

ANALYTIC REPRESENTATION OF PROBABILITY DENSITY FUNCTION OF RANDOM VARIABLE IN DISTRIBUTIONAL SENSE

Abstract. In this paper, we have proved that the probability density function $f(t)$ considered as a distribution has a Cauchy representation in O'_{α} . Additionally, we give some important examples.

Keywords: Space D' , Space O'_{α} , Space D_{L^p} , Cauchy representation, random variable, discrete random variable, probability density, analytic representation, support, spectrum.

1. Introduction

The space D , is the space of all functions ϕ that are infinitely continuously differentiable and that vanish outside some bounded set.

The space of distributions D' is the space of all linear functionals on D that are continuous in the following sense: A functional $T \in D'$ is called continuous in the sense of D if and only if the following condition is satisfied: If $\{\phi_j\}$ is a sequence of functions in D such that the support of every ϕ_j is contained in a fixed compact set K , and if $\{D^p \phi_j\}$ converges uniformly on K for every fixed p , then

$$\lim_{j \rightarrow \infty} \langle T, \phi_j \rangle = \langle T, \lim_{j \rightarrow \infty} \phi_j \rangle$$

The sequence $\{\phi_j\}$ of functions $\phi_j \in C^{\infty}(\square^n)$ converges to $\phi \in C^{\infty}(\square^n)$, at $j \rightarrow \infty$, if for each multindex α , the sequence $\{D_i^{\alpha} \phi_j(t)\}$ converges to $D_i^{\alpha} \phi(t)$ uniformly on any compact subset of \square^n , at $j \rightarrow \infty$.

The space $C^{\infty}(\square^n)$ with this convergence we will denote $E = E(\square^n)$.

The set of all linear, continuous functionals on E is a set of distributions on E and we denote by E' .

The space D is dense in E - for each $\phi \in E$ there is a sequence of functions in D which converges to ϕ in E . E' is dense in D'

$D_{L^p}, 1 \leq p < \infty$ is the space of infinitely differentiable functions φ such that $D^\beta \varphi \in L^p$ for any multi-index β of non-negative integers.

The topology D_{L^p} is given by the norm.

$$\|\varphi\|_{m,p} = \left(\int_{\square^n} |\varphi^{(\beta)}(x)|^p dx \right)^{1/p}, |\beta| \leq m, m = 0, 1, 2, 3, \dots$$

The sequence $\{\varphi_j\}$ of D_{L^p} converges to the function φ , in $D_{L^p}, 1 \leq p < \infty$ at $j \rightarrow \infty$, if each $\varphi_j \in D_{L^p}, \varphi \in D_{L^p}$, and

$$\lim_{j \rightarrow \infty} \|\varphi_j^{(\beta)} - \varphi^{(\beta)}\|_{L^p} = \lim_{j \rightarrow \infty} \left(\int_{\square^n} |\varphi_j^{(\beta)}(x) - \varphi^{(\beta)}(x)|^p dx \right)^{1/p} = 0$$

for each multi-index β .

D is dense in $D_{L^p}, 1 \leq p < \infty$. $D'_{L^p}, 1 \leq p < \infty$ is the space of linear, continuous functional of D_{L^q} where $\frac{1}{p} + \frac{1}{q} = 1$.

Space D'_{L^p} is subspace of D' .

Theorem [2, p.47]. Let $f(t)$ be a (C^m) -function. Let, for $0 \leq k \leq m, f^{(k)}(t) = O(|t|^\alpha)$ for some α less than zero. Let \widehat{f} be the function

$$\widehat{f}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt.$$

Then

$$\lim_{\varepsilon \rightarrow 0^+} [\widehat{f}(x+i\varepsilon) - \widehat{f}(x-i\varepsilon)] = f(x) \text{ for all } x.$$

Let $T \in (E')$ we call

$$\widehat{T}(z) = \frac{1}{2\pi i} \left\langle T, \left(\frac{1}{t-z} \right) \right\rangle \quad (1)$$

the analytic representation of T by means of Cauchy kernel, or Cauchy representation.

