ANALYTIC REPRESENTATION OF PROBABILITY DENSITY FUNCTION OF RANDOM VARIABLE IN DISTRIBUTIONAL SENSE

Abstract. In this paper, we have proved that the probability density function $f(t)$ considered as a distribution has a Cauchy representation in O'_{α}. The distribution space O'_{α} is intermediate space between E' and D'.

Additionally, we give some important examples

Keywords: *SpaceD', Space* O'_{α}, *Space* D_{L^p}, *Cauchy representation, random variable,discrete random variable, probability density, analytic representation, support, spectrum.*

1. Introduction

The space D, is the space of all functions ϕ that are infinitely continuously differentiable and that vanish outside some bounded set.

The space of distributions D' is the space of all linear functionals on D that are continuous in the following sense: A functional $T \in D'$ is called continuous in the sense of D if and only if the following condition is satisfied: If $\{\phi_j\}$ is a sequence of functions in D such that the support of every ϕ_j is contained in a fixed compact set K, and if $\{D^p \phi_j\}$ converges uniformly on K for every fixed p, then

$$\lim_{j\to\infty}\langle T,\phi_j\rangle = \left\langle T, \lim_{j\to\infty}\phi_j \right\rangle$$

The sequence $\{\varphi_j\}$ of functions $\varphi_j \in C^{\infty}(\mathbb{R}^n)$ converges to $\varphi \in C^{\infty}(\mathbb{R}^n)$, at $j \to \infty$, if for each multindex α, the sequence $\{D_t^{\alpha}\varphi_j(t)\}$ converges to $D_t^{\alpha}\varphi(t)$ uniformly on any compact subset of $\mathbb{R}^n$, at $j \to \infty$.

The space $C^{\infty}(\mathbb{R}^n)$ with this convergence we will denote $E = E(\mathbb{R}^n)$.

The set of all linear, continuous functions of E is a set of distributions of E and we denote by E'.

The space D is dense in E- for each $\varphi \in E$ there is a sequence of functions in D which converges to φ in E. E' is dense in D'

$D_{L^p}, 1 \le p < \infty$ is the space of infinitely differentiable functions φ such that $D^\beta \varphi \in L^p$ for any multi-index β of non-negative integers.

The topology D_{L^p} is given by the norm.

$$\|\varphi\|_{m,p} = \left(\int_{\mathbb{R}^n} \left|\varphi^{(\beta)}(x)\right|^p dx \right)^{1/p}, |\beta| \le m, m = 0,1,2,3,...$$

The sequence $\{\varphi_j\}$ of D_{L^p} converges to the function φ, in $D_{L^p}, 1 \le p < \infty$ at $j \to \infty$, if each $\varphi_j \in D_{L^p}, \varphi \in D_{L^p}$, and

$$\lim_{j\to\infty} \left\|\varphi_j^{(\beta)} - \varphi^{(\beta)}\right\|_{L^p} = \lim_{j\to\infty} \left(\int_{\mathbb{R}^n} \left|\varphi_j^{(\beta)}(x) - \varphi^{(\beta)}(x)\right|^p dx \right)^{1/p} = 0$$

for each multi-index β.

D is dense in $D_{L^p}, 1 \le p < \infty$. $D'_{L^p}, 1 \le p < \infty$ is the space of linear, continuous functional of D_{L^q} where $\frac{1}{p} + \frac{1}{q} = 1$.

Space D'_{L^p} is subspace of D'.

Theorem [2, p.47]. Let $f(t)$ be a $\left(C^m\right)$-function. Let, for $0 \le k \le m$, $f^{(k)}(t) = O\left(|t|^\alpha\right)$ for some α less than zero. Let $\hat{f}$ be the function

$$\hat{f}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt$$

Then

$$\lim_{\varepsilon\to 0^+} \left[\hat{f}(x+i\varepsilon) - \hat{f}(x-i\varepsilon) \right] = f(x) \quad \text{for all } x.$$

Let $T \in (E')$ we call

$$\hat{T}(z) = \frac{1}{2\pi i} \left\langle T_t, \left(\frac{1}{t-z} \right) \right\rangle \qquad (1)$$

the analytic representation of T by means of Cauchy kernel, or Cauchy representation.

