THE EQUITABLE PRESENTATION FOR THE EXTENDED QUANTUM ALGEBRA $U_q(f(K,J))$

Abstract. Let $U_q(f(K,J))$ be an extended quantum enveloping algebra associated to sl_2. In this paper, we display an equitable presentation for $U_q(f(K,J))$, in which all generators appear on a more equal footing. We also show that the equitable generators Y and Z are not invertible. Moreover, we give a presentation for the positive even subalgebra of $U_q(f(K,J))$ by generators and relations.

1. Introduction

Quantum enveloping algebras are important examples of Hopf algebra which are neither commutative nor cocommutative. The simplest and most important example of quantum enveloping algebra is $U_q(sl_2)$. Up to now, various generalizations of $U_q(sl_2)$ have been studied by many mathematicians, see [1, 2, 3, 4, 5]. Ji and Wang [2] introduced a class of Hopf algebras $U_q(f(K))$ and studied its representations. Later, Wang, Ji and Yang [3] defined another class of Hopf algebras $U_q(f(K,H))$. The authors described the representation theory and the center of $U_q(f(K,H))$ in [3]. The algebras $U_q(f(K))$ and $U_q(f(K,H))$ are two special kinds of ambiskew polynomial rings. Recently, Wu [4] obtained an extended quantum enveloping algebra $U_{r,t}$ by localizing the weak quantum enveloping algebra of sl_2 with some Ore sets. As a generalization of $U_{r,t}$, Hong [5] defined an extended quantum algebra $U_q(f(K,J))$ by generalizing the Drinfeld double of enveloping algebras of Lie algebras to a double of two Hopf algebras. The Hopf algebra structures theory of $U_q(f(K,J))$ has been studied in [5]. Note that $U_q(f(K,J))$ is isomorphic to $k(G)\otimes U_q(sl_2)$ as algebras, but not as Hopf algebras.

In another aspect, Tatsuro Ito et al [6] introduced an equitable presentation for the quantum group $U_q(sl_2)$. Terwilliger [7] introduced an equitable presentation for the quantum enveloping algebra $U_q(g)$ associated with a symmetrizable Kac-Moody algebra. In the usual Chevalley presentation for $U_q(g)$, the various generators play different roles, while in the equitable presentation, the generators are on a more equal footing. The authors show that $U_q(sl_2)$ has an equitable presentation with generators $x^{\pm 1}$, y, z and relations $xx^{-1}=x^{-1}x=1$,

$$\frac{qxy-q^{-1}yx}{q-q^{-1}}=1, \quad \frac{qyz-q^{-1}zy}{q-q^{-1}}=1, \quad \frac{qzx-q^{-1}xz}{q-q^{-1}}=1.$$

Using this equitable presentation of $U_q(sl_2)$, Alison Gordon Lynch [8] gave a presentation of the positive even subalgebra by generators and relations. The author

2000 *Mathematics Subject Classification.* Primary 17B37, Secondary 17B67.
Key words and phrases. extended quantum algebra; equitable presentation; Hopf algebra.
supported by NSFC (No. 11171296).

also classified up to isomorphism the finite-dimensional irreducible modules under the assumption that q is not a root of unity.

Inspired by the above observation, we would like to generalize some results of $U_q(sl_2)$ to the extended algebra $U_q(f(K,J))$ in this paper. We focus on the equitable presentation for $U_q(f(K,J))$ and $U_{r,t}$, in which the generators are on a roughly equal footing. Then we show that the equitable generators Y and Z are not invertible in $U_{r,t}$. Following the idea of [8], we consider the positive even subalgebra

$$\mathcal{A} = \{X^iY^jZ^k | i,j,k \in \mathbb{N},\ i+j+k \ even\}$$

of $U_q(f(K,J))$. This subalgebra was first discussed in [9]. we will give a presentation for the positive even subalgebra $\mathcal{A}$ by generators and relations.

The contents of this paper are as follows. In Section 2, some results about $U_q(f(K,J))$ and $U_{r,t}$ are reviewed. In Section 3, the equitable presentation for $U_q(f(K,J))$ is introduced, where the generators are on a roughly equal footing. Note that the equitable presentation is not unique. As a example, another equitable presentation for $U_{r,t}$ is also presented. In Section 4, we show that the equitable generator Y (resp. Z) is not invertible in $U_{r,t}$ by displaying an infinite dimensional $U_{r,t}$-module that contains a nonzero null vector for Y (resp. Z). Finally, we give a presentation of the positive even subalgebra of $U_q(f(K,J))$.