The condition $T \in (E')$ is sufficient, but not necessary, for the Cauchy integral of T

$$\widehat{T}(z) = \frac{1}{2\pi i} \left\langle T, \left(\frac{1}{t-z} \right) \right\rangle$$

to exist. If we consider

$$\widehat{T}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt$$

where f is a continuous function with $f(t) = O(|t|^\alpha)$ for some $\alpha < 0$ as $|t| \rightarrow \infty$; then $\widehat{T}(z)$ exists although $T = f(t)$ is not an element of E' . Also $\widehat{T}(z)$ does not exist for all $T \in D'$ since the function $\frac{1}{(t-z)}$ is not a test function in D .

However, for every $T \in D'$ its analytic representation exists i.e., there exists a function $f(z)$, analytic in the z -plane except on K , where K is the support of T , such that, [2]

$$\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} [f(x+i\varepsilon) - f(x-i\varepsilon)] \varphi(x) dx = \langle T, \varphi \rangle \text{ for all } \varphi \in D.$$

The dual space D' is too large for the study of the Cauchy integral $\widehat{T}(z)$ of a distribution T , and the dual space E' is too small. In order to extend the class of distributions which are representable by the Cauchy integral, [2] has introduced the distribution space O'_α which are intermediate space between E' and D' .

Let O_α be the space of all (C^∞) functions $\phi(t)$ on E^n such that $\phi(t) = O(\|t\|^\alpha)$ and $D^k \phi(t) = O(\|t\|^\alpha)$ for all k . Convergence is defined as follows: A sequence $\{\phi_j\}$ is said to be convergent in O_α if and only if the sequence $\{\phi_j\}$ converges uniformly on every compact subset of E^n in any order and if there exists for each k a constant C_k , independent of j , such that

$$|D^k \phi_j(t)| \leq C_k \|t\|^\alpha \text{ for all } t.$$

The space O'_α is the space of all continuous linear functionals on O_α .

For every $T \in O'_\alpha, \alpha \geq -1$ the Cauchy integral

$$\widehat{T}(z) = \frac{1}{2\pi i} \left\langle T, \left(\frac{1}{t-z} \right) \right\rangle$$

is well defined for $z \in \Omega = \{z : \text{Im}(z) \neq 0\}$; in fact, we know that $\widehat{T}(z)$ is an analytic function of z in $\square \setminus \text{supp}(T)$. The analytic representation of elements $T \in O'_\alpha, \alpha \geq -1$ in terms of the Cauchy integral is given by theorem.

Theorem [3]. If $T \in O'_\alpha, \alpha \geq -1$ then

$$\lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{\infty} [\widehat{T}(x+i\varepsilon) - \widehat{T}(x-i\varepsilon)] \varphi(x) dx = \langle T, \varphi \rangle$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{\infty} [\widehat{T}(x+i\varepsilon) + \widehat{T}(x-i\varepsilon)] \varphi(x) dx = -2 \langle T, \tilde{\varphi} \rangle$$

For all $\varphi \in D(\square)$ where $\tilde{\varphi}(t)$ is the principal value integral

$$\tilde{\varphi}(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x-t} dx.$$

Also, in the [1] is proved the existence of the Cauchy representation of the distributions in the intermediate spaces $(D_{L_p}(R^n))' \supset E'(R^n), (1 < p < \infty)$

Theorem [1]. Let $f \in (D_{L_p}(R))', (1 < p < \infty)$ and $F(z)$ be the complex-valued function defined in the region $\Omega = \{z : \text{Im} z \neq 0\}$ by:

$$F(z) = \frac{1}{2\pi i} \left\langle f(t), \frac{1}{t-z} \right\rangle$$

Then $F(z)$ is the Cauchy representation of the generalized function f .

