Original Research Article

Conservation Laws and Travelling Wave Solutions for System of Ion Sound and Langmuir Waves

Abstract

In this paper, system of equations for ion sound and Langmuir waves (ISLWs) is studied to construct novel exact solutions of the coupled nonlinear system. The extended F-expansion method is applied to get exact solutions of ISLWs model. These solutions include many different wave natures expressible in hyperbolic, trigonometric, rational and Jacobi elliptic function solutions, dark and bright solitary wave solutions. Geometrical shape for some of the obtained results are plotted. Furthermore, conservation laws for the system are constructed.

Keywords: Ion sound and Langmuir waves; Jacobi elliptic function; Exact solutions.

1 Introduction

Many physical phenomena that arising in various fields of science can be described by nonlinear evolution equations (NLEEs) for instance optics, plasma wave, solid state physics, chemical physics, and mathematical physics. To understand these nonlinear phenomena, many physicist and mathematicians have made efforts to get various exact solutions of them. The investigation of the exact solutions of NLEEs are important to provide better information, know the mechanism and their applications. With the aid of symbolic computation software such as Mathematica or Maple abundant methods are extensively studied to obtain exact solutions, for example the Bäcklund transform [1, 2], the extended tanh-function method [3], the F-expansion method [4], sine-cosine method [5], the extended F-expansion method [6], Jacobi elliptic function method [7], the Kudryashov method [8], the extended Kudryashov method [9, 10], the lie point symmetry method [11]- [13] and other methods [14]-[19].

In this study, we construct several kinds of exact solutions of the ISLWs model by applying the extended F-expansion method. The ISLWs model [20]-[26] has the form as:

$$\begin{aligned} &i\,E_t + \tfrac{1}{2}\,E_{xx} - n\,E = 0,\\ &n_{tt} - n_{xx} - 2\,(|E|^2)_{xx} = 0, \end{aligned} \tag{1.1}$$

where the electric field of the Langmuir oscillation is $E\,e^{-iw_p t}$ and the density perturbation is n. Recently, many researchers used different techniques to find exact solutions of ISLWs model. Demiray and Bulut [21] applied generalized Kudryashov method to get travelling wave solutions of the ISLWs model. Also, soliton solutions of this system was considered by Seadawy et. al.[22], Alam and Osman [23] and Mohammed et. al. [24]. The physical natures of ISLWs model is useful to seek plasma physics and the effect on incoherent structures. The graphical of some specific solutions are useful to understand the physical phenomena of Eq. (1.1). Moreover, conservation laws are great important in physics and mathematics. Mathematical expressions of physical laws are the coservation laws, such as coservation of mass, energy and momentum. The coservation laws can be used to study the properties of the existance, uniqueness and stability of solutions.

The outline of this paper is as the following : Firstly, we summarized the analytical method that we will use to construct novel exact solutions of the ISLWs model in section 2. In section 3, we get the solutions of the studied model with Jacobi elliptic functions (JEFs) via the extended F-expansion method. The geometrical shape of some solutions in the form of two-dimentional and three-dimentional have been plotted. Furthermore, the Lie point symmetry and the conservation laws for (1.1) are obtained in section 4. Finally, conclusions of the paper are presented in the latest section.

2 Summary of the extended F-expansion method

In this section, the extended F-expansion method [6], will summarize as follows:
Consider NLEEs

$$P(u, u_t, u_x, u_{tt}, u_{xx}, ...) = 0. \tag{2.1}$$

1- Suppose that $u(x,t) = u(\xi)$ and $\xi = c\,t + k\,x + \xi_0$, such that c and k are constants to be evaluated later and ξ_0 is an arbitrary constant. Then (2.1) is transformed to the following equation:

$$G(u,\, u',\, u'', ...) = 0, \tag{2.2}$$

2- Suppose that the solutions of (2.2) in the form

$$u(x,t) = u(\xi) = A_0 + \sum_{i=1}^{N} [A_i F^i(\xi) + B_i\, F^{-i}(\xi)], \tag{2.3}$$

where N is a positive integer and $A_0, A_i,\ B_i\,(i = 1,\ 2, ...,\ N)$ are constants to be determined. The function $F(\xi)$ satisfies the following ordinary differential equation (ODE):

$$(F'(\xi))^2 = q_0 + q_2 F^2(\xi) + q_4 F^4(\xi), \tag{2.4}$$

and the values q_0, q_2 and q_4 are constants.

3- Putting (2.3) with (2.4) in (2.2), we obtain a polynomial in $F(\xi)$. Setting all coefficients of it to zero, we get an algebraic equations for $A_0,\ k,\ A_i,\ B_i(i = 1,\ 2, ...,\ N)$ and c.

4- Setting the values of q_0, q_2, q_4 and the corresponding JEFs $F(\xi)$, we get many exact JEF solutions of Eq. (2.1).

The JEFs can be written as $\mathrm{sn}\xi = \mathrm{sn}(\xi, m)$, $\mathrm{cn}\xi = \mathrm{cn}(\xi, m)$ and $\mathrm{dn}\xi = \mathrm{dn}(\xi, m)$, where $m\,(0 < m < 1)$ is the modulus of the elliptic function. The functions $\mathrm{sn}\xi$, $\mathrm{cn}\xi$ and $\mathrm{dn}\xi$ become $\tanh\xi$, $\mathrm{sech}\xi$ and $\mathrm{sech}\xi$, respectively when $m \longrightarrow 1$. Also, $\mathrm{sn}\xi$, $\mathrm{cn}\xi$ and $\mathrm{dn}\xi$ become $\sin\xi$, $\cos\xi$, 1, respectively when $m \longrightarrow 0$.

3 The exact solution of the ISLWs model by the extended F-expansion method.

Suppose that the solution of (1.1) as the following:

$$E(x,t) = U(\xi)\, e^{i\,\theta}, \quad n(x,t) = V(\xi), \quad \xi = c\,t + k\,x + \xi_0, \quad \theta = \Omega\,x + \mu\,t, \tag{3.1}$$

where c, Ω, k and μ are constants. Putting (3.1) in (1.1) and splitting the imaginary and the real parts, we get

$$(c + \Omega\,k)\,U' = 0 \quad \Rightarrow c = -\Omega\,k, \tag{3.2}$$

$$k^2\,U'' - (2\,\mu + \Omega^2)\,U - 2\,U\,V = 0, \tag{3.3}$$

$$(c^2 - k^2)\,V'' - 2\,k^2\,(U^2)'' = 0. \tag{3.4}$$

By integrating (3.4) twise and putting the integration constant to zero, we obtain

$$V(\xi) = \frac{2\,k^2}{c^2 - k^2}\,U^2(\xi). \tag{3.5}$$

Substituting (3.5) into (3.3), we obtain

$$k^2(\Omega^2 - 1)\,U'' - (\Omega^2 - 1)(\Omega^2 + 2\,\mu)\,U - 4\,U^3 = 0. \tag{3.6}$$

By using balancing procedure, we have $N = 1$. Then (3.6) has solution as

$$U(\xi) = A_0 + A_1\,F(\xi) + \frac{B_1}{F(\xi)}, \tag{3.7}$$

where A_0, A_1, B_1 are undetermined constants. Putting (3.7) into (3.6) and using (2.4), we get a polynomial in the function $F(\xi)$. Setting all coefficients of $F(\xi)$ to zero, we have cases as follows:

Case 1

$$k = \pm\sqrt{\frac{\Omega^2 + 2\,\mu}{q_2}},\ \Omega = \Omega, \mu = \mu, A_0 = 0, A_1 = \pm\sqrt{\frac{q_4\,(\Omega^2 + 2\,\mu)\,(\Omega^2 - 1)}{2\,q_2}},\ B_1 = 0. \tag{3.8}$$

Substituting (3.8) into (3.7), we have

$$U(\xi) = \pm\sqrt{\frac{q_4\,(\Omega^2 + 2\,\mu)\,(\Omega^2 - 1)}{2\,q_2}}\,F(\xi), \quad \xi = \pm\sqrt{\frac{\Omega^2 + 2\,\mu}{q_2}}\,x + c\,t + \xi_0. \tag{3.9}$$