The condition $T \in (E')$ is sufficient, but not necessary, for the Cauchy integral of T

$$\widehat{T}(z) = \frac{1}{2\pi i}\left\langle T_t, \left(\frac{1}{t-z}\right)\right\rangle$$

to exist. If we consider

$$\widehat{T}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}dt$$

where f is a continuous function with $f(t) = O\left(|t|^{\alpha}\right)$ for some $\alpha < 0$ as $|t| \to \infty$; then $\widehat{T}(z)$ exists although $T = f(t)$ is not an element of E'. Also $\widehat{T}(z)$ does not exist for all $T \in D'$ since the function $\frac{1}{(t-z)}$ is not a test function in D.

However, for every $T \in D'$ its analytic representation exists i.e., there exists a function $f(z)$, analytic in the z-plane except on K, where K is the support of T, such that, [2]

$$\lim_{\varepsilon \to 0}\int_{-\infty}^{\infty}\left[f(x+i\varepsilon) - f(x-i\varepsilon)\right]\varphi(x)dx = \langle T, \varphi\rangle \text{ for all } \varphi \in D.$$

The dual space D' is too large for the study of the Cauchy integral $\widehat{T}(z)$ of a distribution T, and the dual space E' is too small. In order to extend the class of distributions which are representable by the Cauchy integral, [2] has introduced the distribution space O'_{α} which are intermediate space between E' and D'.

Let O_{α} be the space of all $\left(C^{\infty}\right)$ functions $\phi(t)$ on E^n such that $\phi(t) = O\left(\|t\|^{\alpha}\right)$ and $D^k\phi(t) = O\left(\|t\|^{\alpha}\right)$ for all k. Convergence is defined as follows: A sequence $\{\phi_j\}$ is said to be convergent in O_{α} if and only if the sequence $\{\phi_j\}$ converges uniformly on every compact subset of E^n in any order and if there exists for each k a constant C_k, independent of j, such that

$$\left|D^k \phi_j(t)\right| \leq C_k \left\|t\right\|^{\alpha} \text{ for all } t.$$

The space O'_{α} is the space of all continuous linear functionals on O_{α}.

For every $T \in O'_{\alpha}, \alpha \geq -1$ the Cauchy integral

$$\widehat{T}(z) = \frac{1}{2\pi i}\left\langle T_t, \left(\frac{1}{t-z}\right)\right\rangle$$

is well defined for $z \in \Omega = \{z : \operatorname{Im}(z) \neq 0\}$; in fact, we know that $\widehat{T}(z)$ is an analytic function of z in $\mathbb{R} \setminus \operatorname{supp}(T)$. The analytic representation of elements $T \in O'_{\alpha}, \alpha \geq -1$ in terms of the Cauchy integral is given by theorem.

Theorem [3]. If $T \in O'_{\alpha}, \alpha \geq -1$ then

$$\lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} \left[\widehat{T}(x+i\varepsilon) - \widehat{T}(x-i\varepsilon)\right]\varphi(x)dx = \langle T, \varphi \rangle$$

and

$$\lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} \left[\widehat{T}(x+i\varepsilon) + \widehat{T}(x-i\varepsilon)\right]\varphi(x)dx = -2\langle T, \tilde{\varphi} \rangle$$

For all $\varphi \in D(\mathbb{R})$ where $\tilde{\varphi}(t)$ is the principal value integral

$$\tilde{\varphi}(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\varphi(x)}{x-t}dx$$

Also, in the [1] is proved the existence of the Cauchy representation of the distributions in the intermediate spaces $\left(D_{L_p}\left(R^n\right)\right)' \supset E'\left(R^n\right), \left(1 < p < \infty\right)$.

Theorem[1]. Let $f \in \left(D_{L_p}(R)\right)', (1 < p < \infty)$ and $F(z)$ be the complex-valued function defined in the region $\Omega = \{z : \operatorname{Im} z \neq 0\}$ by:

$$F(z) = \frac{1}{2\pi i}\left\langle f(t), \frac{1}{t-z}\right\rangle$$

Then $F(z)$ is the Cauchy representation of the generalized function f.

