Method Article

The Modified Variational Iteration Method to Solve Fractional Differential Equations

Abstract:

In this work we conceive a method of how the Lagrange multiplier of modified Variational Iteration Method can be defined from Laplace transform, And we use this technique to solve both differential equations and FDEs with initial value conditions, With Illustrative examples by applying the modified VIM to both Ordinary differential equations and fractional Differential Equations.

Key words: Variational Iteration Method, Lagrange Multiplier, Laplace Transform, Fractional Differential Equations, Caputo derivative

1 Introduction:

Many of Method have been used to solve a number of nonlinear problems which arise in mathematical physics and another related areas like [4], [15],[16],[17] The technique of Lagrange multipliers [11-16] was widely used to solve a number of nonlinear problems and another related areas, and it was developed into a powerful analytical method, Like the varianional iteration method [1, 2], [3],[5-10], for solving differential equations. The method has been applied to initial boundary problems [12-14], fractal initial value problems [18], etc. Generally, in applications of variational iteration method to initial value problems of differential equations, one usually follows the fallowing three steps:

(a) Establishing the correction functional

(b) Identifying the Lagrange multipliers

(c) Determining the initial iteration

The step (b) is very decisive. Applications of the method to fractional differential equations (FDEs) mainly and directly used the Lagrange multipliers in ordinary differential equations (ODEs).the present article conceives a method how the Lagrange multiplier has to be defined from Laplace transform. This technique can be easily extended to solve both differential equations and FDEs with initial value conditions.

2. Basics of the variation iteration method:

To illustrate the basic idea of the technique, consider the following general nonlinear system:

$$\frac{d^m u(t)}{dt^m} + R[u(t)] + N[u(t)] = g(t) \tag{1}$$

Where *R* is a linear operator and *N* is a nonlinear operator and *g(t)* is a given continuous function and $\frac{d^m u}{dt^m}$ is the term of the highest-order derivative.

The basic concept of the method is to construct a correction functional for the system (1), which reads

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^{t} \lambda(t,\tau)\{Ru_n(\tau) + Nu_n(\tau) - g(\tau)\}d\tau, \qquad (2)$$

Where $\lambda(t,\tau)$ is a general Lagrange multipliers [7,8,11] that can be identified optimally via variational theory, u_n is the n^{th} approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e. $\delta u_n = 0$, where δ is the variational derivative.

To illustrate how restricted variation works in the variational iteration method.

3. New identification of the Lagrange multipliers:

Now we revisit the original idea of the Lagrange multipliers in the case of an algebraic equation. Firstly, an iteration formula for finding the solution of the algebraic equation $f(x) = 0$ can be constructed as:

$$x_{n+1} = x_n + \lambda f(x_n) \qquad (3)$$

The optimality condition for the extreme $\frac{\delta x_{n+1}}{\delta x_n} = 0$ leads to

$$\lambda = -\frac{1}{f'(x_n)} \qquad (4)$$

Where δ is the classical variational operator. From (3) and (4), for a given initial value x_0 we can find the approximate solution x_{n+1} by the iterative scheme for (4)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ f'(x_n) \neq 0, n = 0,1,2,\ldots \qquad (5)$$

The algorithm is well known as the Newton-Raphson method and has quadratic convergence.

Now, we extend this idea to finding the unknown Lagrange multiplier. The main step is to first take the Laplace transform to Equation (1), then the linear part is transformed into an algebraic equation as follows:

$$s^m u(s) - u^{(m-1)}(0) - \cdots - s^{(m-1)}u(0) + L[R[u]] + L[N[u]] - L[g(t)] = 0 \qquad (6)$$

where

$$u(s) = L[u(t)] = \int_0^\infty e^{-st}u(t)dt.$$

The iteration formula of (6) can be used to suggest the main iterative scheme involving the Lagrange multiplier as:

$$u_{n+1}(s) = u_n(s) + \lambda(s)\left[s^m u_n(s) - u^{(m-1)}(0) - \cdots - s^{(m-1)}u(0) + L\left(R[u_n] + N[u_n] - g(t)\right)\right] \qquad (7)$$

