the absolute error between the function needed to be approximated and the restrictive approximation we can classify the modification for two types according to if the function be convex or concave.

<u>Case 1: Convex Functions</u>

From Fig. 2.b the RA lies up the function so the error will be negative so we subtract the modification term.

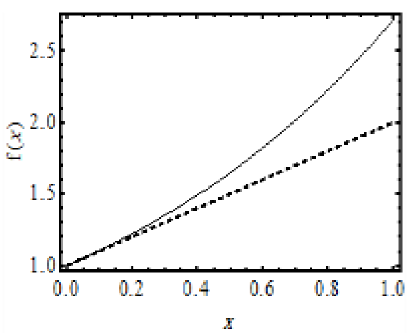


Fig. 2.a

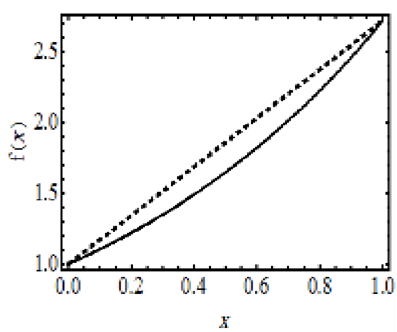


Fig. 2.b

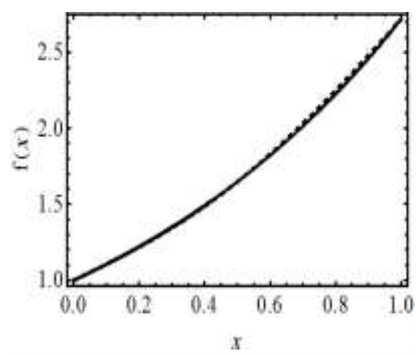


Fig. 2.c

<u>Case 2: Concave Functions</u>

From Fig. 3.b the RA lies below the function so the error will be negative so we will adding the modification term.

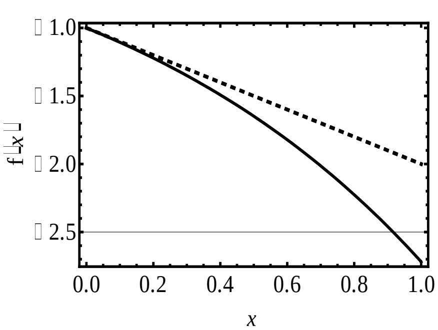


Fig. 3.a

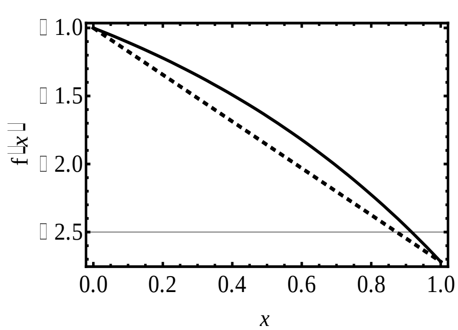


Fig. 3.b

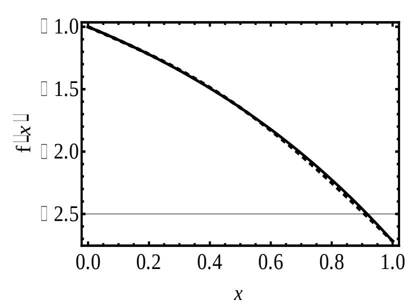


Fig. 3.c

We choose the modification term to satisfy three conditions:

1) At $x = 0$ the modification term vanish (equal zero) see point A in Fig. 4
2) At the point B have the maximum error (L_∞ error norm) the modification term has maximum value and equal to the L_∞ error norm thus forced the new approximations to be near zero around the point have the L_∞ error norm.
3) At x_a where we calculate the restrictive term the error equal to zero so we does not need to the modification term at point C in Fig. 4