Let X be a random variable taking values in the interval $(-\infty, \infty)$ and let $F(t)$ be a probability distribution function for the random variable X, i.e., $F(t)$ is defined as the probability that X takes the values in the interval $(-\infty, t)$. As it is well known, the function $F(t)$ satisfies the following properties:

(i) $0 \leq F(t) \leq 1$;

(ii) $\lim_{t\to-\infty} F(t) = 0$ and $\lim_{t\to\infty} F(t) = 1$;

(iii) $F(t_1) < F(t_2)$ for $t_1 < t_2$.

Let X takes the values $x_1, x_2, \ldots$ with the probabilities $P(X = x_k) = p_k$ for $k = 1, 2, \ldots$, respectively. A random variable defined in this way that receives a countable number of values (consequently, finally many values) is called a *discrete random variable*.

The function $F(t) = \sum_{x_k \leq t} p_k$ is called a value distribution function or a cumulative function.

From the definition we also get that

$$P(a < X \leq b) = F(b) - F(a)$$

If the function $F(t)$ is such that $F'(t) = f(t)$, except perhaps in a finite number of points, then $f(t)$ is called the *density function* or the *probability density function* for the random variable X and it, also, holds

$$F(t) = \int_{-\infty}^{t} f(x)dx.$$

The density function belongs to the L^1 space, and satisfies the condition

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Also, $P(a < X \le b) = \int_a^b f(t)dt$.

If X is a discrete random variable with a distribution function $F(t)$, then $f(t)$ does not exist in the ordinary sense. However, the probability density defines a generalized function on some space of test functions. For example, suppose that it is certain that the random variable x takes the value x_0. Then,

$$F(t) = \begin{cases} 0 & \text{for } t < x_0 \\ 1 & \text{for } t > x_0 \end{cases}$$

Thus

$$F(t) = H(t - x_0)$$

where H is the Heaviside step function. In this case the probability density $f(t)$ does not exist in the ordinary sense. If, however, we admit generalized functions, then,

$$f(t) = \frac{d}{dt} H(t - x_0) = \delta(t - x_0)$$

is the Dirac delta function.

As a matter of fact, every probability density $f(t)$ belongs to a certain space of generalized functions. This fact is proved in the following theorem.

Theorem [1]. Every probability density $f(t)$ defines a generalized function on the space $D_{L_p}(R), (1 < p < \infty)$ of test functions.

Corollary [1]. For every probability density $f(t)$, the Cauchy representation, as defined in (1), exists.

Discrete case. Suppose a random variable takes values $t_1, t_2, \ldots, t_n$ with probabilities $p_1, \ldots, p_n, \sum_j p_j = 1$. Then $f(t)$ is a multiple step function and

$$f(t) = F'(t) = \sum_{i=1}^{n} p_k \delta(t - x_k)$$

And the Cauchy representation of $f(t)$ equals

$$\widehat{f}(t) = \frac{1}{2\pi i} \sum_{k=1}^{n} p_k \frac{1}{t_k - z}$$

The support of $f(t)$ is called the *spectrum* of the random variable.

2. Main results

2.1 We will give an interesting theorem of the Cauchy representation of the density function in O'_α.

Theorem. Let X be a discrete random variable with sets of values $t_1, t_2, \ldots, t_k, \ldots$ and probabilities $P(X = t_k) = p_k$ for $k = 1, 2, 3, \ldots$. Then the density function $f(t)$ considered as a distribution has a Cauchy representation in the space O'_α.

Proof: Let $F(t)$ be a function of the probability distribution for the random variable X.

As we know, it is a step function. It does not have derivation in the ordinary sense. It has a density function in the distributive sense, i.e.

$$F'(t) = \sum_k p_k \delta(t - t_k) = f(t)$$

We will show that $f(t)$ has Cauchy representation in the space O'_α.