Throughout, k is a field with characteristic zero and q is an invertible element in k satisfying $q^2 \neq 1$; $\mathbb{N}$ is the set of natural numbers; $\mathbb{Z}$ is the set of all integers. We will work over the field $k(q)$.

2. The extended quantum algebra $U_q(f(K,J))$

For the reader's convenience, we list some notations and basic facts of $U_{r,t}$ and $U_q(f(K,J))$ by referring to [4, 5], we describe it here as follows.

Definition 2.1 ([4]) *Let r, t be two fixed non-negative integers. $U_{r,t} = U_{r,t}(sl(2))$ is the $k(q)$-algebra generated by six variables $J^{\pm 1}$, $K^{\pm 1}$, E and F, where $J^{\pm 1}$ are in the center of $U_{r,t}$, with the relations:*

$$KK^{-1} = K^{-1}K = JJ^{-1} = J^{-1}J = 1, \tag{2.1}$$

$$KEK^{-1} = q^2E, \qquad KFK^{-1} = q^{-2}F, \tag{2.2}$$

$$EF - FE = \frac{K - K^{-1}J^r}{q - q^{-1}}. \tag{2.3}$$

In particular, if $r = 0$, then the algebra $U_{r,t}$ is isomorphic to a tensor product of the algebra of infinite cyclic group and $U_q(sl_2)$ as Hopf algebras.

Definition 2.2 ([5]) *Let G be an Abel group, and g_1, g_2, h_1, h_2 be four fixed elements of G. As a vector space, $U_q(f(K,J))$ is isomorphic to the the tensor product $k(G) \otimes U_q(sl_2)$. Any element of $k(G)$ and $J^{\pm 1}$ are in the center of $U_q(f(K,J))$. The extended quantum algebra $U_q(f(K,J))$ is generated by g_1, g_2, h_1, h_2 and E, F, $K^{\pm 1}$, $J^{\pm 1}$ with the relations:*

$$KK^{-1} = K^{-1}K = JJ^{-1} = J^{-1}J = 1, \tag{2.4}$$

$$KEK^{-1} = q^2E, \qquad KFK^{-1} = q^{-2}F, \tag{2.5}$$

$$EF - FE = \frac{Kh_1h_2 - K^{-1}g_1g_2}{q - q^{-1}}. \tag{2.6}$$

The quantum algebra $U_q(f(K,J))$ can be equipped with a Hopf algebra structure given by:

$$\Delta(J^{\pm 1}) = J^{\pm 1} \otimes J^{\pm 1}, \quad \Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \tag{2.7}$$

$$\Delta(E) = g_1 \otimes E + E \otimes K h_1, \qquad \Delta(F) = K^{-1} g_2 \otimes F + F \otimes h_2, \tag{2.8}$$

$$\Delta(a) = a \otimes a, \ a \in G, \tag{2.9}$$

$$\varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(J^{\pm 1}) = \varepsilon(K^{\pm 1}) = \varepsilon(a) = 1, \ a \in G, \tag{2.10}$$

$$S(J) = J^{-1}, \quad S(J^{-1}) = J, \quad S(K) = K^{-1}, \quad S(K^{-1}) = K, \tag{2.11}$$

$$S(E) = -EK^{-1} g_1^{-1} h_1^{-1}, \quad S(F) = -KF g_2^{-1} h_2^{-1}. \tag{2.12}$$

$$S(a) = a^{-1}, \ a \in G. \tag{2.13}$$

Obviously, $U_{r,t}$ is a special case of $U_q(f(K,J))$. Note that $U_q(f(K,J))$ is isomorphic to $k(G) \otimes U_q(sl_2)$ as algebras, but not as Hopf algebras. Since the coproduct of $U_q(f(K,J))$ is not the usual coproduct of $k(G) \otimes U_q(sl_2)$. Neither is the antipode.

Similar to Theorem 2.4 of [4], we have the following conclusion.

Proposition 2.3 *The algebra $U_q(f(K,J))$ is a Noetherian domain. Moreover, the set*

$$\{K^i E^j F^k J^l g_1^m g_2^n h_1^u h_2^v | j,k \in \mathbb{N}, \ i,l,m,n,u,v \in \mathbb{Z}\}$$

is a basis for $U_q(f(K,J))$.