Considering $L(R[u_n] + N[u_n])$ as restricted terms, one can derive a Lagrange multiplier as:

$$\delta u_{n+1}(s) = \delta u_n(s) + s^m \delta\lambda(s) u_n(s)$$
$$0 = 1 + s^m \lambda(s)$$
$$\lambda = -\frac{1}{s^m} \tag{8}$$

With Equation (8) and the inverse-Laplace transform L^{-1},
The iteration formula (7) can be explicitly given as:

$$u_{n+1}(t) = u_n(t) - L^{-1}\left[\frac{1}{s^m}\left[s^m u_n(s) - u^{(m-1)}(0) - \cdots - s^{(m-1)}u(0) + L\big(R[u_n] + N[u_n] - g(t)\big)\right]\right]$$

$$= L^{-1}\left(\frac{1}{s^m}u^{(m-1)}(0) + \cdots + \frac{u(0)}{s} - \frac{1}{s^m}L\big(R[u_n] + N[u_n] - g(t)\big)\right) \tag{9}$$

Where the initial iteration $u_0(t)$ can be determined by

$$u_0(t) = L^{-1}\left(\frac{1}{s^m}u^{(m-1)}(0) + \cdots + \frac{u(0)}{s}\right)$$
$$= u(0) + u'(0)t + \cdots + \frac{u^{(m-1)}t^{m-1}}{(m-1)!} \tag{10}$$

Equation (10) also explained why the initial iteration in the classical VIM is determined by the Taylor series.
So Consequently, the solution

$$u(t) = \lim_{n\to\infty} u_n(t).$$

4. Illustrative examples:

We now consider the applications of the modified VIM to both ODEs and FDEs.

4.1 Ordinary Differential Equation:

Example 1:

Consider the following differential equation:

$$\frac{du}{dt} + u = 0, \qquad u(0) = u_0,$$

This has the exact solution:

$$u(t) = u_0 e^{-t}$$

We can obtain the successive approximate solutions as:

$$u_{n+1}(t) = L^{-1}\left(\frac{u^{(m-1)}(0)}{s^m} + \cdots + \frac{u(0)}{s} - \frac{1}{s^m}L[u_n(t)]\right)$$

$$= L^{-1}\left(\frac{u_0}{s} - \frac{1}{s}L[u_n]\right)$$

$$u_0(t) = u(0) = u_0$$

$$u_1(t) = L^{-1}\left(\frac{u_0}{s} - \frac{1}{s}L[u_0]\right)$$

$$= L^{-1}\left(\frac{u_0}{s} - \frac{u_0}{s}L[1]\right) = L^{-1}\left(\frac{u_0}{s} - \frac{u_0}{s^2}\right)$$

$$= u_0L^{-1}\left(\frac{1}{s} - \frac{1}{s^2}\right) = u_0(1-t)$$

$$u_2(t) = L^{-1}\left(\frac{u_0}{s} - \frac{1}{s}L[u_1]\right)$$

$$= L^{-1}\left(\frac{u_0}{s} - \frac{1}{s}L\big(u_0(1-t)\big)\right) = u_0L^{-1}\left(\frac{1}{s} - \frac{1}{s}L[1-t]\right)$$

$$= u_0L^{-1}\left(\frac{1}{s} - \frac{1}{s}\left(\frac{1}{s} - \frac{1}{s^2}\right)\right) = u_0L^{-1}\left(\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3}\right)$$

$$= u_0\left(1 - t + \frac{t^2}{2}\right)$$

.
.
.

$$= u_0L^{-1}\left(\frac{1}{s} - \frac{1}{s^2} + \cdots + (-1)^{n+1}\frac{1}{s^{n+1}}\right)$$

$$u_n(t) = u_0\left[\sum_{k=1}^{n}\frac{(-t)^n}{n!}\right]$$

For $n \to \infty$, $u_n(t)$ tends to the exact solution u_0e^{-t}.