Let $\phi \in D$ be an arbitrary function and let $supp\phi \subset [-\delta, \delta]$.

We consider the integral

$$\int_{-\infty}^{\infty} \left[\widehat{f}(x+iy) - \widehat{f}(x-iy) \right] \phi(x) dx \text{ ,where } \widehat{f}(x+iy) = \frac{1}{2\pi i} \sum_{k=1}^{n} \frac{p_k}{t_k - z}, \quad z = x + iy .$$

We first consider the integral

$$\int_{\mathbb{R}} \hat{f}(x+iy)\phi(x)dx = \frac{1}{2\pi i}\int_{\mathbb{R}} \sum_{k=1}^{n} \frac{p_k}{t_k - z}\phi(x)dx =$$
$$= \frac{1}{2\pi i}\sum_{k=1}^{n}\left[\int_{|x-t_k|\le\delta} \frac{p_k}{t_k - x - iy}\phi(x)dx + \int_{|x-t_k|>\delta} \frac{p_k}{t_k - x - iy}\phi(x)dx\right] =$$
$$= \frac{1}{2\pi i}\sum_{k=1}^{n}\left[I_1 + I_2\right]$$

Now

$$I_1 = p_k \int_{|x-t_k|\le\delta} \frac{\phi(x)dx}{t_k - x - iy} = p_k \int_{|x-t_k|\le\delta} \frac{\phi(x)-\phi(t_k)}{t_k - x - iy}dx + p_k \int_{|x-t_k|\le\delta} \frac{\phi(t_k)dx}{t_k - x - iy} = I_1' + I_1'',$$

where

$$I_1' = p_k \int_{|x-t_k|\le\delta} \frac{\phi(x)-\phi(t_k)}{t_k - x - iy}dx .$$

Since the function ϕ is continuous in t_k for given $\varepsilon > 0$ there exists $\delta > 0$, such that

$|\phi(x)-\phi(t_k)| < \varepsilon$ for $|x-t_k| < \delta$. So, $I_1' \to 0$ for $y \to 0^+$.

We now consider the integral I_1''.

$$I_1'' = p_k \int_{|x-t_k|\le\delta} \frac{\phi(t_k)dx}{t_k - x - iy} = p_k \cdot \phi(t_k)\ln(t_k - x - iy)\Big|_{|t_k - x|\le\delta} =$$
$$= p_k\phi(t_k)\left(\ln|t_k - x - iy| + i\arg\left(t_k - x - iy\right)\right)\Big|_{|t_k - x|\le\delta} \to p_k\phi(t_k)i\pi \text{ if } y \to 0^+.$$

It remains to consider the following integral

$$I_2 = \int_{|x-t_k|>\delta} \frac{\phi(x)}{t_k - x - iy}dx.$$

Because the function ϕ is bounded, there exists $M > 0$ such that $|\phi(x)| \le M$, from where we get that

$$\left|\int_{|x-t_k|>\delta} \frac{\phi(x)dx}{t_k - x - iy}\right| \le M\int_{\mathbb{R}} \frac{dx}{t_k - x - iy} \to 0 \text{ when } y \to 0 .$$

Similarly, we get for the second integral

$$\int_{\mathbb{R}} \hat{f}(x-iy)\phi(x)dx = \frac{1}{2\pi i}\sum_{k=1}^{n}\int_{\mathbb{R}} \frac{p_k\phi(x)}{t_k - x + iy}dx$$

Finally, we get that

$$\lim_{y\to 0^+}\int_{\mathbb{R}}\left[\hat{f}(x+iy) - \hat{f}(x-iy)\right]\phi(x)dx = \sum_{k=1}^{n} p_k\phi(t_k) = \sum_{k=1}^{n} p_k\left\langle \delta(t-t_k),\phi\right\rangle = \left\langle f,\phi\right\rangle$$

Example 1. The Poisson's distribution $P(X=k) = \frac{\lambda^k}{k!}e^{-\lambda}$ has a Cauchy representation in the space O'_α.

$$\hat{f}(z) = \frac{1}{2\pi i}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda}\cdot\frac{1}{k-z}.$$