3. The equitable presentation for $U_q(f(K,J))$ and $U_{r,t}$

From Definition 2.2, we can see the generators of $U_q(f(K,J))$ play a very different role. In the following, we will introduce a presentation for $U_q(f(K,J))$, whose generators are on a more equal footing.

Theorem 3.1 *The algebra $U_q(f(K,J))$ is isomorphic to an associative algebra B_q. The algebra B_q is generated by $X^{\pm 1}$, Y, Z, $\bar{g}_i$, $\bar{h}_i$ $(i = 1,2)$ and $C^{\pm 1}$, where $C^{\pm 1}$, $X^{\pm 1}$ and $\bar{g}_i, \bar{h}_i$ $(i = 1,2)$ are in the center of B_q, satisfying the following relations:*

$$XX^{-1} = X^{-1}X = CC^{-1} = C^{-1}C = 1, \tag{3.1}$$

$$\frac{qXY - q^{-1}YX}{q - q^{-1}} = 1, \tag{3.2}$$

$$\frac{qYZ - q^{-1}ZY}{q - q^{-1}} = \bar{h}_1 \bar{h}_2, \tag{3.3}$$

$$\frac{qZX - q^{-1}XZ}{q - q^{-1}} = \bar{g}_1 \bar{g}_2. \tag{3.4}$$

An isomorphism $\psi : U_q(f(K,J)) \to B_q$ is defined as follows:

$$C^{\pm 1} \to J^{\pm 1},$$
$$X^{\pm 1} \to K^{\pm 1},$$
$$Y \to K^{-1} + F(q - q^{-1}),$$
$$Z \to g_1 g_2 K^{-1} - K^{-1} E(q^2 - 1),$$

$$g_i \to \bar{g}_i, \quad h_i \to \bar{h}_i,\ i = 1, 2.$$

The inverse of ψ *is* $\rho : B_q \to U_q(f(K, J))$ *given by:*

$$J^{\pm 1} \to C^{\pm 1},$$
$$K^{\pm 1} \to X^{\pm 1},$$
$$E \to (\bar{g}_1\bar{g}_2 - XZ)(q^2 - 1)^{-1},$$
$$F \to (Y - X^{-1})(q - q^{-1})^{-1},$$
$$\bar{g}_i \to g_i, \quad \bar{h}_i \to h_i,\ i = 1, 2.$$

Proof From (3.2)-(3.4), it is easy to get

$$XY = q^{-2}YX + (1 - q^{-2}), \tag{3.5}$$

$$XZ = q^2 ZX + (1 - q^2)\bar{g}_1\bar{g}_2, \tag{3.6}$$

$$ZY = q^2 YZ + (1 - q^2)\bar{h}_1\bar{h}_2. \tag{3.7}$$

Using these relations, we can prove that ψ preserves the relations (3.1)-(3.4). We only check ψ preserves (3.4).

Since

$$\begin{aligned} \psi(EF - FE) &= \tfrac{(\bar{g}_1\bar{g}_2 - XZ)(Y - X^{-1}) - (Y - X^{-1})(\bar{g}_1\bar{g}_2 - XZ)}{q(q - q^{-1})^2} \\ &= \tfrac{XZX^{-1} - Z + YXZ - XZY}{q(q - q^{-1})^2} \\ &= \tfrac{\bar{h}_1\bar{h}_2 X - \bar{g}_1\bar{g}_2 X^{-1}}{q - q^{-1}} \\ &= \psi(\tfrac{h_1h_2K - g_1g_2K^{-1}}{q - q^{-1}}). \end{aligned}$$

Then ψ preserves (3.4). So ψ is a homomorphism from $U_q(f(K, J))$ to B_q. Furthermore, we can check $\rho\psi = \psi\rho = id$ for each generator of $U_q(f(K, J))$. Thus, ρ is the inverse of ψ.

Definition 3.2 *The presentation given in Theorem 3.1 is called the equitable presentation for* $U_q(f(K, J))$. *We call the generators* $C^{\pm 1}$, $X^{\pm 1}$, Y, Z, $\bar{g}_i$, $\bar{h}_i$ $(i = 1, 2)$ *the equitable generators.*

Since B_q is isomorphic to $U_q(f(K, J))$ as algebras, we can regard $U_q(f(K, J))$ as an algebra generated by $C^{\pm 1}$, $X^{\pm 1}$, Y, Z and $\bar{g}_i$, $\bar{h}_i$ $(i = 1, 2)$ with the relations (3.1)-(3.4). Due to Proposition 2.3 and Theorem 3.1, the algebra $U_q(f(K, J))$ has another kind of PBW basis as follows.