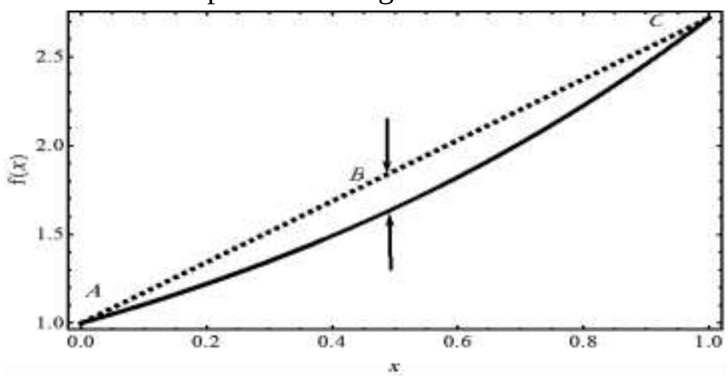


Fig. 4

5. Numerical Examples

In this section we introduce three examples solved by Taylor approximation (TA), Padé approximation (PA), Restrictive Taylor approximation (RTA) and Restrictive Padé approximation (RPA). Then comparing this result by the two Modified restrictive Taylor approximation (MRTA) and Restrictive Padé approximation (MRPA), finally graph both expansion and error.

Example 1.

$$f(x) = Exp(cx) \tag{20}$$

For $c = 1$ the restrictive terms are calculated at For $x = 1$ for both RTA and RPA. We expand the expantion domain to [0, 1] and plot TA, PA, RTA,RPA, MRTA and MRPA in Fig. 5. Also the errors are in Fig. 6.

$$TA_1 = 1 + x$$
$$RTA_1 = 1 + \varepsilon x$$

Where the ε is the restrictive term to be determine by

$$RTA_1 = 1 + 1.718281828459045x$$

$$MRTA_1 = RTA_1 - L_\infty \operatorname{Sin}(x)$$

$$MRTA_1 = 1 + 1.718281828459045x - 0.21041964352939435 \operatorname{Sin}(x)$$

	TA_1	RTA_1	$MRTA_1$
L_2	1.0915436502789486	0.488301679070951	0.039990197968343945
L_∞	0.7182818284590451	0.21041964352939435	0.02540297101555078

Table 2 error norm for taylor expantions for ex 1

$$PA_1 = \frac{1+\frac{x}{2}}{1-\frac{x}{2}} \qquad RPA_1 = \frac{1+\varepsilon x}{1-b_1 x}$$

Where the ε is the restrictive term to be determine by

$$RPA_1 = \frac{1+0.5819767068693265x}{1-0.41802329313067355x}$$

$$MRPA_1 = RTA_1 + L_\infty \operatorname{Sin}(x)$$

$$MRPA_1 = \frac{1+0.5819767068693265x}{1-0.41802329313067355x} + 0.02419074111857622 \operatorname{Sin}(x)$$

	PA1	RPA1	MPTA1
L_2	0.35755397907149267	0.048375725339978484	0.028051263981545418
L_∞	0.2817181715409549	0.02419074111857622	0.015226505423012967

Table 3 error norm for Pade expantions for ex 1

x	TA_1	TA_2	RTA_1	RTA_2	$MRTA_1$	$MRTA_1$
0	0	0	0	0	0	0
0.1	0.005170918075	0.00017091807564	0.066657264770	0.00201190020894	0.0016340189693555	0.009794214106360721
0.2	0.021402758160	0.00140275816016	0.122253607531	0.00732851497819	0.0014279557275782	0.015128048925197923
0.5	0.148721270700	0.02372127070012	0.210419643529	0.03084918641463	0.	0.007356202059822792
0.7	0.313752707470	0.06875270747047	0.189044572450	0.03820538847445	0.0188115048852566	0.007296579921924451
0.9	0.559603111156	0.15460311115695	0.086850534456	0.02220516989487	0.0218272886552890	0.010399055579572991
1	0.718281828459	0.21828182845904	0	0	0	0