Putting q_0, q_2, q_4 and the JEFs solution for $F(\xi)$ in (3.9) with (3.5) and (3.1). Therefore, we construct the exact solutions of (1.1) as follows:

Case 1.1: When $q_4 = m^2$, $q_2 = -1 - m^2$, $q_0 = 1$ and $F(\xi) = \mathrm{sn}\xi$. So, periodic wave solutions of (1.1) given as,

$$E(x,t) = \pm\, m\sqrt{\frac{(\Omega^2 + 2\,\mu)\,(1 - \Omega^2)}{2\,(1 + m^2)}}\,e^{i\Theta}\,\mathrm{sn}\xi,\ \ n(x,t) = \frac{-m^2\,(\Omega^2 + 2\,\mu)}{(1 + m^2)}\,\mathrm{sn}^2\xi, \tag{3.10}$$

Case 1.2: Setting $q_4 = -m^2$, $q_2 = 2m^2 - 1$, $q_0 = 1 - m^2$ and $F(\xi) = \mathrm{cn}\xi$, we have the JEFs solutions of (1.1)

$$E(x,t) = \pm\, m\sqrt{\frac{(\Omega^2 + 2\,\mu)\,(\Omega^2 - 1)}{2\,(1 - 2\,m^2)}}\,e^{i\Theta}\,\mathrm{cn}\xi, \quad n(x,t) = \frac{m^2\,(\Omega^2 + 2\,\mu)}{(1 - 2\,m^2)}\,\mathrm{cn}^2\xi, \tag{3.11}$$

Case 1.3: Putting $q_4 = -1$, $q_2 = 2 - m^2$, $q_0 = m^2 - 1$ and $F(\xi) = \mathrm{dn}\xi$. Thus, the periodic wave solutions of (1.1) obtain as follows:

$$E(x,t) = \pm\sqrt{\frac{(\Omega^2 + 2\,\mu)\,(\Omega^2 - 1)}{2\,(m^2 - 2)}}\,e^{i\Theta}\,\mathrm{dn}\xi, \quad n(x,t) = \frac{(\Omega^2 + 2\,\mu)}{(m^2 - 2)}\,\mathrm{dn}^2\xi, \tag{3.12}$$

Case 1.4: Putting $q_0 = \frac{1}{4}$, $q_2 = \frac{m^2-2}{2}$, $q_4 = \frac{m^4}{4}$, we have $F(\xi) = \frac{\mathrm{sn}\xi}{1 \pm \mathrm{dn}\xi}$. Thus, we construct solutions of (1.1) expressed by rational expressions of JEFs:

$$E(x,t) = \pm\,\frac{m^2}{2}\sqrt{\frac{(\Omega^2 + 2\,\mu)\,(\Omega^2 - 1)}{(m^2 - 2)}}\,\frac{\mathrm{sn}\xi}{1 \pm \mathrm{dn}\xi}\,e^{i\Theta}, \quad n(x,t) = \frac{m^4\,(\Omega^2 + 2\,\mu)}{2\,(m^2 - 2)}\left(\frac{\mathrm{sn}\xi}{1 \pm \mathrm{dn}\xi}\right)^2, \tag{3.13}$$

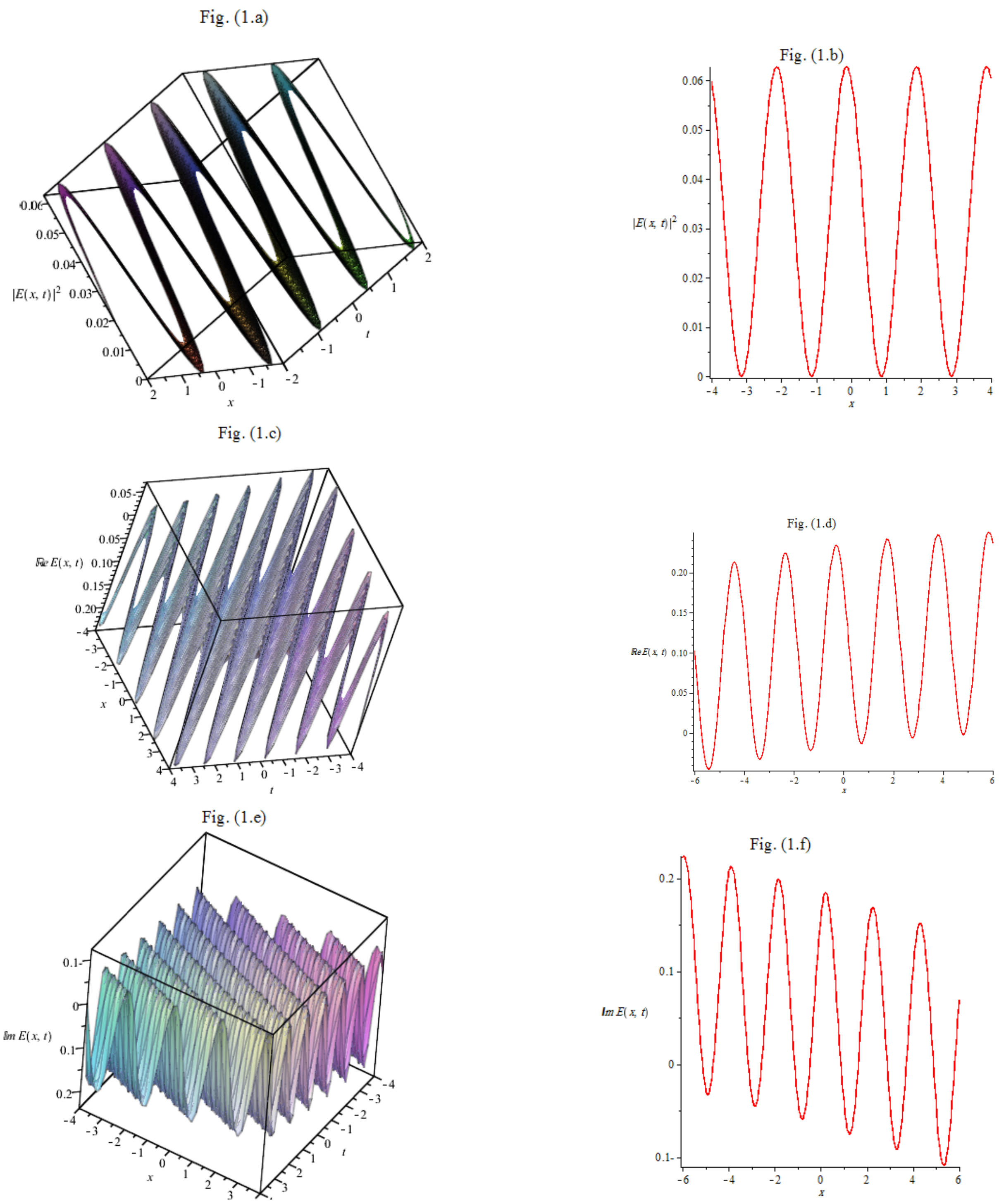

Figure 1: (a-f) 3D and 2D the periodic wave solution (3.10) are plotted with the parameters $\Omega = 1.5, m = 0.2, \mu = -2.43$ and $\xi_0 = 1$ for 3D figure and $t = 1$ for 2D figure.