Proposition 3.3 *The set*

$$\{X^i Y^j Z^k C^l \bar{g}_1^m \bar{g}_2^n \bar{h}_1^u \bar{h}_2^v | j, k \in \mathbb{N},\ i, l, m, n, u, v \in \mathbb{Z}\}$$

is a basis of $U_q(f(K, J))$.

To make the algebra isomorphisms in Theorem 3.1 into isomorphisms of Hopf algebras, we need to define the Hopf algebra structure for $U_q(f(K, J))$ in terms of the equitable generators as follows.

Theorem 3.4 *With refer to Proposition 2.3 and Theorem 3.1, the comultiplication* Δ *satisfies*

$$\Delta(C^{\pm 1}) = C^{\pm 1} \otimes C^{\pm 1}, \quad \Delta(X^{\pm 1}) = X^{\pm 1} \otimes X^{\pm 1},$$

$$\Delta(\bar{g}_i) = \bar{g}_i \otimes \bar{g}_i, \quad \Delta(\bar{h}_i) = \bar{h}_i \otimes \bar{h}_i,\ i = 1, 2,$$

$$\Delta(Y) = X^{-1} \otimes X^{-1} + \bar{g}_2 X^{-1} \otimes (Y - X^{-1}) + (Y - X^{-1}) \otimes \bar{h}_2,$$

$$\Delta(Z) = \bar{g}_1\bar{g}_2 X^{-1} \otimes \bar{g}_1\bar{g}_2 X^{-1} - \bar{g}_1 X^{-1} \otimes (\bar{g}_1\bar{g}_2 X^{-1} - Z) - (\bar{g}_1\bar{g}_2 X^{-1} - Z) \otimes \bar{h}_1.$$

The counit ε satisfies

$$\varepsilon(C^{\pm1}) = \varepsilon(X^{\pm1}) = \varepsilon(Y) = \varepsilon(Z) = 1, \quad \varepsilon(\bar{g}_i) = \varepsilon(\bar{h}_i) = 1,\ i = 1, 2.$$

The antipode S satisfies

$$S(C^{\pm1}) = C^{\mp1}, \qquad S(X^{\pm}) = X^{\mp1},$$
$$S(\bar{g}_i) = \bar{g}_i^{-1}, \qquad S(\bar{h}_i) = \bar{h}_i^{-1},\ i = 1, 2,$$
$$S(Y) = X - (XY - 1)\bar{g}_2^{-1}\bar{h}_2^{-1}, \qquad S(Z) = \bar{g}_1^{-1}\bar{g}_2^{-1}X + q^2(\bar{g}_1\bar{g}_2 - XZ)\bar{g}_1^{-1}\bar{h}_1^{-1}.$$

Proof Using Proposition 2.3 and Theorem 3.1, it is just a computation.

corollary 3.5 *If one takes $\bar{h}_1 = \bar{h}_2 = \bar{g}_1 = 1$ and $\bar{g}_2 = C^r$, then the presentation given in Theorem 3.1 is an equitable presentation for $U_{r,t}$.*

Proof Since the algebra $U_{r,t}$ is a special case of $U_q(f(K,J))$ provided that $h_1 = h_2 = g_1 = 1$ and $g_2 = J^r$.

We remark that the equitable presentation of $U_{r,t}$ is not unique. Next, we will present another equitable presentation of $U_{r,t}$.

Theorem 3.6 *The algebra $U_{r,t}$ is isomorphic to the associative algebra A_q as Hopf algebras. The algebra A_q generated by six generators $C^{\pm1}$, $X^{\pm1}$, Y, Z, where $C^{\pm1}$ are in the center of A_q, with the following relations:*

$$XX^{-1} = X^{-1}X = CC^{-1} = C^{-1}C = 1, \tag{3.8}$$

$$\frac{qZX - q^{-1}XZ}{q - q^{-1}} = 1, \tag{3.9}$$

$$\frac{qXY - q^{-1}YX}{q - q^{-1}} = 1, \tag{3.10}$$

$$\frac{qYZ - q^{-1}ZY}{q - q^{-1}} = C^{-r}. \tag{3.11}$$

The algebra A_q has a Hopf algebra structure with co-multiplication, counit and antipode defined by