Table 4 errorfor example 1

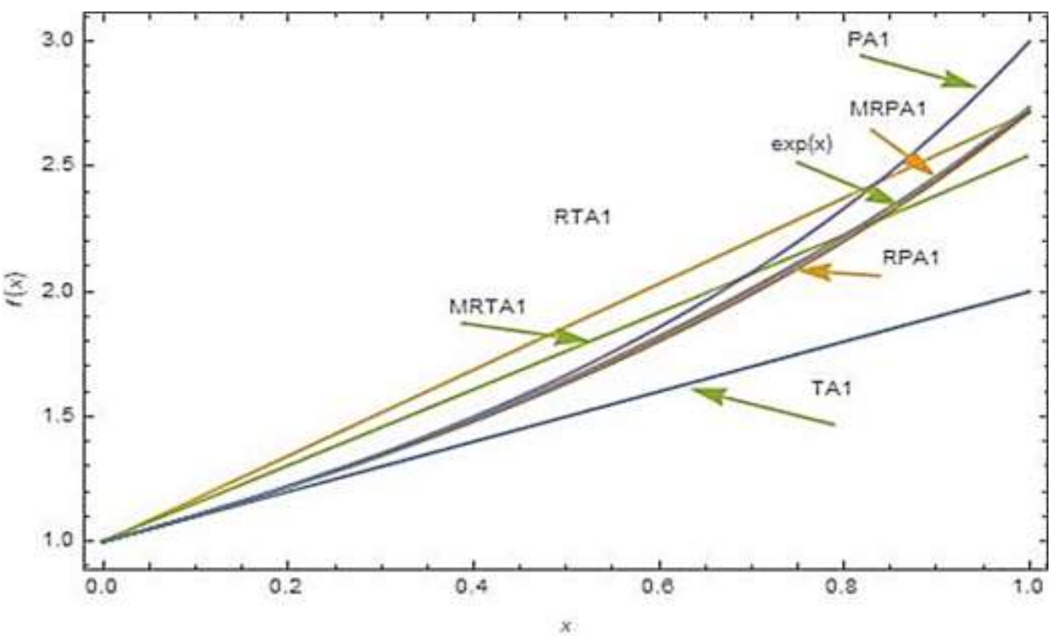


Fig. 5. Expansion of example 1

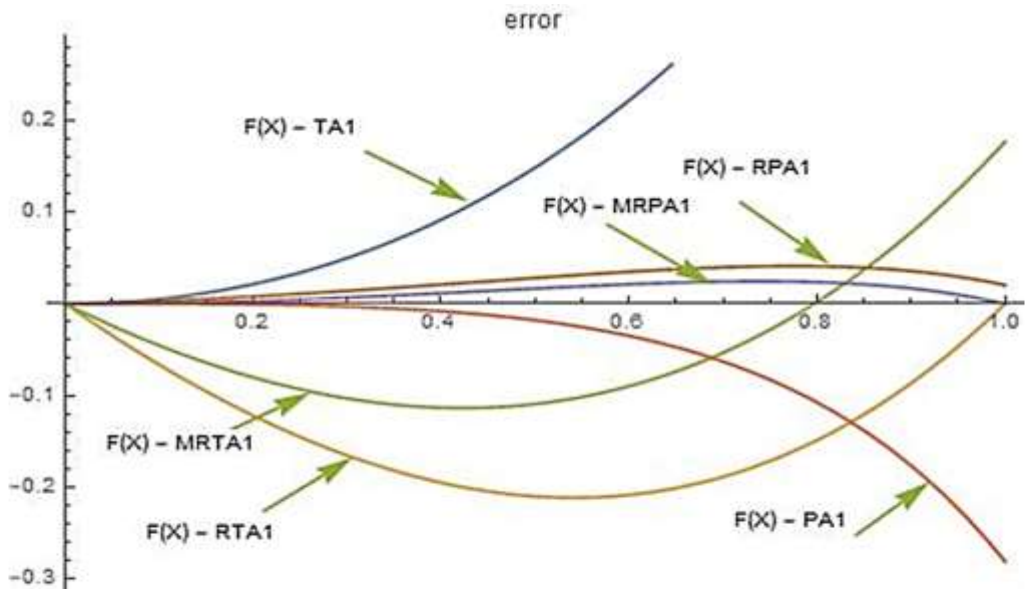


Fig. 6. The errors of example 1

Example 2.