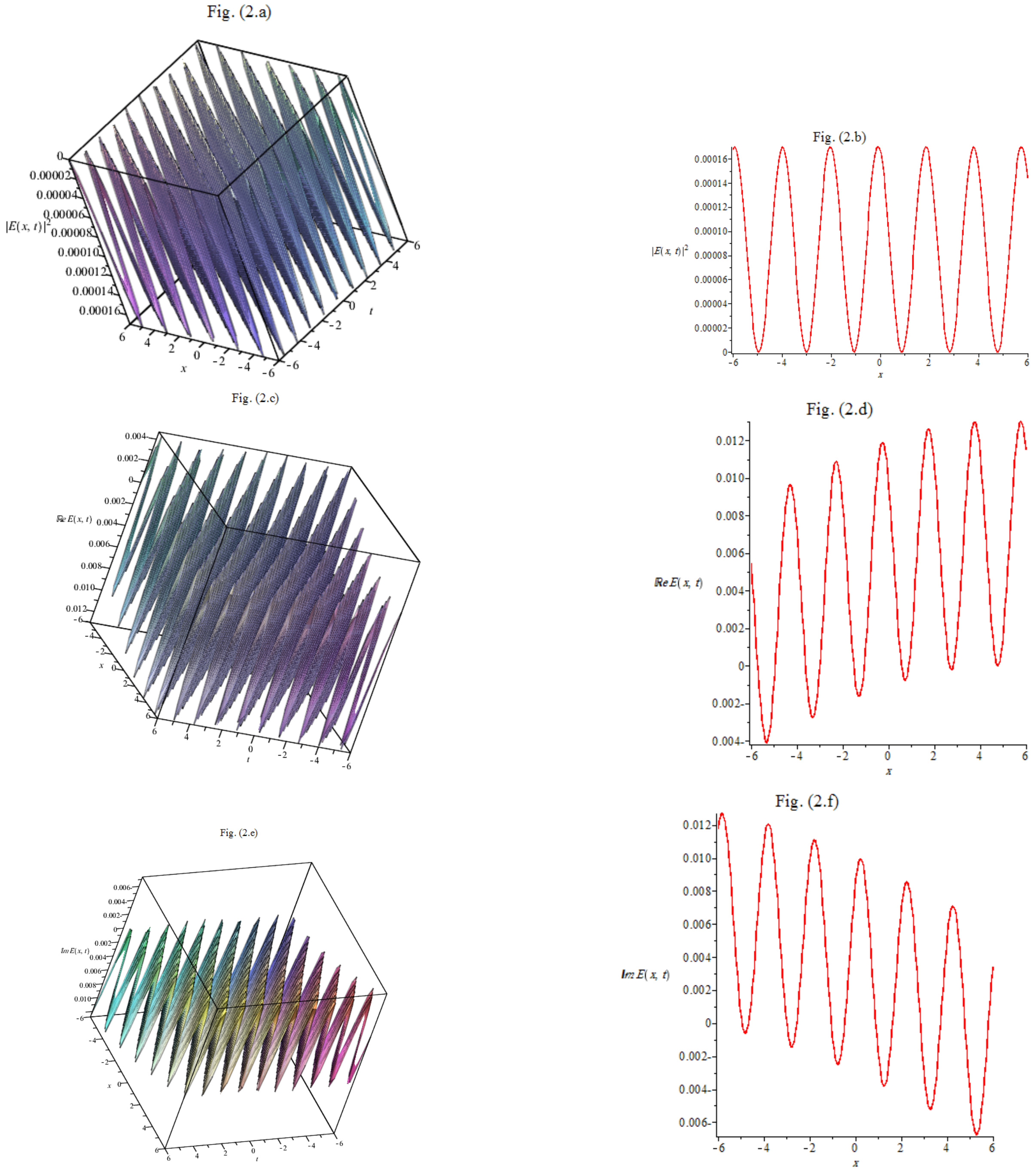
Fig. (2.a) Fig. (2.b) Fig. (2.c) Fig. (2.d) Fig. (2.e) Fig. (2.f)

Figure 2: The periodic wave solution (3.13) are plotted when (+) sign is taken for (a-f) 3D and 2D with the choice of parameters $\Omega = 1.5, m = 0.2, \mu = -2.43$ and $\xi_0 = 1$ for 3D figure and $t = 1$ for 2D figure.

Case 1.5: If $q_0 = \frac{m^2-1}{4}$, $q_2 = \frac{m^2+1}{2}$, $q_4 = \frac{m^2-1}{4}$ we get $F(\xi) = \frac{\mathrm{dn}\xi}{1 \pm m\,\mathrm{sn}\xi}$. Then, we have the solutions of (1.1) as

$$
\begin{aligned}
E(x,t) &= \pm\,\tfrac{1}{2}\sqrt{\tfrac{(m^2-1)(\Omega^2+2\,\mu)\,(\Omega^2-1)}{(m^2+1)}}\;\frac{\mathrm{dn}\xi}{1\pm m\,\mathrm{sn}\xi}\;e^{i\Theta},\\
n(x,t) &= \tfrac{(m^2-1)(\Omega^2+2\,\mu)}{2\,(m^2+1)}\left(\frac{\mathrm{dn}\xi}{1\pm m\,\mathrm{sn}\xi}\right)^2,
\end{aligned}
\tag{3.14}
$$

Case 1.6: When $q_0 = \frac{1-m^2}{4}$, $q_2 = \frac{m^2+1}{2}$, $q_4 = \frac{1-m^2}{4}$ and $F(\xi) = \frac{\mathrm{cn}\xi}{1\pm\mathrm{sn}\xi}$, we obtain

$$
\begin{aligned}
E(x,t) &= \pm\,\tfrac{1}{2}\sqrt{\tfrac{(1-m^2)(\Omega^2+2\,\mu)\,(\Omega^2-1)}{(m^2+1)}}\;\frac{\mathrm{cn}\xi}{1\pm\mathrm{sn}\xi}\;e^{i\Theta},\\
n(x,t) &= \tfrac{(1-m^2)(\Omega^2+2\,\mu)}{2\,(m^2+1)}\left(\frac{\mathrm{cn}\xi}{1\pm\mathrm{sn}\xi}\right)^2,
\end{aligned}
\tag{3.15}
$$

Case 1.7: When $q_4 = \frac{-1}{4}$, $q_2 = \frac{m^2+1}{2}$, $q_0 = \frac{-(1-m^2)^2}{4}$ and $F(\xi) = (m\,\mathrm{cn}\xi \pm \mathrm{dn}\xi)$, we have

$$
\begin{aligned}
E(x,t) &= \pm\,\tfrac{1}{2}\sqrt{\tfrac{(\Omega^2+2\,\mu)\,(1-\Omega^2)}{(m^2+1)}}\;(m\,\mathrm{cn}\xi \pm \mathrm{dn}\xi)\,e^{i\Theta},\\
n(x,t) &= \tfrac{-(\Omega^2+2\,\mu)}{2\,(m^2+1)}\left(m\,\mathrm{cn}\xi \pm \mathrm{dn}\xi\right)^2,
\end{aligned}
\tag{3.16}
$$

Case 1.8: When $q_0 = \frac{1}{4}$, $q_2 = \frac{m^2+1}{2}$, $q_4 = \frac{(1-m^2)^2}{4}$ and $F(\xi) = \frac{\mathrm{sn}\xi}{\mathrm{cn}\xi\pm\mathrm{dn}\xi}$, thus the solutions of (1.1) are

$$
\begin{aligned}
E(x,t) &= \pm\,\tfrac{1-m^2}{2}\sqrt{\tfrac{(\Omega^2+2\,\mu)\,(\Omega^2-1)}{(m^2+1)}}\;\frac{\mathrm{sn}\xi}{\mathrm{cn}\xi\pm\mathrm{dn}\xi}\;e^{i\Theta},\\
n(x,t) &= \tfrac{(1-m^2)^2\,(\Omega^2+2\,\mu)}{2\,(m^2+1)}\left(\frac{\mathrm{sn}\xi}{\mathrm{cn}\xi\pm\mathrm{dn}\xi}\right)^2,
\end{aligned}
\tag{3.17}
$$

Case 2

$$
k = \pm\sqrt{\frac{\Omega^2+2\,\mu}{q_2}},\ \Omega=\Omega, \mu=\mu, A_0=0, A_1=0,\ B_1=\sqrt{\frac{q_0\,(\Omega^2+2\,\mu)\,(\Omega^2-1)}{2\,q_2}}. \tag{3.18}
$$

Substituting (3.18) into (3.7), we obtain the general solutions as,

$$
U(\xi) = \pm\sqrt{\frac{q_0\,(\Omega^2+2\,\mu)\,(\Omega^2-1)}{2\,q_2}}\;\frac{1}{F(\xi)},\quad \xi = \pm\sqrt{\frac{\Omega^2+2\,\mu}{q_2}}\;x + c\,t + \xi_0. \tag{3.19}
$$

Putting q_0, q_2, q_4 and the JEFs solution for $F(\xi)$ in (3.19) with (3.5) and (3.1). Therefore, we have the exact solutions of (1.1) as