$$\Delta(C^{\pm1}) = C^{\pm1} \otimes C^{\pm1}, \quad \Delta(X^{\pm1}) = X^{\pm1} \otimes X^{\pm1}, \tag{3.12}$$

$$\Delta(Y) = X^{-1} \otimes X^{-1} + C^{-rt}X^{-1} \otimes (Y - X^{-1}) + (Y - X^{-1}) \otimes C^{rt}, \tag{3.13}$$

$$\Delta(Z) = X^{-1} \otimes X^{-1} - C^{rt}X^{-1} \otimes (X^{-1} - Z) - (X^{-1} - Z) \otimes C^{-r(t+1)}, \tag{3.14}$$

$$\varepsilon(C) = \varepsilon(C^{-1}) = \varepsilon(X) = \varepsilon(X^{-1}) = \varepsilon(Y) = \varepsilon(Z) = 1, \tag{3.15}$$

$$S(C) = C^{-1}, \quad S(C^{-1})) = C, \quad S(X) = X^{-1}, \quad S(X^{-1}) = X, \tag{3.16}$$

$$S(Y) = X - XY - 1, \quad S(Z) = X + q^2C^r(1 - XZ). \tag{3.17}$$

Proof Let us define $\psi : U_{r,t} \to A_q$ satisfying

$$\psi(J^{\pm1}) = C^{\mp1}, \quad \psi(K) = C^{-r}X, \quad \psi(K^{-1}) = C^rX^{-1},$$
$$\psi(E) = (1 - XZ)(q^2 - 1)^{-1}, \quad \psi(F) = (Y - X^{-1})(q - q^{-1})^{-1}.$$

The inverse of ψ is $\rho : A_q \to U_{r,t}$ given by:

$$\rho(C^{\pm1}) = J^{\mp1}, \quad \rho(X) = J^{-r}K, \quad \rho(X^{-1}) = J^rK^{-1},$$

$$\rho(Y) = K^{-1}J^r + F(q - q^{-1}), \quad \rho(Z) = K^{-1}J^r - K^{-1}J^r E(q^2 - 1).$$

Similar to Theorem 3.1, we can check ρ and ψ are isomorphisms of Hopf algebras.

Remark 3.7 *The equitable presentation for $U_{r,t}$ given in Theorem 3.6 is not unique. In fact, there exists an algebra automorphism ω_s of $U_{r,t}$ that satisfies $\omega_s(J^{\pm 1}) = J^{\pm 1}$, $\omega_s(K) = K^{-1}J^r$, $\omega_s(K^{-1}) = KJ^{-r}$, $\omega_s(E) = FJ^s$, $\omega_s(F) = EJ^{-s}$ for an integer s.*

4. Y and Z are not invertible in $U_{r,t}$

In this section, we show that the equitable generator Y (resp. Z) is not invertible in $U_{r,t}$ by displaying an infinite dimensional $U_{r,t}$-module that contains a nonzero null vector for Y (resp. Z). Similar to Lemma 3.1-3.4 of [6], we have the following results.

Lemma 4.1 *There exists an $U_{r,t}$-module $\Gamma_Y = \{u_{ij}\}_{i\in\mathbb{Z},\ j\in\mathbb{N}}$ with the following properties:*

$$C^m u_{ij} = q^{2m} u_{ij},\ m \in \mathbb{Z}, \tag{4.1}$$

$$X u_{ij} = u_{i+1,j}, \quad X^{-1} u_{ij} = u_{i-1,j}, \tag{4.2}$$

$$Y u_{ij} = q^{-r}(q^{2i} - q^{2i-2j}) u_{i,j-1} - (q^{2i} - 1) u_{i-1,j}, \tag{4.3}$$

$$Z u_{ij} = q^{-r} q^{-2i} u_{i,j+1} + (1 - q^{-2i}) u_{i-1,j}, \tag{4.4}$$

for all $i \in \mathbb{Z},\ j \in \mathbb{N}$. In the above equations, $u_{i,-1} = 0$ for $i \in \mathbb{Z}$.

Proof Obviously, the given actions satisfy the relation (3.8). We have to check that these actions satisfy the relations (3.9)-(3.11) in the following.