Let X be a random variable taking values in the interval $(-\infty, \infty)$ and let $F(t)$ be a probability distribution function for the random variable X , i.e., $F(t)$ is defined as the probability that X takes the values in the interval $(-\infty, t)$. As it is well known, the function $F(t)$ satisfies the following properties:

- (i) $0 \leq F(t) \leq 1$;
- (ii) $\lim_{t \rightarrow -\infty} F(t) = 0$ and $\lim_{t \rightarrow \infty} F(t) = 1$;
- (iii) $F(t_1) < F(t_2)$ for $t_1 < t_2$.

Let X takes the values x_1, x_2, \dots with the probabilities $P(X = x_k) = p_k$ for $k = 1, 2, \dots$, respectively. A random variable defined in this way that receives a countable number of values (consequently, finally many values) is called a *discrete random variable*.

The function $F(t) = \sum_{x_k \leq t} p_k$ is called a value distribution function or a cumulative function.

From the definition we also get that

$$P(a < X \leq b) = F(b) - F(a)$$

If the function $F(t)$ is such that $F'(t) = f(t)$, except perhaps in a finite number of points, then $f(t)$ is called the *density function* or the *probability density function* for the random variable X and it, also, holds

$$F(t) = \int_{-\infty}^t f(x) dx.$$

The density function belongs to the L^1 space, and satisfies the condition

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

Also, $P(a < X \leq b) = \int_a^b f(t) dt$.

If X is a discrete random variable with a distribution function $F(t)$, then $f(t)$ does not exist in the ordinary sense. However, the probability density defines a generalized function on some space of test functions. For example, suppose that it is certain that the random variable X takes the value x_0 . Then,

$$F(t) = \begin{cases} 0 & \text{for } t < x_0 \\ 1 & \text{for } t > x_0 \end{cases}$$

Thus

$$F(t) = H(t - x_0)$$

where H is the Heaviside step function. In this case the probability density $f(t)$ does not exist in the ordinary sense. If, however, we admit generalized functions, then,

$$f(t) = \frac{d}{dt} H(t - x_0) = \delta(t - x_0)$$

is the Dirac delta function.

As a matter of fact, every probability density $f(t)$ belongs to a certain space of generalized functions. This fact is proved in the following theorem.

Theorem [1]. Every probability density $f(t)$ defines a generalized function on the space $D_{L_p}(R)$, ($1 < p < \infty$) of test functions.

Corollary [1]. For every probability density $f(t)$, the Cauchy representation, as defined in (1), exists.

Discrete case. Suppose a random variable takes values t_1, t_2, \dots, t_n with probabilities p_1, \dots, p_n , $\sum_j p_j = 1$. Then $f(t)$ is a multiple step function and

$$f(t) = F'(t) = \sum_{i=1}^n p_i \delta(t - x_i).$$

And the Cauchy representation of $f(t)$ equals

$$\widehat{f}(t) = \frac{1}{2\pi i} \sum_{k=1}^n p_k \frac{1}{t_k - z}.$$

The support of $f(t)$ is called the *spectrum* of the random variable.

2. Main results

2.1 We will give an interesting theorem of the Cauchy representation of the density function in O'_α .

Theorem. Let X be a discrete random variable with sets of values $t_1, t_2, \dots, t_k, \dots$ and probabilities $P(X=t_k) = p_k$ for $k = 1, 2, 3, \dots$. Then the density function $f(t)$ considered as a distribution has a Cauchy representation in the space O'_α .

Proof: Let $F(t)$ be a function of the probability distribution for the random variable X .

As we know, it is a step function. It does not have derivation in the ordinary sense. It has a density function in the distributive sense, i.e.

$$F'(t) = \sum_k p_k \delta(t - t_k) = f(t).$$

We will show that $f(t)$ has Cauchy representation in the space O'_α .

Let $\phi \in D$ be an arbitrary function and let $\text{supp}\phi \subset [-\delta, \delta]$.