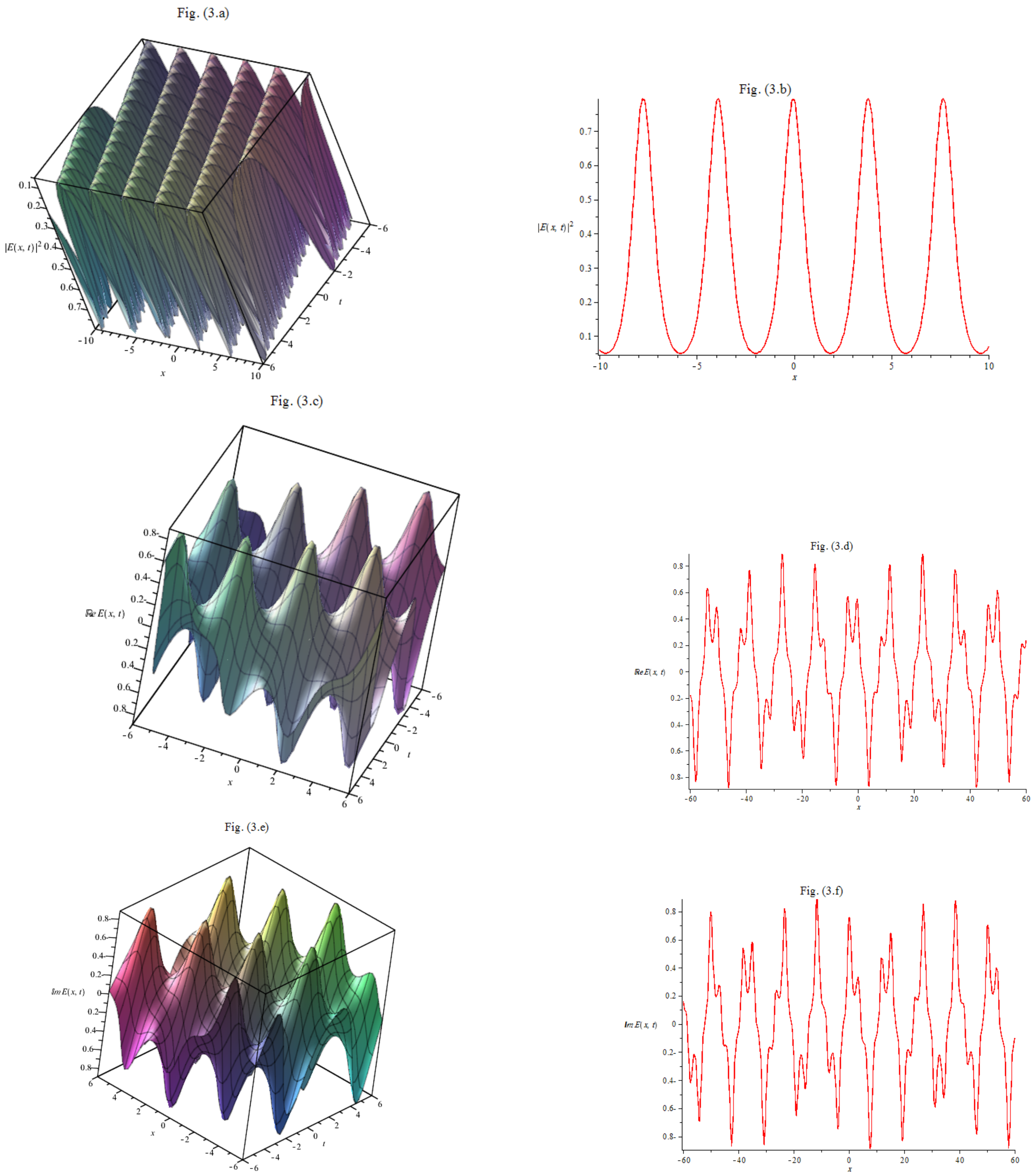

Figure 3: The periodic wave solution (3.16) are plotted when (+) sign is taken for (a-f) 3D and 2D with the choice of parameters $\Omega = 0.5, m = 0.6, \mu = 1$ and $\xi_0 = 1$ for 3D figure and $t = 1$ for 2D figure.

Case 2.1: When $q_0 = 1$, $q_2 = -1 - m^2$, $q_4 = m^2$, and $F(\xi) = \text{sn}\xi$. So, the periodic wave solutions of (1.1) given as

$$E(x,t) = \pm \frac{\sqrt{(\Omega^2 + 2\,\mu)\,(1-\Omega^2)}}{\sqrt{2\,(1+m^2)}}\, e^{i\Theta}\ \ \mathrm{ns}\xi,\ \ n(x,t) = \frac{-(\Omega^2 + 2\,\mu)}{(1+m^2)}\,\mathrm{ns}^2\xi, \tag{3.20}$$

Case 2.2: When $q_0 = 1 - m^2$, $q_2 = 2m^2 - 1$, $q_4 = -m^2$, $F(\xi) = \mathrm{cn}\xi$, then, the exact JEFs solutions of Eq. (1.1) are

$$E(x,t) = \pm \frac{\sqrt{(1-m^2)(\Omega^2 + 2\,\mu)\,(\Omega^2-1)}}{\sqrt{2\,(2\,m^2-1)}}\, e^{i\Theta}\,\mathrm{nc}\xi,\ n(x,t) = \frac{(1-m^2)(\Omega^2 + 2\,\mu)}{(2\,m^2-1)}\,\mathrm{nc}^2\xi, \tag{3.21}$$

Case 2.3: If $q_4 = -1$, $q_2 = 2 - m^2$, $q_0 = m^2 - 1$ and $F(\xi) = \mathrm{dn}\xi$, thus yeilds the periodic wave solutions of Eq. (1.1) as follows:

$$E(x,t) = \pm \frac{\sqrt{(m^2-1)(\Omega^2 + 2\,\mu)\,(\Omega^2-1)}}{\sqrt{2\,(2-m^2)}}\, e^{i\Theta}\ \mathrm{nd}\xi,\ n(x,t) = \frac{(m^2-1)(\Omega^2 + 2\,\mu)}{(2-m^2)}\ \mathrm{nd}^2\xi, \tag{3.22}$$

Case 2.4: Putting $q_0 = \frac{1}{4}$, $q_2 = \frac{m^2-2}{2}$, $q_4 = \frac{m^4}{4}$ and $F(\xi) = \frac{\mathrm{sn}\xi}{1\pm\mathrm{dn}\xi}$. We construct solutions of (1.1) expressed by rational expressions of JEFs

$$E(x,t) = \pm \sqrt{\frac{(\Omega^2 + 2\,\mu)\,(\Omega^2-1)}{m^2-2}}\,\frac{1\pm\mathrm{dn}\xi}{2\,\mathrm{sn}\xi}\, e^{i\Theta},\quad n(x,t) = \frac{(\Omega^2 + 2\,\mu)}{2\,(m^2-2)}\left(\frac{1\pm\mathrm{dn}\xi}{\mathrm{sn}\xi}\right)^2, \tag{3.23}$$

Case 2.5: If $q_0 = \frac{m^2-1}{4}$, $q_2 = \frac{m^2+1}{2}$, $q_4 = \frac{m^2-1}{4}$, then $F(\xi) = \frac{\mathrm{dn}\xi}{1\pm m\,\mathrm{sn}\xi}$. Thus, we have the solutions of (1.1) as

$$\begin{aligned} E(x,t) &= \pm\sqrt{\tfrac{(m^2-1)(\Omega^2+2\,\mu)\,(\Omega^2-1)}{m^2+1}}\ \tfrac{(1\pm m\,\mathrm{sn}\xi)}{2\,\mathrm{dn}\xi}\, e^{i\Theta}, \\ n(x,t) &= \tfrac{(m^2-1)(\Omega^2+2\,\mu)}{2\,(m^2+1)}\left(\tfrac{1\pm m\,\mathrm{sn}\xi}{\mathrm{dn}\xi}\right)^2, \end{aligned} \tag{3.24}$$