For (3.9), since

$$\begin{aligned} qZXu_{ij} &= qZu_{i+1,j} = q^{-r}q^{-2i-1}u_{i+1,j+1} + (q - q^{-2i-1})u_{ij}, \\ q^{-1}XZu_{ij} &= q^{-1}q^{-r}q^{-2i}u_{i+1,j+1} + q^{-1}(1 - q^{-2i})u_{i,j} \\ &= q^{-r}q^{-2i-1}u_{i+1,j+1} + (q^{-1} - q^{-2i-1})u_{ij}. \end{aligned}$$

Then

$$(qZX - q^{-1}XZ)u_{ij} = (q - q^{-1})u_{ij}.$$

For (3.10), since

$$\begin{aligned} qXYu_{ij} &= q^{-r}q(q^{2i} - q^{2i-2j})u_{i+1,j-1} - q(q^{2i} - 1)u_{ij} \\ &= q^{-r}(q^{2i+1} - q^{2i-2j+1})u_{i+1,j-1} - (q^{2i+1} - q)u_{ij}, \\ q^{-1}YXu_{ij} &= q^{-1}Yu_{i+1,j} \\ &= q^{-r}(q^{2i+1} - q^{2i+1-2j})u_{i+1,j-1} - (q^{2i+1} - q^{-1})u_{ij}. \end{aligned}$$

Then we obtain

$$(qXY - q^{-1}YX)u_{ij} = (q - q^{-1})u_{ij}.$$

For (3.11), we have

$$\begin{aligned} qYZu_{ij} &= q^{-r}q^{-2i+1}Yu_{i,j+1} + (q - q^{1-2i})Yu_{i-1,j} \\ &= q^{-r}q^{-2i+1}[q^{-r}(q^{2i} - q^{2i-2j-2})u_{ij} - (q^{2i} - 1)u_{i-1,j+1}] \\ &\quad + (q - q^{1-2i})[q^{-r}(q^{2i-2} - q^{2i-2j-2})u_{i-1,j-1} - (q^{2i-2} - 1)u_{i-2,j}] \\ &= q^{-2r}(q - q^{-2j-1})u_{ij} - q^{-r}(q - q^{1-2i})u_{i-1,j+1} \\ &\quad + q^{-r}(1 - q^{-2i})(q^{2i-1} - q^{2i-2j-1})u_{i-1,j-1} - (q^{2i-1} - q - q^{-1} - q^{1-2i} - q)u_{i-2,j}. \end{aligned}$$

Similarly, we get

$$\begin{aligned} q^{-1}ZYu_{ij} &= q^{-2r}(q^{-1}-q^{-2j-1})u_{ij} - q^{-r}(q-q^{1-2i})u_{i-1,j+1} \\ &\quad + q^{-r}(1-q^{-2i})(q^{2i-1}-q^{2i-2j-1})u_{i-1,j-1} - (q^{2i-1}-q-q^{-1}-q^{1-2i}-q)u_{i-2,j}. \end{aligned}$$

Then

$$(qYZ - q^{-1}YZ)u_{ij} = q^{-2r}(q-q^{-1})u_{ij} = C^r(q-q^{-1})u_{ij}.$$

Thus, the given actions satisfy the relations (3.8)-(3.11).

Proposition 4.2 *The following (i)-(iii) hold.*

(i) $Yu_{00} = 0$, *where the vector* $u_{00} \in \Gamma_Y$,

(ii) Y *is not invertible on* Γ_Y,

(iii) Y *is not invertible in* $U_{r,t}$.

Proof It is easy to see by Lemma 4.1.

Similarly to Lemma 4.1 and Proposition 4.2, we also have the following results.

Lemma 4.3 *There exists an* $U_{r,t}$*-module* $\Gamma_Z = \{v_{ij}\}_{i\in\mathbb{Z},\ j\in\mathbb{N}}$ *with the following properties:*

$$C^m v_{ij} = q^{2m}v_{ij},\ m\in\mathbb{Z}, \tag{4.5}$$

$$Xv_{ij} = v_{i+1,j}, \quad X^{-1}v_{ij} = v_{i-1,j}, \tag{4.6}$$

$$Yv_{ij} = q^{-r}q^{2i}v_{i,j+1} - (q^{2i}-1)v_{i-1,j}, \tag{4.7}$$

$$Zv_{ij} = q^{-r}(q^{-2i}-q^{2j-2i})u_{i,j-1} - (q^{-2i}-1)v_{i-1,j}, \tag{4.8}$$

for all $i\in\mathbb{Z}, j\in\mathbb{N}$. *In the above equations,* $v_{i,-1} = 0$ *for* $i\in\mathbb{Z}$.