$$f(x) = \sqrt{\frac{1+\frac{1}{2}x}{1+2x}} \tag{21}$$

Baker and Morris [1] show the accurate of PA via TA in this example we introduce a batter methods (RTA and RPA) as shown in Fig. 7 and the error in Fig. 8.

$$TA_1 = 1 - 0.75\,x \qquad RTA_1 = 1 + \varepsilon x$$

Where the ε is the restrictive term to be determine by

$$RTA_1 = 1 - 0.2928932188x \qquad MRTA_1 = RTA_1 - L_\infty \operatorname{Sin}(x)$$

$$MRTA_1 = 1 - 0.2928932188x - 0.06634613154689295 \operatorname{Sin}(x)$$

	TA1	RTA1	MRTA1
L_2	0.8045311072982965	0.1522780428123673	0.02842734952458836
L_∞	0.4571067811865476	0.06634613154689295	0.016018818139471036

Table 5 error norm for Taylor expantions for ex 2

	PA1	RPA1	MRPA1
L_2	0.012674300329633452	0.002665666175542447	0.0001662632781343035
L_∞	0.007178933099166618	0.001204511854803969	0.00008268011675605091

Table 6 error norm for Pade expantions for ex 2

$$PA_1 = \frac{1 + 0.875x}{1 + 1.625x}$$

$$RPA_1 = \frac{1 + \ \varepsilon x}{1 - b_1 x}$$

Where the ε is the restrictive term to be determine by

$$RPA_1 = \frac{1 + 0.8106601718x}{1 + 1.560660172x}$$

$$MRPA_1 = RTA_1 + \ L_\infty \ \text{Sin}(x)$$

$$MRPA_1 = \frac{1 + 0.8106601718x}{1 + 1.560660172x} + 0.0012045118548 03969 \ \text{Sin}(x)$$

X	TA1	TA2	RTA1	RTA2	MRTA1	MRTA1
0	0	0	0	0	0	0
0.1	0.0104143466	0.0017731533065	0.03529633142	0.005843278881	0.014794249266	0.0108892724538
0.2	0.0364052604	0.0123447395720	0.05501609580	0.018120989180	0.016018818139	0.013706214783
0.5	0.1655694150	0.1391180849579	0.06298397555	0.051292719745	0.003362155995	0.002854953813
0.7	0.275	0.3221875000000	0.04497474683	0.051017677218	0.008700401101	0.007211289103
0.9	0.3946229171	0.5925645828710	0.01677318593	0.024366424367	0.003728896220	0.007633873032
1	0.4571067811	0.761643218813	$1.11022302462 \times 10^{-16}$	$1.110223024625 \times 10^{-16}$	$1.110223024625 \times 10^{-16}$	$1.110223024625 \times 10^{-16}$