Case 2.6: When $q_0 = \frac{1-m^2}{4}$, $q_2 = \frac{m^2+1}{2}$, $q_4 = \frac{1-m^2}{4}$, $F(\xi) = \frac{\mathrm{cn}\xi}{1\pm\mathrm{sn}\xi}$, we obtain

$$\begin{aligned} E(x,t) &= \pm\sqrt{\tfrac{(1-m^2)(\Omega^2+2\,\mu)\,(\Omega^2-1)}{m^2+1}}\ \tfrac{(1\pm\mathrm{sn}\xi)}{2\,\mathrm{cn}\xi}\, e^{i\Theta}, \\ n(x,t) &= \tfrac{(1-m^2)(\Omega^2+2\,\mu)}{2\,(m^2+1)}\left(\tfrac{(1\pm\mathrm{sn}\xi)}{\mathrm{cn}\xi}\right)^2, \end{aligned} \tag{3.25}$$

Case 2.7: If $q_0 = \frac{-(1-m^2)^2}{4}$, $q_2 = \frac{m^2+1}{2}$, $q_4 = \frac{-1}{4}$, and $F(\xi) = (m\,\mathrm{cn}\xi \pm \mathrm{dn}\xi)$. Then, the exact solutions of Eq. (1.1) given as,

$$\begin{aligned} E(x,t) &= \pm\tfrac{(1-m^2)\sqrt{(\Omega^2+2\,\mu)\,(1-\Omega^2)}}{2\,\sqrt{(m^2+1)}\ (m\,\mathrm{cn}\xi\pm\mathrm{dn}\xi)}\, e^{i\Theta}, \\ n(x,t) &= \tfrac{-(1-m^2)^2\,(\Omega^2+2\,\mu)}{2\,(m^2+1)\ (m\,\mathrm{cn}\xi\pm\mathrm{dn}\xi)^2}, \end{aligned} \tag{3.26}$$

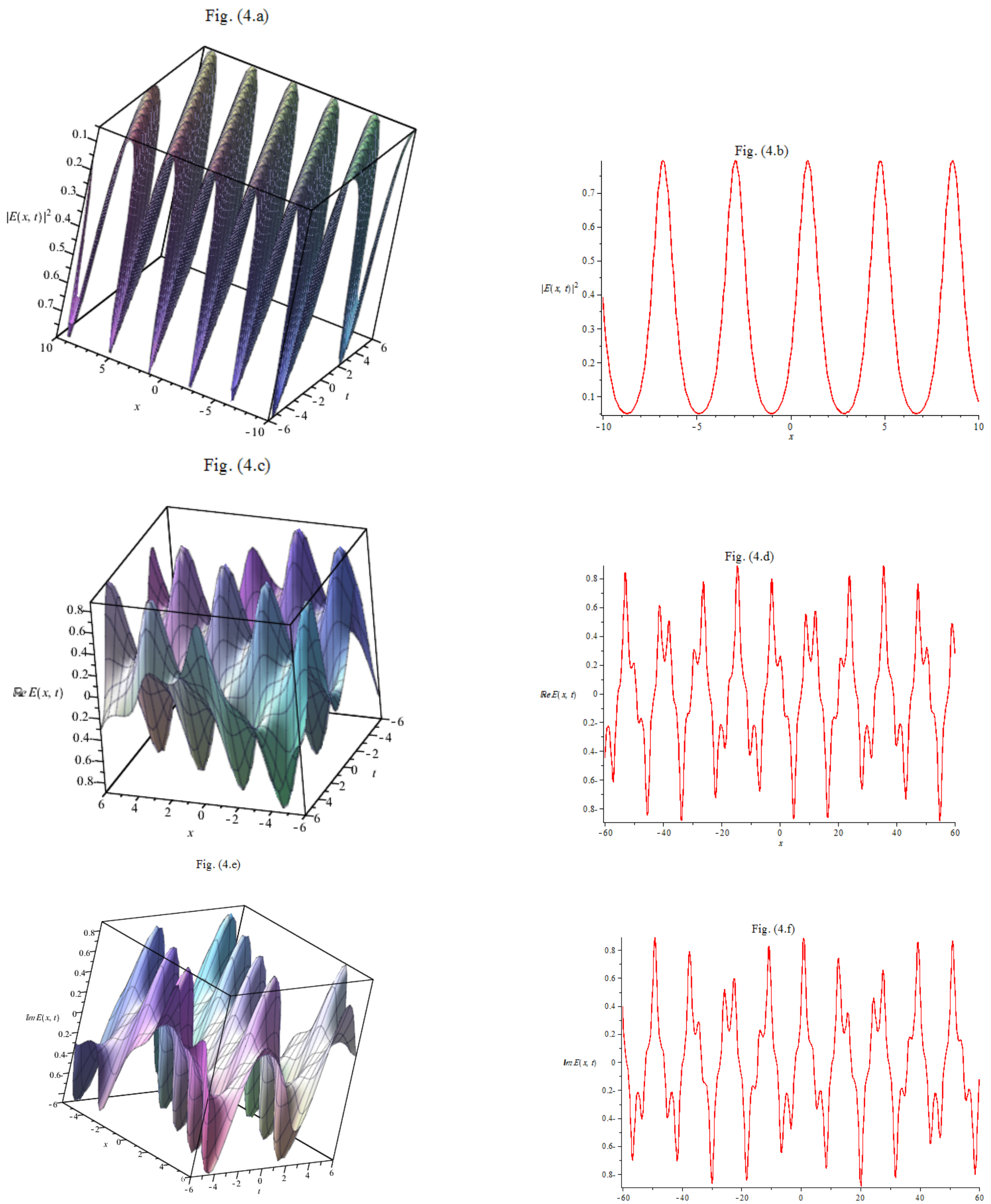

Figure 4: (a-f) 3D and 2D are plotted for the periodic wave solution (3.24) when (+) sign is taken with the choice of parameters $\Omega = 0.5$, $m = 0.6$, $\mu = 1$ and $\xi_0 = 1$ for 3D plots and $t = 1$ for 2D plots.

Case 3

$$k=\pm\sqrt{\frac{\Omega^2+2\,\mu}{q_2\pm 6\,\sqrt{q_0\,q_4}}},\ \Omega=\Omega,\ \mu=\mu,\ A_0=0,\ A_1=\pm\sqrt{\frac{q_4\,(\Omega^2+2\,\mu)\,(\Omega^2-1)}{2\,(q_2\pm 6\,\sqrt{q_0\,q_4})}}$$

,

$$B_1=\mp\sqrt{\frac{q_0\,(\Omega^2+2\,\mu)\,(\Omega^2-1)}{2\,(q_2\pm 6\,\sqrt{q_0\,q_4})}}. \tag{3.27}$$

Substituting (3.27) into (3.7), we obtain the general solutions in the form,

$$U(\xi)=\pm\sqrt{\frac{q_4\,(\Omega^2+2\,\mu)\,(\Omega^2-1)}{2\,(q_2\pm 6\,\sqrt{q_0\,q_4})}}\,F(\xi)\mp\sqrt{\frac{q_0\,(\Omega^2+2\,\mu)\,(\Omega^2-1)}{2\,(q_2\pm 6\,\sqrt{q_0\,q_4})}}\,\frac{1}{F(\xi)},$$

$$\xi=\pm\sqrt{\frac{\Omega^2+2\,\mu}{q_2\pm 6\,\sqrt{q_0\,q_4}}}\,x+c\,t+\xi_0. \tag{3.28}$$

We can find many JEF solutions of Eq. (3.28) by setting q_0, q_2, q_4 and the JEFs solution for $F(\xi)$. Hence, the solutions of Eq. (1.1) written as the following:

Case 3.1: When $q_4=m^2$, $q_2=-1-m^2$, $q_0=1$ and $F(\xi)=\mathrm{sn}\xi$. So, we obtain the following combined JEFs solutions of (1.1):

$$\begin{aligned} E(x,t)&=\sqrt{\tfrac{(\Omega^2+2\,\mu)\,(\Omega^2-1)}{2\,[-(1+m^2)\pm 6\,m]}}\left(\pm\,m\,\mathrm{sn}\xi\mp\mathrm{ns}\xi\right)e^{i\Theta},\\ n(x,t)&=\tfrac{(\Omega^2+2\,\mu)}{[-(1+m^2)\pm 6\,m]}\left(m\,\mathrm{sn}\xi-\mathrm{ns}\xi\right)^2, \end{aligned} \tag{3.29}$$