Proposition 4.4 *The following (i)-(iii) hold.*

(i) $Zv_{00} = 0$, *where the vector* $v_{00} \in \Gamma_Z$,

(ii) Z *is not invertible on* Γ_Z,

(iii) Z *is not invertible in* $U_{r,t}$.

5. The positive even subalgebra of $U_q(f(K,J))$

In this section, we consider a positive even subalgebra of $U_q(f(K,J))$. Let the subalgebra

$$\mathcal{A} = \{X^iY^jZ^k | i,j,k\in\mathbb{N},\ i+j+k \ even\} \tag{5.1}$$

be the positive even subalgebra of $U_q(f(K,J))$. We give a presentation for $\mathcal{A}$ by generators and relations in the following.

The relations (3.1)-(3.4) can be reformulated as follows:

$$q(1-XY) = q^{-1}(1-YX), \tag{5.2}$$

$$q(\bar{g}_1\bar{g}_2 - ZX) = q^{-1}(\bar{g}_1\bar{g}_2 - XZ), \tag{5.3}$$

$$q(\bar{h}_1\bar{h}_2 - YZ) = q^{-1}(\bar{h}_1\bar{h}_2 - ZY). \tag{5.4}$$

Definition 5.1 *Let* $\tilde{X}, \tilde{Y}, \tilde{Z}$ *be defined as the following elements of* $U_q(f(K,J))$,

$$\tilde{X} = q(\bar{h}_1\bar{h}_2 - YZ) = q^{-1}(\bar{h}_1\bar{h}_2 - ZY), \tag{5.5}$$

$$\tilde{Y} = q(\bar{g}_1\bar{g}_2 - ZX) = q^{-1}(\bar{g}_1\bar{g}_2 - XZ), \tag{5.6}$$

$$\tilde{Z} = q(1-XY) = q^{-1}(1-YX). \tag{5.7}$$

Proposition 5.2 *The following relations hold in $U_q(f(K,J))$:*

$$X\tilde{Y} = q^2\tilde{Y}X, \qquad X\tilde{Z} = q^{-2}\tilde{Z}X, \tag{5.8}$$

$$Y\tilde{Z} = q^2\tilde{Z}Y, \qquad Y\tilde{X} = q^{-2}\tilde{X}Y, \tag{5.9}$$

$$Z\tilde{X} = q^2\tilde{X}Z, \qquad Z\tilde{Y} = q^{-2}\tilde{Y}Z. \tag{5.10}$$

Proof Immediate from the relations (5.5)-(5.7).

We now display some relations satisfied by $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ in $U_q(f(K,J))$.

Lemma 5.3 *The following relations hold in $U_q(f(K,J))$:*

$$XY = 1 - q^{-1}\tilde{Z}, \quad YX = 1 - q\tilde{Z}, \tag{5.11}$$

$$ZX = \bar{g}_1\bar{g}_2 - q^{-1}\tilde{Y}, \quad XZ = \bar{g}_1\bar{g}_2 - q\tilde{Y}, \tag{5.12}$$

$$YZ = \bar{h}_1\bar{h}_2 - q^{-1}\tilde{X}, \quad ZY = \bar{h}_1\bar{h}_2 - q\tilde{X}. \tag{5.13}$$

Proof These equations are reformulations of (5.5)-(5.7).

Lemma 5.4 *The following relations hold in $U_q(f(K,J))$:*

$$X^2 = \bar{g}_1\bar{g}_2 - \frac{\tilde{Y}\tilde{Z} - q^{-1}\tilde{Z}\tilde{Y}}{(q-q^{-1})}, \tag{5.14}$$

$$Y^2 = \bar{h}_1\bar{h}_2 - \frac{\tilde{Z}\tilde{X} - q^{-1}\tilde{X}\tilde{Z}}{(q-q^{-1})}, \tag{5.15}$$

$$Z^2 = \bar{g}_1\bar{g}_2\bar{h}_1\bar{h}_2 - \frac{\tilde{X}\tilde{Y} - q^{-1}\tilde{Y}\tilde{X}}{(q-q^{-1})}. \tag{5.16}$$