We consider the integral

$$\int_{-\infty}^{\infty} [\widehat{f}(x+iy) - \widehat{f}(x-iy)] \phi(x) dx, \text{ where } \widehat{f}(x+iy) = \frac{1}{2\pi i} \sum_{k=1}^n \frac{p_k}{t_k - z}, \quad z = x + iy.$$

We first consider the integral

$$\begin{aligned}
\int_{\square} \widehat{f}(x+iy)\phi(x)dx &= \frac{1}{2\pi i} \int_{\square} \sum_{k=1}^n \frac{p_k}{t_k - z} \phi(x)dx = \\
&= \frac{1}{2\pi i} \sum_{k=1}^n \left[\int_{|x-t_k| \leq \delta} \frac{p_k}{t_k - x - iy} \phi(x)dx + \int_{|x-t_k| > \delta} \frac{p_k}{t_k - x - iy} \phi(x)dx \right] = \\
&= \frac{1}{2\pi i} \sum_{k=1}^n [I_1 + I_2]
\end{aligned}$$

Now

$$I_1 = p_k \int_{|x-t_k| \leq \delta} \frac{\phi(x)dx}{t_k - x - iy} = p_k \int_{|x-t_k| \leq \delta} \frac{\phi(x) - \phi(t_k)}{t_k - x - iy} dx + p_k \int_{|x-t_k| \leq \delta} \frac{\phi(t_k)dx}{t_k - x - iy} = I_1' + I_1'',$$

where

$$I_1' = p_k \int_{|x-t_k| \leq \delta} \frac{\phi(x) - \phi(t_k)}{t_k - x - iy} dx.$$

Since the function ϕ is continuous in t_k for given $\varepsilon > 0$ there exists $\delta > 0$, such that

$$|\phi(x) - \phi(t_k)| < \varepsilon \text{ for } |x - t_k| < \delta. \text{ So, } I_1' \rightarrow 0 \text{ for } y \rightarrow 0^+.$$

We now consider the integral I_1'' .

$$\begin{aligned}
I_1'' &= p_k \int_{|x-t_k| \leq \delta} \frac{\phi(t_k)dx}{t_k - x - iy} = p_k \cdot \phi(t_k) \ln(t_k - x - iy) \Big|_{|t_k - x| \leq \delta} = \\
&= p_k \phi(t_k) (\ln|t_k - x - iy| + i \arg(t_k - x - iy)) \Big|_{|t_k - x| \leq \delta} \rightarrow p_k \phi(t_k) i\pi \text{ if } y \rightarrow 0^+.
\end{aligned}$$

It remains to consider the following integral

$$I_2 = \int_{|x-t_k| > \delta} \frac{\phi(x)}{t_k - x - iy} dx.$$

Because the function ϕ is bounded, there exists $M > 0$ such that $|\phi(x)| \leq M$, from where we get that

$$\left| \int_{|x-t_k| > \delta} \frac{\phi(x)dx}{t_k - x - iy} \right| \leq M \int_{\square} \frac{dx}{t_k - x - iy} \rightarrow 0 \text{ when } y \rightarrow 0.$$

Similarly, we get for the second integral

$$\int_{\square} \widehat{f}(x-iy)\phi(x)dx = \frac{1}{2\pi i} \sum_{k=1}^n \int_{\square} \frac{p_k \phi(x)}{t_k - x + iy} dx.$$

Finally, we get that

$$\lim_{y \rightarrow 0^+} \int_{\square} [\widehat{f}(x+iy) - \widehat{f}(x-iy)]\phi(x)dx = \sum_{k=1}^n p_k \phi(t_k) = \sum_{k=1}^n p_k \langle \delta(t-t_k), \phi \rangle = \langle f, \phi \rangle.$$

Example 1. The Poisson's distribution $P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$ has a Cauchy representation in the space O'_α .

$$\widehat{f}(z) = \frac{1}{2\pi i} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} \cdot \frac{1}{k-z}.$$

The function $\widehat{f}(z)$ has a singular (isolated singularity at the points $z = k$)

$$\text{Res}(\widehat{f}; k) = \frac{1}{2\pi i} \cdot \frac{\lambda^k}{k!} e^{-\lambda}.$$

Example 2. The geometric distribution $P(X=k) = q^{k-1} \cdot p$, $p+q=1$ has a representation

$$\widehat{f}(z) = \frac{1}{2\pi i} \sum_{k=1}^{\infty} \frac{q^{k-1} \cdot p}{k-z}.$$