Case 3.2: Putting $q_4=-m^2$, $q_2=2m^2-1$, $q_0=1-m^2$ and $F(\xi)=\mathrm{cn}\xi$. Thus, the exact JEFs solutions of (1.1) are

$$\begin{aligned} E(x,t)&=\sqrt{\tfrac{(\Omega^2+2\,\mu)\,(1-\Omega^2)}{2\left(2\,m^2-1\pm 6\,m\,\sqrt{m^2-1}\right)}}\left(\pm\,m\,\mathrm{cn}\xi\mp\sqrt{m^2-1}\ \mathrm{nc}\xi\right)e^{i\Theta},\\ n(x,t)&=\tfrac{-(\Omega^2+2\,\mu)}{\left(2\,m^2-1\pm 6\,m\,\sqrt{m^2-1}\right)}\left(m\,\mathrm{cn}\xi-\sqrt{m^2-1}\ \mathrm{nc}\xi\right)^2, \end{aligned} \tag{3.30}$$

Case 3.3: If $q_4=-1$, $q_2=2-m^2$, $q_0=m^2-1$ and $F(\xi)=\mathrm{dn}\xi$, we obtain the combined solutions of (1.1) as

$$\begin{aligned} E(x,t)&=\sqrt{\tfrac{(\Omega^2+2\,\mu)\,(1-\Omega^2)}{2\left(2-m^2\pm 6\,\sqrt{1-m^2}\right)}}\left(\pm\,\mathrm{dn}\xi\mp\sqrt{1-m^2}\ \mathrm{nd}\xi\right)e^{i\Theta},\\ n(x,t)&=\tfrac{-(\Omega^2+2\,\mu)}{\left(2-m^2\pm 6\,\sqrt{1-m^2}\right)}\left(\mathrm{dn}\xi-\sqrt{1-m^2}\ \mathrm{nd}\xi\right)^2, \end{aligned} \tag{3.31}$$

Case 3.4: If $q_0 = \frac{1}{4}$, $q_2 = \frac{m^2-2}{2}$, $q_4 = \frac{m^4}{4}$, we get $F(\xi) = \frac{\mathrm{sn}\xi}{1\pm \mathrm{dn}\xi}$, this yeilds the exact solutions of (1.1) as

$$\begin{aligned} E(x,t) &= \tfrac{1}{2}\sqrt{\tfrac{(\Omega^2+2\,\mu)\,(\Omega^2-1)}{(m^2-2\pm 3\,m^2)}}\left(\pm\, \tfrac{m^2\mathrm{sn}\xi}{1\pm \mathrm{dn}\xi} \mp \tfrac{1\pm \mathrm{dn}\xi}{\mathrm{sn}\xi}\right) e^{i\Theta}, \\ n(x,t) &= \tfrac{(\Omega^2+2\,\mu)}{2\,(m^2-2\pm 3\,m^2)}\left(\tfrac{m^2\mathrm{sn}\xi}{1\pm \mathrm{dn}\xi} - \tfrac{1\pm \mathrm{dn}\xi}{\mathrm{sn}\xi}\right)^2, \end{aligned} \tag{3.32}$$

Case 3.5: When $q_0 = \frac{m^2-1}{4}$, $q_2 = \frac{m^2+1}{2}$, $q_4 = \frac{m^2-1}{4}$, we have $F(\xi) = \frac{\mathrm{dn}\xi}{1\pm m\,\mathrm{sn}\xi}$. Thus, the rational JEFs solutions of (1.1) are

$$\begin{aligned} E(x,t) &= \tfrac{1}{2}\sqrt{\tfrac{(m^2-1)(\Omega^2+2\,\mu)\,(\Omega^2-1)}{[m^2+1\pm 3\,(m^2-1)]}}\left(\pm\, \tfrac{\mathrm{dn}\xi}{1\pm m\,\mathrm{sn}\xi} \mp \tfrac{1\pm m\,\mathrm{sn}\xi}{\mathrm{dn}\xi}\right) e^{i\Theta}, \\ n(x,t) &= \tfrac{(m^2-1)(\Omega^2+2\,\mu)}{2\,[m^2+1\pm 3\,(m^2-1)]}\left(\tfrac{\mathrm{dn}\xi}{1\pm m\,\mathrm{sn}\xi} - \tfrac{1\pm m\,\mathrm{sn}\xi}{\mathrm{dn}\xi}\right)^2, \end{aligned} \tag{3.33}$$

Case 3.6: When $q_0 = \frac{1-m^2}{4}$, $q_2 = \frac{m^2+1}{2}$, $q_4 = \frac{1-m^2}{4}$, $F(\xi) = \frac{\mathrm{cn}\xi}{1\pm \mathrm{sn}\xi}$, we obtain

$$\begin{aligned} E(x,t) &= \tfrac{1}{2}\sqrt{\tfrac{(1-m^2)(\Omega^2+2\,\mu)\,(\Omega^2-1)}{[m^2+1\pm 3\,(1-m^2)]}}\left(\tfrac{\mathrm{cn}\xi}{1\pm \mathrm{sn}\xi} \mp \tfrac{1\pm \mathrm{sn}\xi}{\mathrm{cn}\xi}\right) e^{i\Theta}, \\ n(x,t) &= \tfrac{(1-m^2)(\Omega^2+2\,\mu)}{2\,[m^2+1\pm 3\,(1-m^2)]}\left(\tfrac{\mathrm{cn}\xi}{1\pm \mathrm{sn}\xi} - \tfrac{1\pm \mathrm{sn}\xi}{\mathrm{cn}\xi}\right)^2 \end{aligned} \tag{3.34}$$

Case 3.7: If $q_4 = \frac{-1}{4}$, $q_2 = \frac{m^2+1}{2}$, $q_0 = \frac{-(1-m^2)^2}{4}$, and $F(\xi) = (m\,\mathrm{cn}\xi \pm \mathrm{dn}\xi)$. Then, the exact solutions of Eq. (1.1) are

$$\begin{aligned} E(x,t) &= \tfrac{1}{2}\sqrt{\tfrac{(\Omega^2+2\,\mu)\,(1-\Omega^2)}{[m^2+1\pm 3\,(1-m^2)]}}\left(\pm\, (m\,\mathrm{cn}\xi \pm \mathrm{dn}\xi) \mp \tfrac{1-m^2}{m\,\mathrm{cn}\xi\pm \mathrm{dn}\xi}\right) e^{i\Theta}, \\ n(x,t) &= \tfrac{-(\Omega^2+2\,\mu)}{2\,[m^2+1\pm 3\,(1-m^2)]}\left((m\,\mathrm{cn}\xi \pm \mathrm{dn}\xi) - \tfrac{1-m^2}{m\,\mathrm{cn}\xi\pm \mathrm{dn}\xi}\right)^2. \end{aligned} \tag{3.35}$$

We can obtain other JEFs solutions, but we ignored here for simplicity. If $m \to 1$, then the travelling wave solutions given as

$$E(x,t) = \pm\frac{1}{2}\sqrt{(\Omega^2+2\,\mu)\,(1-\Omega^2)}\,\frac{\tanh\xi}{1\pm \mathrm{sech}\xi}\,e^{i\Theta},\ \ n(x,t) = \frac{-(\Omega^2+2\,\mu)}{2}\left(\frac{\tanh\xi}{1\pm \mathrm{sech}\xi}\right)^2, \tag{3.36}$$

$$E(x,t) = \pm\frac{1}{2}\sqrt{(\Omega^2+2\,\mu)\,(1-\Omega^2)}\,\frac{1\pm \mathrm{sech}\xi}{\tanh\xi}\,e^{i\Theta},\ \ n(x,t) = \frac{-(\Omega^2+2\,\mu)}{2}\left(\frac{1\pm \mathrm{sech}\xi}{\tanh\xi}\right)^2, \tag{3.37}$$

$$E(x,t) = \pm\sqrt{\frac{(\Omega^2+2\,\mu)\,(1-\Omega^2)}{2}}\,\mathrm{sech}\xi\,e^{i\Theta},\quad n(x,t) = -(\Omega^2+2\,\mu)\,\mathrm{sech}^2\xi, \tag{3.38}$$

$$E(x,t) = \pm\frac{1}{2}\sqrt{(\Omega^2+2\,\mu)\,(1-\Omega^2)}\,\tanh\xi\,e^{i\Theta},\quad n(x,t) = \frac{-(\Omega^2+2\,\mu)}{2}\,\tanh^2\xi, \tag{3.39}$$