We notice that the integration by parts is not used and the calculation of the Lagrange multiplier here is much simpler. Furthermore, the VIM can be easily extended to FDEs and this is the main purpose of this paper

4.2 Fractional Differential Equations:

Let us consider the FDE

$$\frac{du}{dt} + {}_0^cD_t^\alpha u = g(t,u), \qquad 0 < t, \qquad 0 < \alpha < 1,$$

And the variational iteration formula is given as

$$u_{n+1}(t) = u_n + \int_0^t \lambda(t,\tau)\left(\frac{du_n}{dt} + {}_0^cD_t^\alpha u_n - g(\tau,u_n)\right)d\tau,$$

Where ${}_0^cD_t^\alpha u$ is the Caputo derivative and $g(\tau,u_n)$ is a nonlinear term:

$${}_0^cD_t^\alpha u + R[u] + N[u] = g(t)$$

$$u^{(k)}(0) = a_k, \qquad 0 < t, 0 < \alpha, m = [\alpha] + 1, k = 0, \dots, m-1. \tag{11}$$

then, we consider the application of the modified VIM.
The following Laplace transform of the term ${}_0^cD_t^\alpha u$ holds

$$L[{}_0^cD_t^\alpha u] = s^\alpha u(s) - \sum_{k=0}^{m-1} u^{(k)}(0)s^{\alpha-1-k}, m-1 < \alpha \le m. \tag{12}$$

Taking the above Laplace transform to both sides of (11), the iteration formula of equation (11) can be constructed as:

$$u_{n+1}(s) = u_n(s) + \lambda(s) \times$$

$$\left[s^\alpha u_n(s) - \sum_{k=0}^{m-1} u^{(k)}(0)s^{\alpha-k-1} + L\big(R[u_n] + N[u_n] - g(t)\big)\right]$$

Then consequently, after the identification of a Lagrange multiplier

$\lambda(s) = -\frac{1}{s^\alpha}$,

then**:**

$$u_{n+1}(t) = u_n(t) - L^{-1}\left[\frac{1}{s^\alpha}\left[s^\alpha u_n(s) - \sum_{k=0}^{m-1} u^{(k)}(0)s^{\alpha-k-1} + L\big(R[u_n] + N[u_n] - g(t)\big)\right]\right]$$

$$= L^{-1}\left(\sum_{k=0}^{m-1} u^{(k)}(0)s^{\alpha-k-1} - \frac{1}{s^\alpha}L\big(R[u_n] + N[u_n] - g(t)\big)\right),$$

$$m-1 < \alpha \le m \tag{13}$$

and

$$u_0(t) = L^{-1}\left(\sum_{k=0}^{m-1} u^{(k)}(0)s^{\alpha-k-1}\right)$$

$$= u(0) + u'(0) + \cdots + \frac{u^{(m-1)}(0)t^{m-1}}{(m-1)!} \tag{14}$$

Let us apply the above VIM to solve FDEs of Caputo type.

Example 2:

Consider the linear initial value problem

$$D^{\alpha}u + u = 0,\ u(0) = 1, u'(0) = 0, 0 < \alpha < 2. \quad (15)$$

After taking the Laplace transform to both sides of Equation (15) we get the following iteration formula:

$$u_{n+1}(s) = u_n(s) + \lambda(s^{\alpha}u_n(s) - u(0)s^{\alpha-1} - u'(0)s^{\alpha-2} + L[u_n(t)]) \quad (16)$$