Table 7 errorfor example 2

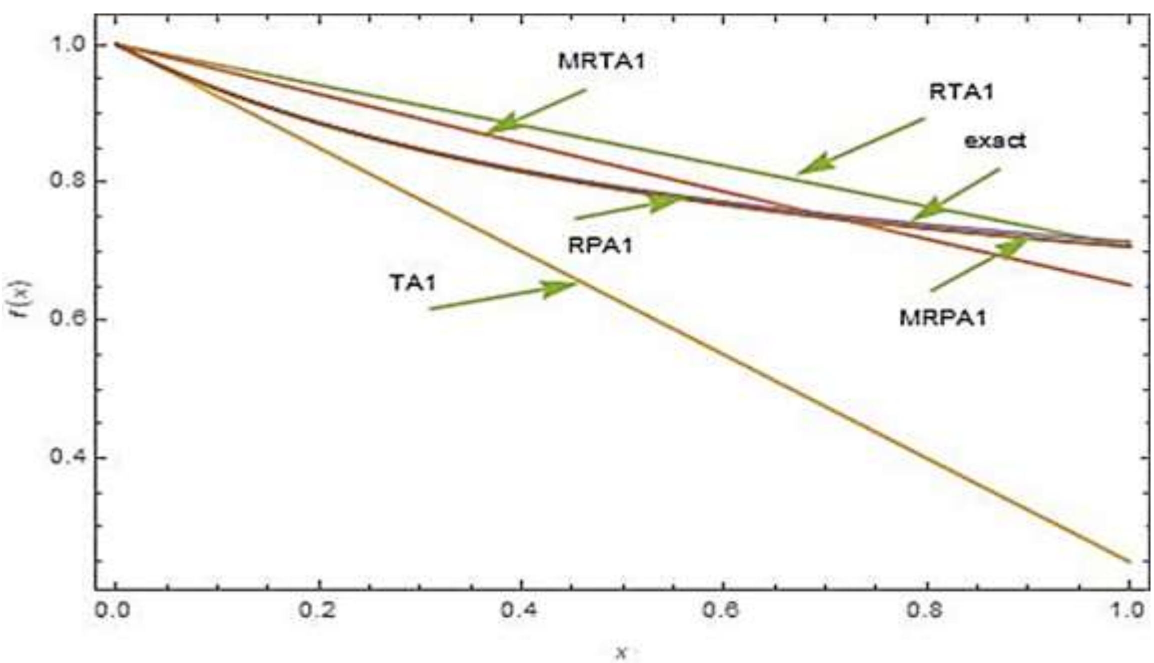


Fig.7. Expansion of example 2

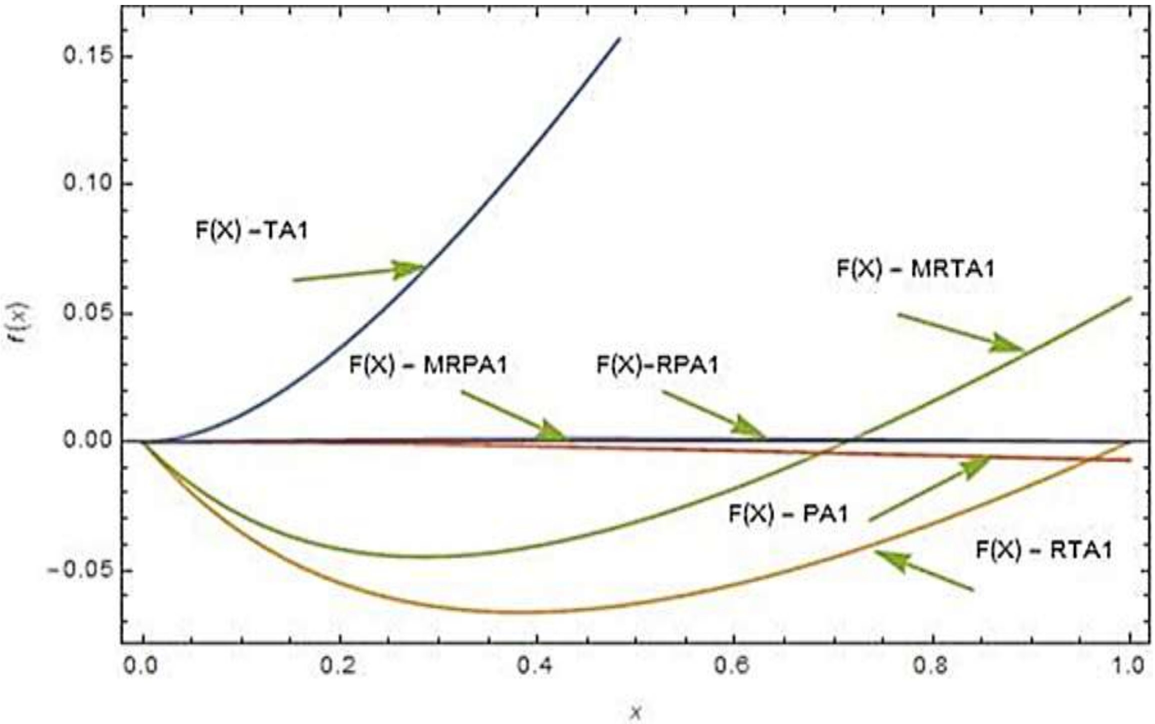


Fig.8. the errors of example 2

6. Results and Conclusion

The numerical examples show the high accurate of Modified Restrictive Taylor approximation (MRTA) and Modified Restrictive Padé approximation (MRPA) than Restrictive Taylor approximation (RTA) and Restrictive Padé approximation (RPA).We know that TA and PA gives the exact solution at $x = 0$ also at the point of calculate the restrictive term as shown Fig 2.b or Fig 3.b. As we show in example 1&2 (Fig 5 ,6.7 and 8) we force RTA and RPA close to the exact curve by adding modified terms to decrease the error close to zero.

References

[1] G.A. Baker, P. Graves-Morris, Pad´e Approximants. Part I, Encyclopedia of Mathematics and its Applications, Vol. 13, Addison-Wesley Publishing Co., 1981.
[2] G.A. Baker, P. Graves-Morris, Pad´e Approximants. Part II, Encyclopedia of Mathematics and its Applications, Vol. 14, Addison-Wesley Publishing Co., 1981.
[3] G.A. Baker Jr., Essentials of Padé Approximants, Academic Press, NewYork, 1975.
[4] J.L Walsh, Padé approximants as limits of rational functions of best approximation, real domain, J. Approx. Theory, Vol. 11, (3), (1974) 225-230.
[5] C. Brezinski, Rational Approximation to Formal Power Series, J. Approx. Theory 25, (1979) 295-317
[6] M.A.H. Khan, High-order differential approximants, Journal of Computational and Applied Mathematics, Vol. 149,(2), (2002) 457-468.

[7] B de la Calle Ysern, G López Lagomasino, Convergence of Multipoint Padé-type Approximants, J. Approx. Theory, Vol. 109 (2), (2001) 257-278