The function $\widehat{f}(z)$ has a singular (isolated singularity at the points $z = k$)

$$\text{Res}(\widehat{f}(z); k) = \frac{1}{2\pi i} q^{k-1} \cdot p.$$

Example 3. Let the random variable X has the Poisson distribution, $\frac{x^k}{k!} e^{-x}$. We will show that the function $T(z)$ is an analytical representation of the Poisson distribution, in the sense of distributions, where

$$T(z) = \frac{z^k}{k!} e^{-z} \log z, \quad z \in \mathbb{C} \setminus \mathbb{R}^-, \quad -\pi \leq \arg z \leq \pi.$$

Let $\varphi \in \mathcal{D}$ be a function with support in $[-a, a]$, where $a > 0$. Then, we have

$$\begin{aligned} \int_{-\infty}^{\infty} T(z)\varphi(x)dx &= \int_{-\infty}^{\infty} \frac{z^k}{k!} e^{-z} \log z \varphi(x)dx = \frac{1}{k!} \int_{\mathbf{R}} (x+iy)^k e^{-(x+iy)} \log(x+iy) \varphi(x)dx = \\ &= \frac{1}{k!} \int_{\mathbf{R}} (x+iy)^k e^{-(x+iy)} \left[\ln \sqrt{x^2+y^2} + i \arg(x+iy) \right] \varphi(x)dx = \\ &= \frac{1}{k!} \int_{\mathbf{R}} \sum_{n=0}^k \binom{k}{n} x^n (iy)^{k-n} e^{-x} e^{-iy} \left[\ln \sqrt{x^2+y^2} + i \arg(x+iy) \right] \varphi(x)dx = \\ &= \frac{1}{k!} \sum_{n=0}^k \binom{k}{n} \int_{\mathbf{R}} x^n (iy)^{k-n} e^{-x} e^{-iy} \left[\ln \sqrt{x^2+y^2} + i \arg(x+iy) \right] \varphi(x)dx = \\ &= \frac{1}{k!} \sum_{n=0}^k \binom{k}{n} \int_{\mathbf{R}} x^n (iy)^{k-n} e^{-(x+iy)} \left[\ln \sqrt{x^2+y^2} + i \arg(x+iy) \right] dx \end{aligned}$$

Let us consider the integrand

$$x^n (iy)^{k-n} e^{-(x+iy)} \left[\ln \sqrt{x^2+y^2} + i \arg(x+iy) \right]$$

For $x < 0$, if $y \rightarrow 0^+$ then $\arg(x+iy) \rightarrow \pi$, so the integrand tends to $x^n e^{-x} (\ln|x| + i\pi)$. For

$x > 0$, if $y \rightarrow 0^+$ then $\arg(x+iy) \rightarrow 0$. Therefore, we get that

$$\begin{aligned} \lim_{y \rightarrow 0^+} \frac{1}{k!} \sum_{n=0}^k \binom{k}{n} \int_{\mathbf{R}} x^n (iy)^{k-n} e^{-(x+iy)} \left[\ln \sqrt{x^2+y^2} + i \arg(x+iy) \right] \varphi(x)dx &= \\ &= \frac{1}{k!} \sum_{n=0}^k \binom{k}{n} \int_{\mathbf{R}} x^n 0^{k-n} e^{-x} (\ln|x| + i\pi) \varphi(x)dx = \\ &= \frac{1}{k!} \int_{\mathbf{R}} \sum_{n=0}^k \binom{k}{n} x^n 0^{k-n} e^{-x} (\ln|x| + i\pi) \varphi(x)dx = \frac{1}{k!} \int_{\mathbf{R}} x^k e^{-x} (\ln|x| + i\pi) \varphi(x)dx \end{aligned}$$