The function $\hat{f}(z)$ has a singular (isolated singularity at the points $z=k$)

$$\operatorname{Res}\left(\hat{f};k\right) = \frac{1}{2\pi i}\cdot\frac{\lambda^k}{k!}e^{-\lambda}.$$

Example 2. The geometric distribution $P(X=k) = q^{k-1}\cdot p,\ p+q=1$ has a representation

$$\hat{f}(z) = \frac{1}{2\pi i}\sum_{k=1}^{\infty}\frac{q^{k-1}\cdot p}{k-z}.$$

The function $\hat{f}(z)$ has a singular (isolated singularity at the points $z=k$)

$$\operatorname{Res}\left(\hat{f}(z);k\right) = \frac{1}{2\pi i}q^{k-1}\cdot p.$$

Example 3. Let the random variable X has the Poisson distribution, $\frac{x^k}{k!}e^{-x}$. We will show that the function $T(z)$ is an analytical representation of the Poisson distribution, in the sense of distributions, where

$$T(z)=\frac{z^k}{k!}e^{-z}\log z, \quad z\in \mathbb{C}/\mathbb{R}, \quad -\pi \le \arg z \le \pi .$$

Let $\varphi \in \mathcal{D}$ be a function with support in $[-a,a]$, where $a>0$. Then, we have

$$\int_{-\infty}^{\infty} T(z)\varphi(x)dx=\int_{-\infty}^{\infty}\frac{z^k}{k!}e^{-z}\log z\varphi(x)dx=\frac{1}{k!}\int_{\mathbf{R}}(x+iy)^k e^{-(x+iy)}\log(x+iy)\varphi(x)dx=$$
$$=\frac{1}{k!}\int_{\mathbf{R}}(x+iy)^k e^{-(x+iy)}\left[\ln\sqrt{x^2+y^2}+i\arg\left(x+iy\right)\right]\varphi(x)dx=$$
$$=\frac{1}{k!}\int_{\mathbf{R}}\sum_{n=0}^{k}\binom{k}{n}x^n(iy)^{k-n}e^{-x}e^{-iy}\left[\ln\sqrt{x^2+y^2}+i\arg\left(x+iy\right)\right]\varphi(x)dx=$$
$$=\frac{1}{k!}\sum_{n=0}^{k}\binom{k}{n}\int_{\mathbf{R}}x^n(iy)^{k-n}e^{-x}e^{-iy}\left[\ln\sqrt{x^2+y^2}+i\arg\left(x+iy\right)\right]\varphi(x)dx=$$
$$=\frac{1}{k!}\sum_{n=0}^{k}\binom{k}{n}\int_{\mathbf{R}}x^n(iy)^{k-n}e^{-(x+iy)}\left[\ln\sqrt{x^2+y^2}+i\arg\left(x+iy\right)\right]dx$$

Let us consider the integrand

$$x^n(iy)^{k-n}e^{-(x+iy)}\left[\ln\sqrt{x^2+y^2}+i\arg\left(x+iy\right)\right]$$

For $x<0$, if $y\to 0^+$ then $\arg\left(x+iy\right)\to\pi$, so the integrand tends to $x^n e^{-x}\left(\ln|x|+i\pi\right)$. For $x>0$, if $y\to 0^+$ then $\arg\left(x+iy\right)\to 0$. Therefore, we get that

$$\lim_{y\to 0+}\frac{1}{k!}\sum_{n=0}^{k}\binom{k}{n}\int_{\mathbf{R}}x^n(iy)^{k-n}e^{-(x+iy)}\left[\ln\sqrt{x^2+y^2}+i\arg(x+iy)\right]\varphi(x)dx=$$
$$=\frac{1}{k!}\sum_{n=0}^{k}\binom{k}{n}\int_{\mathbf{R}}x^n 0^{k-n}e^{-x}\left(\ln|x|+i\pi\right)\varphi(x)dx=$$
$$=\frac{1}{k!}\int_{\mathbf{R}}\sum_{n=0}^{k}\binom{k}{n}x^n 0^{k-n}e^{-x}\left(\ln|x|+i\pi\right)\varphi(x)dx=\frac{1}{k!}\int_{\mathbf{R}}x^k e^{-x}\left(\ln|x|+i\pi\right)\varphi(x)dx$$