Setting $L[u_n(t)]$ as a restricted variation, $\lambda(s)$ can be identified as

$$\lambda(s) = -\frac{1}{s^{\alpha}} \quad (17)$$

The approximate solution of Equation (4.15) can be given as:

$$u_{n+1}(t) = u_n(t) - L^{-1}\left[\frac{1}{s^{\alpha}}(s^{\alpha}u_n(s) - u(0)s^{\alpha-1} - u'(0)s^{\alpha-2} + L[u_n(t)])\right]$$

$$= L^{-1}\left(\frac{1}{s^{\alpha}}(u(0)s^{\alpha-1} - L[u_n(t)])\right)$$

$$u_{n+1}(t) = L^{-1}\left(\frac{1}{s} - \frac{1}{s^{\alpha}}L[u_n(t)]\right)$$

$$u_0(t) = 1$$

$$u_1(t) = L^{-1}\left(\frac{1}{s} - \frac{1}{s^{\alpha}}L[u_0(t)]\right) = L^{-1}\left(\frac{1}{s} - \frac{1}{s^{\alpha}}L[1]\right)$$

$$= L^{-1}\left(\frac{1}{s} - \frac{1}{s^{\alpha+1}}\right) = 1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$u_2(t) = L^{-1}\left(\frac{1}{s} - \frac{1}{s^{\alpha}}L[u_1(t)]\right)$$

$$= L^{-1}\left(\frac{1}{s} - \frac{1}{s^{\alpha}}L\left[1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right]\right)$$

$$= L^{-1}\left(\frac{1}{s} - \frac{1}{s^{\alpha+1}} + \frac{1}{s^{2\alpha+1}}\right)$$

$$u_2(t) = 1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$u_3(t) = L^{-1}\left(\frac{1}{s} - \frac{1}{s^{\alpha}}L[u_2(t)]\right)$$

$$= L^{-1}\left(\frac{1}{s} - \frac{1}{s^{\alpha}} L\left[1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\right)$$
$$= L^{-1}\left(\frac{1}{s} - \frac{1}{s^{\alpha}}\left(\frac{1}{s} - \frac{1}{s^{\alpha+1}} + \frac{1}{s^{2\alpha+1}}\right)\right)$$
$$= L^{-1}\left(\frac{1}{s} - \frac{1}{s^{\alpha+1}} + \frac{1}{s^{2\alpha+1}} - \frac{1}{s^{3\alpha+1}}\right)$$

$$u_3(t) = 1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

Then $u_n(t)$ rapidly tends to the exact solution for $n \to \infty$:

$$u(t) = \mathrm{E}_{\alpha}(-t)^{\alpha}$$

Since E_{α} is the Mittag-Leffler function defind as :

$$\mathrm{E}_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}$$

4.3 Solution of Equation (15) by Using Laplace Transform:

$$L[D^{\alpha} f(t)] = \frac{s^m F(s) - s^{m-1}F(0) - \cdots - F^{(m-1)}(0)}{s^{m-\alpha}}$$

$$m - 1 < \alpha \le m.$$

Then equation (15) will be:

$$\frac{s^2 u(s) - su(0)}{s^{2-\alpha}} + u(s) = 0$$

$$\frac{su(s) - 1}{s^{1-\alpha}} + u(s) = 0$$
$$s^{\alpha} u(s) - s^{\alpha-1} + u(s) = 0$$

$$u(s)[s^{\alpha} + 1] = s^{\alpha-1}$$
$$u(s) = \frac{s^{\alpha-1}}{s^{\alpha}+1} = \frac{1}{s}\left(\frac{1}{1 + \frac{1}{s^{\alpha}}}\right)$$

$$= \frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{-1}{s^{\alpha}}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{s^{n\alpha+1}}$$

By taking the inverse Laplace transform for both two sides give:

$$u(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{n\alpha}}{\Gamma(n\alpha + 1)} = \sum_{n=0}^{\infty} \frac{(-t)^{n\alpha}}{\Gamma(n\alpha + 1)} = \mathrm{E}_\alpha(-t)^\alpha$$

Conclusion:

Variational iteration method has been known as a powerful tool for solving many functional equations such as ordinary, partial differential equations, integral equations and so many other equations. In this paper, we have presented the modified variation iteration method which included Lagrange multiplier that easily identify by Laplace transform to give an analytical solutions of fractional differential equations, All examples showed that the results of the modified variational iteration method are in excellent agreement with those obtained by the Laplace transform method, but the results showed that the modified variational iteration method is more effective than the results of Laplace transform method because the inverse of Laplace transform some-times may be difficult to compute.