[8] H.N.A.Ismail, E. M.E. Elbarbary, Restrictive Taylor approximation and Parabolic Partial Differential Equations, Int. J. Computer Math.78 (2001) 73–82.
[9] H. N.A. Ismail, E. M. E. Elbarbary, G. S.E. Salem, Restrictive Taylor Approximation for Solving Convection – Diffusion Equation, Appl. Math.Comput. 147 (2004) 355–363 .
[10] H. N. Ismail, T. M. Rageh, G. S. Salem, Modified approximation for the kdv-Burgers equation, Appl. Math.Comput. 234 (2014) 58–62.
[11] T. M. Rageh, G. Salem and F. A. El-Salam, Restrictive Taylor Approximation for Gardner and KdV Equations, Int. J. Adv. Appl. Math. and Mech. 1(3) (2014) 1 – 10 .
[12] H. N. A. Ismail, I. K. Youssef, and Tamer M. Rageh. "New approaches for Taylor and Padé approximations." International Journal of Advances in Applied Mathematics and Mechanics 2.3 (2015): 78-86.
[13] H. N.A. Ismail, Elbarbary E. M. and Elbeetar A. A., Restrictive Padé Approximation for the Solution of Schrodinger Equation", Int. J. Computer Math. Vol. 79 (5), (2002) 603-613.
[14] H. N.A. Ismail, Elsady Z. and Abd Rabboh A. A., Restrictive Padé Approximation for Solution of Generalized Burger's Equation" J. Inst. Math. & Comp. Sc. Vol. 14 No. 2, (2003) 31-35
[15] H. N.A. Ismail, G. S.E. Salem, A. A. Abd Rabboh, Comparison Study between Restrictive Padé, Restrictive Taylor Approximations and Adomain Decomposition Methods for the Solitary Wave Solution of the General KdV Equation, Appl. Math.Comput. 167 (2005) 849–869.
[16] H. N.A. Ismail, On the convergence of the restrictive Pade approximation to the exact solutions of IBVP of parabolic and hyperbolic types ", Appl. Math. & Computation 162, (2005)1055-1064.
[17] H. N.A. Ismail, Unique solvability of restrictive Pade and restrictive Taylors approximations types , Appl. Math. & Computation 152, (2004) 89-97.