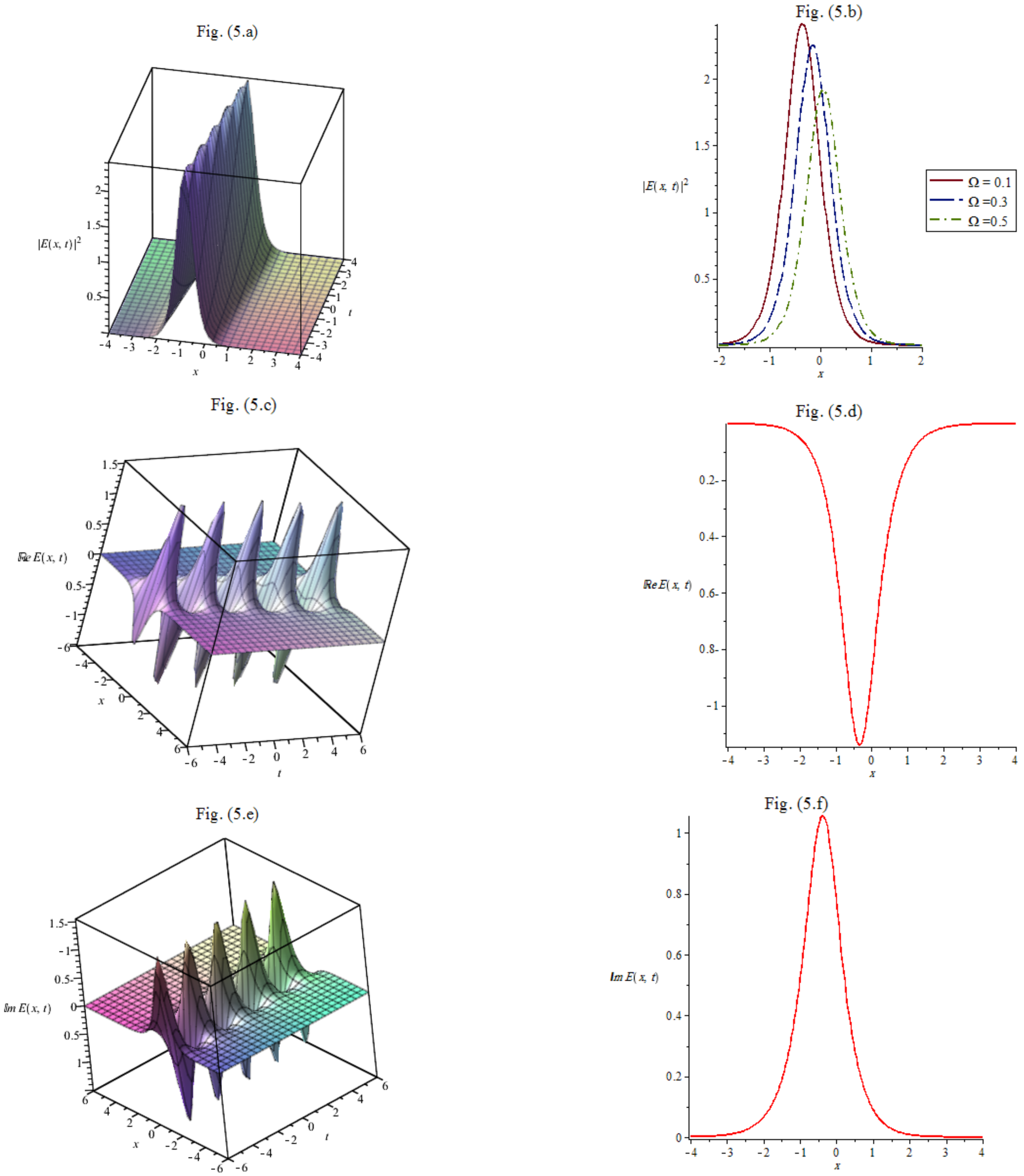

Fig. (5.a)

Fig. (5.b)

Fig. (5.c)

Fig. (5.d)

Fig. (5.e)

Fig. (5.f)

Figure 5: The bright solitary wave solution (3.38) are plotted in (a-f) 3D and 2D with the parameters $\Omega = 0.1$, $\mu = 2.43$ and $\xi_0 = 1$ for 3D figure and $t = 1$ for 2D figure. In Fig. (5.b), we discuss the intensity profile at different values of Ω.

$$E(x,t) = \pm\frac{1}{2}\sqrt{(\Omega^2 + 2\,\mu)\,(1 - \Omega^2)}\,\coth\xi\, e^{i\Theta}, \quad n(x,t) = \frac{-(\Omega^2 + 2\,\mu)}{2}\coth^2\xi, \tag{3.40}$$

$$E(x,t) = \pm\sqrt{\frac{(\Omega^2 + 2\,\mu)\,(\Omega^2 - 1)}{2}}\,\mathrm{csch}\xi\, e^{i\Theta}, \quad n(x,t) = (\Omega^2 + 2\,\mu)\,\mathrm{csch}^2\xi. \tag{3.41}$$

Also, if $m \to 0$, then we can obtain the triangular function solutions for (1.1) , but we omitted these solutions for simplicity. The solutions (3.38) and (3.39) are called bright and dark soliton solutions, respectively. The bright solitary wave solution (3.38) are plotted in Fig. 5 with the parameters $\Omega = 0.1$, $\mu = 2.43$ and $\xi_0 = 1$ for 3D figure and $t = 1$ for 2D figure. Moreover, in Fig. (5.b), we have studied the intensity profile at different values of the parameter Ω. We get that with the increase of Ω the width increases and the amplitude decreases.

In addition, multiplying Eq. (3.6) by U' and integrating once, we get

$$\frac{1}{2}\,(U')^2 = C + \frac{(\Omega^2 + 2\,\mu)}{2\,k^2}\,U^2 + \frac{U^4}{k^2\,(\Omega^2 - 1)},$$

where C is an arbitrary constant. We can be written this equation like as an energy equation of classical particle as $\frac{1}{2}\,(U')^2 + f(U) = 0$, where $f(U)$ is the potential energy and it is given by $f(U) = -[\,C + \frac{(\Omega^2+2\,\mu)}{2\,k^2}\,U^2 + \frac{U^4}{k^2\,(\Omega^2-1)}\,]$. The physical solution to exist it must satisfy $f(U) = 0$ and $\frac{df(U)}{dU} = 0$ at $U = 0$. It is clear that $(U')^2 \geq 0$, this means $f(U) \leq 0$. So, there exist a point U_c such that $f(U_c) = 0$, i.e.,

$$C + \frac{(\Omega^2 + 2\,\mu)}{2\,k^2}\,U_c^2 + \frac{U_c^4}{k^2\,(\Omega^2 - 1)} = 0.$$

When $C = 0$, the amplitude of solitary wave $f(U_c) = 0$ is $U_c = \pm\sqrt{\frac{(\Omega^2+2\,\mu)\,(1-\Omega^2)}{2}}$.