References:

[1] Abbasbandy, S: A new application of He’ s variational iteration method for quadratic Riccati differential equation by using Adomian’s polynomials .J. Comput. Appl. Math. 207(1), 59-63 (2007)

[2] L. Debnath, D. Bhatta, Solutions to few linear fractional inhomogeneous partial differential equations in fluid mechanics, Frac. Calc. Appl. Anal. 7 (2004) 21-36.

[3-] A. Hanyga, Multidimensional solutions of time-fractional diffusion-wave equations, Proc. R. Soc. London. A 458 (2002) 933-957.

[4]Hasan, Faeza L., and Mohamed A. Abdoon. "The generalized (2+ 1) and (3+ 1)-dimensional with advanced analytical wave solutions via computational applications." *International Journal of Nonlinear Analysis and Applications* 12.2 (2021): 1213-1241.

[5] J.H. He, Nonlinear oscillation with fractional derivative and its applications, in: International Conference on Vibrating Engineering'98, Dalian, China,1998, pp. 288-291.

[6] J.H. He, Some applications of nonlinear fractional differential equations and their approximations, Bull. Sci. Technol. 15 (2) (1999) 86-90.

[7] J.H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods Appl. Mech. Engrg. 167 (1998) 57-6

[8] J.H. He, Variational iteration method- a kind of non-linear analytical technique: Some examples, Int. J. Nonlinear Mech. 34 (1999) 699-708.

[9] J.H. He, Variational iteration method-Some recent results and new interpretations, J. Comput. Appl. Math. 207 (1) (2007) 3-17.

[10] J.H. He, X.H. Wu, Variational iteration method: New development and applications, Comput. Math. Appl. 54 (7-8) (2007) 881-894.

[11] Inokuti, M, Sekine, H, Hura, T:General use of the lagrange multiplier ion non linear Mathematical physics.In:Nemat-Nasser,S (ed) Variational Methods in the Mechanics of Solids, pp. 156-162.Pregman Press, New York(1978)

[12] S. Momani, Analytic and approximate solutions of the space- and time-fractional telegraph equations, Appl. Math. Comput. 170 (2) (2005) 1126-1134
[13] Noor, MA,Mohyud-Din, ST:Variational iteration technique for solving higher order boundary value problems. App.Math.Comput. 189 (2), 1929-1942 (2007)
[14] Noor, MA, Mohyud-Din, ST: Variational iteration method for solving higher-order nonlinear boundary value problems using H’s polynomials. Int . J. Nonlinear Sci. Mumer. Simul. 9(2), 141-156 (2008)
[15] Elzaki, Tarig M., and Hwajoon Kim. "The solution of radial diffusivity and shock wave equations by Elzaki variational iteration method." *Int. J. of Math. Anal* 9 (2015): 1065-1071..
[16] Mohamed, Mohamed Z., and Tarig M. Elzaki. "Comparison between the Laplace decomposition method and Adomian decomposition in time-space fractional nonlinear fractional differential equations." *Applied Mathematics* 9.04 (2018): 448.
[17]Mohamed, M. Z., Elzaki, T. M., Algolam, M. S., Abd Elmohmoud, E. M., & Hamza, A. E. (2021). New modified variational iteration Laplace transform method compares Laplace adomian decomposition method for solution time-partial fractional differential equations. *Journal of Applied Mathematics*, *2021*.
[18] Wu, GC: New trends in variational iteration method .Commun. Fract. Calc. 2(2), 59-75 (2011)