Let us now consider the second integral

$$\begin{aligned}
\int_{\mathbf{R}} T(x-iy)\varphi(x)dx &= \int_{\mathbf{R}} \frac{(x-iy)^k}{k!} e^{-(x-iy)} \log(x-iy)\varphi(x) = \\
&= \int_{\mathbf{R}} \frac{1}{k!} \sum_{n=0}^k \binom{k}{n} x^n (-iy)^{k-n} e^{-(x-iy)} \left(\ln \sqrt{x^2+y^2} + i \arg(x-iy) \right) \varphi(x) dx = \\
&= \frac{1}{k!} \sum_{n=0}^k \binom{k}{n} \int_{\mathbf{R}} x^n (-iy)^{k-n} e^{-(x-iy)} \left(\ln \sqrt{x^2+y^2} + i \arg(x-iy) \right) \varphi(x) dx = \\
&= \frac{1}{k!} \sum_{n=0}^k \binom{k}{n} \int_{\mathbf{R}} x^n (-iy)^{k-n} e^{-x} e^{iy} \left(\ln \sqrt{x^2+y^2} + i \arg(x-iy) \right) \varphi(x) dx.
\end{aligned}$$

For $x < 0$, if $y \rightarrow 0^+$ then $\arg(x-iy) \rightarrow -\pi$, so the integrand tends to $x^n e^{-x} (\ln|x| - i\pi)$. For

$x > 0$, if $y \rightarrow 0^+$ then $\arg(x-iy) \rightarrow 0$. Therefore, we get that

$$\begin{aligned}
\lim_{y \rightarrow 0^+} \frac{1}{k!} \sum_{n=0}^k \binom{k}{n} \int_{\mathbf{R}} x^n (iy)^{k-n} e^{-x} e^{iy} \left(\ln \sqrt{x^2+y^2} + i \arg(x-iy) \right) \varphi(x) dx = \\
= \frac{1}{k!} \int_{\mathbf{R}} x^k e^{-x} (\ln|x| - i\pi) \varphi(x) dx.
\end{aligned}$$

So, we finally have that

$$\begin{aligned}
\lim_{y \rightarrow 0^+} \int_{\mathbf{R}} [T(x+iy) - T(x-iy)] \varphi(x) dx &= \\
= \lim_{y \rightarrow 0^+} \int_{\mathbf{R}} \left[\frac{(x+iy)^k}{k!} e^{-(x+iy)} \log(x+iy) - \frac{(x-iy)^k}{k!} e^{-(x-iy)} \log(x-iy) \right] \varphi(x) dx = \\
= \int_{\mathbf{R}} \left[\frac{x^k}{k!} e^{-x} (\ln x + i\pi) - \frac{x^k}{k!} e^{-x} (\ln x - i\pi) \right] \varphi(x) dx = \\
= \int_{\mathbf{R}} \frac{x^k}{k!} e^{-x} [\ln x + i\pi - \ln x + i\pi] \varphi(x) dx = \\
= \int_{\mathbf{R}} \frac{x^k}{k!} e^{-x} 2i\pi \varphi(x) dx = \\
= 2\pi i \int_{\mathbf{R}} \frac{x^k}{k!} e^{-x} \varphi(x) dx.
\end{aligned}$$

References

- [1]. M. Aslam Chaudhry: *Cauchy representation of distributions and applications to probability*, Bulletin of the Australian Mathematical Society, Volume 42, Issue 1, August 1990, pp. 125 – 131.
<https://doi.org/10.1017/S0004972700028227>
- [2]. H. Bremermann: *Distributions, Complex Variable and Fourier Transforms*, Addison Wesley, 1965
- [3]. R. Carmichael, D. Mitrovic: *Distributions and analytic functions*, New York, 1989.
- [4]. L. Jantcher: *Distributionen*, Walter de Gruyter, Berlin, New York, 1971.
- [5]. N. Reckoski: *One proof for the analytic representation of distributions*, Matematicki Bilten 28 (LIV), Skopje, 2004 (19-30).
- [6]. W. Rudin: *Functional Analysis*, North-Holland publishing company, Amsterdam, 1974.
- [7]. A.H. Zemanian: *Distribution Theory and Transform Analysis*, McGraw-Hill Company, New York, 1965.