Proof We only verify (5.14). From Definition 5.1, we obtain

$$\begin{aligned} q\tilde{X}\tilde{Y} - q^{-1}\tilde{Y}\tilde{X} &= q(\bar{h}_1\bar{h}_2 - ZY)(\bar{g}_1\bar{g}_2 - ZX) - q^{-1}(\bar{g}_1\bar{g}_2 - ZX)(\bar{h}_1\bar{h}_2 - ZY) \\ &= (q-q^{-1})(\bar{g}_1\bar{g}_2\bar{h}_1\bar{h}_2 - \bar{h}_1\bar{h}_2ZX - \bar{g}_1\bar{g}_2ZY) + qZYZX - q^{-1}ZXZY. \end{aligned}$$

Using (3.5)-(3.7), we find

$$\begin{aligned} qZYZX - q^{-1}ZXZY &= q^{-1}ZZ(YX) + (q-q^{-1})\bar{h}_1\bar{h}_2ZX - q^{-1}ZXZY \\ &= qZ(ZX)Y - (q-q^{-1})Z^2 + (q-q^{-1})\bar{h}_1\bar{h}_2ZX - q^{-1}ZXZY \\ &= (q-q^{-1})\bar{h}_1\bar{h}_2ZX - (q-q^{-1})Z^2 + (q-q^{-1})\bar{g}_1\bar{g}_2ZY. \end{aligned}$$

Thus, we have

$$q\tilde{X}\tilde{Y} - q^{-1}\tilde{Y}\tilde{X} = (q-q^{-1})(\bar{g}_1\bar{g}_2\bar{h}_1\bar{h}_2 - Z^2).$$

So the relation (5.14) holds. The remaining relations are similarly obtained.

Theorem 5.5 *The subalgebra $\mathcal{A}$ is generated by $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ and $\bar{g}_1\bar{g}_2$, $\bar{h}_1\bar{h}_2$ with the relations (5.11)-(5.16).*

Proof Let W denote the subalgebra of $U_q(f(K,J))$ generated by $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ and $\bar{g}_1\bar{g}_2$, $\bar{h}_1\bar{h}_2$ with the relations (5.11)-(5.16). It is obviously that $W \subseteq \mathcal{A}$. We now show that $\mathcal{A} \subseteq W$. Let $Y^iZ^jX^k$ be an element of $\mathcal{A}$. Since $i+j+k$ is even, then any element of $\mathcal{A}$ is spanned by

$$X^2,\ Y^2,\ Z^2,\ XY,\ XZ,\ YZ. \tag{5.17}$$

By Lemma 5.3 and Lemma 5.4, each term in (5.17) is contained in W. Therefore each element of $\mathcal{A}$ is contained in W. Consequently, $W = \mathcal{A}$.

References

[1] S.P. Smith. A class of algebras similar to the enveloping algebra of $sl(2)$. Trans. Amer. Math. Soc., 1990, 322: 285–314.
[2] Ji Qingzhong, Wang Dingguo. Finite-dimensional representations of quantum groups $U_q(f(K))$. East-West J. Math., 2000, 2(2): 201–213.
[3] Wang Dingguo, Ji Qingzhong, Yang Shilin, Finite-dimensional representations of quantum groups $U_q(f(K,H))$, Comm Algebra, 2002, 30: 2191–2211.
[4] Wu Zhixiang. Extended quantum enveloping algebra of $sl(2)$. Galsgow Math. J., 2009, 51: 441–465.
[5] Hong Yanyong, Wu Zhixiang. Extension of quantum enveloping algebras. Appl. Math. J. Chinese Univ., 2012, 27(3): 290–304.
[6] Tatsuro Ito, Paul Terwilliger, Chih-wen Weng. The quantum algebra $U_q(sl_2)$ and its equitable presentation. J. Algebra, 2006, 298(1): 284–301.
[7] P. Terwilliger. The equitable presentation for the quantum group $U_q(g)$ associated with a symmetrizable Kac-Moody algebra g. J. Algebra, 2006, 298(1): 302–319.
[8] Alison Gordon Lynch. The positive even subalgebra of $U_q(sl_2)$ and its finite-dimensional irreducible modules. arXiv:1506.02086v1 [math.QA] 5 Jun 2015.
[9] S. Bockting-Conrad, P. Terwilliger. The algebra $U_q(sl_2)$ in disguise. Linear Algebra Appl., 2013, 459: 548–585.