Let us now consider the second integral

$$\int_{\mathbf{R}} T(x-iy)\varphi(x)dx = \int_{\mathbf{R}} \frac{(x-iy)^k}{k!} e^{-(x-iy)} \log(x-iy)\varphi(x) =$$

$$= \int_{\mathbf{R}} \frac{1}{k!} \sum_{n=0}^{k} \binom{k}{n} x^n (-iy)^{k-n} e^{-(x-iy)} \left(\ln\sqrt{x^2+y^2} + i\arg\left(x-iy\right)\right)\varphi(x)dx =$$

$$= \frac{1}{k!} \sum_{n=0}^{k} \binom{k}{n} \int_{\mathbf{R}} x^n (-iy)^{k-n} e^{-(x-iy)} \left(\ln\sqrt{x^2+y^2} + i\arg\left(x-iy\right)\right)\varphi(x)dx =$$

$$= \frac{1}{k!} \sum_{n=0}^{k} \binom{k}{n} \int_{\mathbf{R}} x^n (-iy)^{k-n} e^{-x} e^{iy} \left(\ln\sqrt{x^2+y^2} + i\arg\left(x-iy\right)\right)\varphi(x)dx.$$

For $x<0$, if $y \to 0^+$ then $\arg\left(x-iy\right) \to -\pi$, so the integrand tends to $x^n e^{-x}\left(\ln|x| - i\pi\right)$. For $x>0$, if $y \to 0^+$ then $\arg\left(x-iy\right) \to 0$. Therefore, we get that

$$\lim_{y\to 0^+} \frac{1}{k!} \sum_{n=0}^{k} \binom{k}{n} \int_{\mathbf{R}} x^n (iy)^{k-n} e^{-x} e^{iy} \left(\ln\sqrt{x^2+y^2} + i\arg\left(x-iy\right)\right)\varphi(x)dx =$$

$$= \frac{1}{k!} \int_{\mathbf{R}} x^k e^{-x} \left(\ln|x| - i\pi\right)\varphi(x)dx.$$

So, we finally have that

$$\lim_{y\to 0^+} \int_{\mathbf{R}} \left[T(x+iy) - T(x-iy)\right]\varphi(x)dx =$$

$$= \lim_{y\to 0^+} \int_{\mathbf{R}} \left[\frac{(x+iy)^k}{k!} e^{-(x+iy)} \log(x+iy) - \frac{(x-iy)^k}{k!} e^{-(x-iy)} \log(x-iy) \right]\varphi(x)dx =$$

$$= \int_{\mathbf{R}} \left[\frac{x^k}{k!} e^{-x} \left(\ln x + i\pi\right) - \frac{x^k}{k!} e^{-x} \left(\ln x - i\pi\right) \right]\varphi(x)dx =$$

$$= \int_{\mathbf{R}} \frac{x^k}{k!} e^{-x} \left[\ln x + i\pi - \ln x + i\pi\right]\varphi(x)dx =$$

$$= \int_{\mathbf{R}} \frac{x^k}{k!} e^{-x} 2i\pi\varphi(x)dx =$$

$$= 2\pi i \int_{\mathbf{R}} \frac{x^k}{k!} e^{-x}\varphi(x)dx.$$

3 CONCLUSIONS

The Cauchy representation in the space O'_α exists, but the Cauchy representation is not valid for every Schwartz distribution $f \in D$. In [3] is proved that the Cauchy representation of distributions in D_{L^p}, $1 \leq p < \infty$ exist, and that every probability density defines a generalized function on the space D_{L^p}, $1 \leq p < \infty$ of test functions.

In this work we have proved that the probability density function $f(t)$ in distributional sense has Cauchy representation in the space O'_α. We found analytical representation of Poisson distribution, in sense of distributions and also, we give the prove of that and some other exercises. So, other researchers can find other spaces, for which this applies.

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