4 Lie symmetry analysis and conservation laws

In this section, we derive Lie symmetry analysis of the model (1.1) and investigate the conservation laws by Ibragimove's theorem [27]. For this, we consider the transformation

$$E(x,t) = u(x,t) + i\,v(x,t). \tag{4.1}$$

Substituting (4.1) in (1.1) and splitting the result into imaginary and real parts, we get the following system

$$\begin{cases} F^1 = u_t + \frac{1}{2}\,v_{xx} - n\,v = 0, \\ F^2 = v_t - \frac{1}{2}\,u_{xx} + n\,u = 0, \\ F^3 = n_{tt} - n_{xx} - 4\left(u_x^2 + v_x^2 + u\,u_{xx} + v\,v_{xx}\right) = 0. \end{cases} \tag{4.2}$$

We can be written this equation in the form

$$n_{tt}-n_{xx}-4\left[u_x^2+v_x^2+2\left(n\,(u^2+v^2)+u\,v_t-v\,u_t\right)\right]=0. \tag{4.3}$$

The Lie point symmetries for (4.3) is generated by a vector field in the form

$$\begin{aligned}X=\xi^1(x,t,u,v,n)\,\partial_x+\xi^2(x,t,u,v,n)\,\partial_t+\eta^1(x,t,u,v,n)\,\partial_u+\eta^2(x,t,u,v,n)\,\partial_v+\\ \eta^3(x,t,u,v,n)\,\partial_n.\end{aligned} \tag{4.4}$$

Appling the prolongation $\mathrm{Pr}^{(2)}X$ to (4.3), we have system of linear partial differential equations(PDEs). Solving it by Maple, we get the infinitesmals as follows:

$$\xi^1=c_3,\ \xi^2=c_4,\ \eta^1=c_1\,t+c_2,\ \eta^2=\frac{1}{2}\,v\,t^2\,c_1+v\,t\,c_2+v\,c_5,\ \eta^3=-\frac{1}{2}\,u\,t^2\,c_1-u\,t\,c_2-u\,c_5. \tag{4.5}$$

where c_1, c_2, c_3, c_4 and c_5 are constants. Eq. (4.3) admits the algebra of Lie point symmetries generated as

$$\begin{aligned}&X_1=t\,\partial_u+\tfrac{1}{2}\,v\,t^2\partial_v-\tfrac{1}{2}\,u\,t^2\partial_n,\ \ X_2=\partial_u+v\,t\,\partial_v-u\,t\,\partial_n,\ \ X_3=\partial_x,\ \ X_4=\partial_t,\\ &X_5=v\,\partial_v-u\,\partial_n.\end{aligned} \tag{4.6}$$

For simplicity, suppose that a sth-order system of PDEs of r dependent variables $u=(u^1,u^2,...,u^r)$ and k independent variables $x=(x^1,x^2,...,x^k)$, define as

$$F_\alpha(x,u,u_{(1)},...,u_{(s)})=0,\ \ \alpha=1,2,...,r, \tag{4.7}$$

where, $u_{(1)},u_{(2)},...,u_{(s)}$ denote the collections of all first, second,...,sth-order partial derivatives. This means that, $u_i^\alpha=D_i(u^\alpha)$, $u_{ij}^\alpha=D_jD_i(u^\alpha),...$, respectively, where the total derivative operator with respect to x^i given as

$$D_i=\frac{\partial}{\partial x^i}+u_i^\alpha\,\frac{\partial}{\partial u^\alpha}+u_{ij}^\alpha\,\frac{\partial}{\partial u_j^\alpha}+...,\ \ i=1,2,...,k. \tag{4.8}$$

Also, we can define the symmetry operator and the adjoint equation for the system (4.7), respectively as

$$X=\xi^i\,\frac{\partial}{\partial x^i}+\eta^\alpha\,\frac{\partial}{\partial u^\alpha}, \tag{4.9}$$

$$F_\alpha^*(x,u,v,u_{(1)},v_{(1)},...,u_{(s)},v_{(s)})=\frac{\delta(v^iF^i)}{\delta u^\alpha}=0,\ \ \alpha=1,2,...,r. \tag{4.10}$$

Theorem [27]: Any Lie point, Lie-Bäcklund and non-local symmetry X, that is define in (4.9) admitted by the system (4.7) provides a conservation law for (4.7) and its adjoint (4.10), then T^i that is called the conserved vector are calculated by

$$\begin{aligned} T^i = {} & \xi^i\, L + W^\alpha \left[\frac{\partial L}{\partial u_i^\alpha} - D_j\left(\frac{\partial L}{\partial u_{ij}^\alpha}\right) + D_j D_k\left(\frac{\partial L}{\partial u_{ijk}^\alpha}\right) - ...\right] + \\ & D_j(W^\alpha)\left[\frac{\partial L}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial L}{\partial u_{ijk}^\alpha}\right) + D_k D_r\left(\frac{\partial L}{\partial u_{ijkl}^\alpha}\right) - ...\right] + \\ & D_j D_k(W^\alpha)\left[\frac{\partial L}{\partial u_{ijk}^\alpha} - D_r\left(\frac{\partial L}{\partial u_{ijkl}^\alpha}\right) + ...\right] + ..., \end{aligned} \tag{4.11}$$

where, $W^\alpha = \eta_\alpha - \xi^i u_i^\alpha$ and $L = \sum_{i=1}^{r} v^i F^i$ are the Lie characteristic function and the formal lagrangian, respectively. Now we will obtain the conservation laws for (1.1), first we can define the Lagrangian formal for the system (1.1) as

$$L = \bar{u}\left(u_t + \frac{1}{2}\, v_{xx} - n\, v\right) + \bar{v}\left(v_t - \frac{1}{2}\, u_{xx} + n\, u\right) + \bar{n}\left[n_{tt} - n_{xx} - 4\left(u_x^2 + v_x^2 + u\, u_{xx} + v\, v_{xx}\right)\right], \tag{4.12}$$

where $\bar{u}$, $\bar{v}$ and $\bar{n}$ are new dependent variables. By using (4.11) and (4.12), we get

$$\begin{cases} T^1 = \xi^1\, L + W^1(-8\,\bar{n}\, u_x + \frac{1}{2}\bar{v}_x + 4\,\bar{n}_x\, u + 4\,\bar{n}\, u_x) - (\frac{1}{2}\,\bar{v} + 4\,\bar{n}\, u)\, D_x(W^1) + (\frac{1}{2}\,\bar{u} - 4\,\bar{n}\, v) \\ D_x(W^2) + W^2(-8\,\bar{n}\, v_x - \frac{1}{2}\bar{u}_x + 4\,\bar{n}_x\, v + 4\,\bar{n}\, v_x) + \bar{n}_x\, W^3 - \bar{n}\, D_x(W^3), \\ T^2 = \xi^2\, L + W^1\,\bar{u} + W^2\,\bar{v} - \bar{n}_t\, W^3 + \bar{n}\, D_t(W^3). \end{cases} \tag{4.13}$$

From the symmetry operators given in (4.6) with (4.13), we get the following cases for the conservation laws:

Case 1: We consider the symmetry operator $X_1 = t\,\partial_u + \frac{1}{2}\, v\, t^2 \partial_v - \frac{1}{2}\, u\, t^2 \partial_n$, we have $\xi^1 = \xi^2 = 0$, $\eta^1 = t$, $\eta^2 = \frac{1}{2}\, v\, t^2$, $\eta^3 = -\frac{1}{2}\, u\, t^2$ and the Lie characteristic functions corresponding to this symmetry are $W^1 = t$, $W^2 = \frac{1}{2}\, v\, t^2$ and $W^3 = \frac{1}{2}\, u\, t^2$. Thus, the associated conserved vectors are

$$\begin{cases} T^1 = \bar{n}\, u_x\,(\frac{1}{2}\, t^2 - 4\, t) + t^2\, v_x(\frac{1}{4}\,\bar{u} - 4\,\bar{n}\, v) + \frac{1}{2}\, t(\bar{v}_x - \frac{1}{2}\, t\, v\,\bar{u}_x) + \bar{n}_x[4\, u + t^2\,(2\, v^2 - \frac{1}{2}\, u)], \\ T^2 = \frac{1}{2} t^2\,(v\,\bar{v} + u\,\bar{n}_t) + t\,(\bar{u} - \bar{n}\, u), \end{cases} \tag{4.14}$$

Case 2: Using the symmetry operator $X_2 = \partial_u + v\, t\,\partial_v - u\, t\,\partial_n$, we have $\xi^1 = \xi^2 = 0$, $\eta^1 = 1$, $\eta^2 = v\, t$, $\eta^3 = -u\, t$. Then $W^1 = 1$, $W^2 = v\, t$ and $W^3 = -u\, t$. Thus, the associated conserved vectors given as

$$\begin{cases} T^1 = \bar{n}\, u_x\,(t - 4) + t\, v_x\,(\frac{1}{2}\,\bar{u} - 8\, v\,\bar{n}) + \frac{1}{2}\,\bar{v}_x + \frac{1}{2}\, v\, t\,\bar{u}_x + \bar{n}_x[4\, u + t\,(4\, v^2 - u)], \\ T^2 = \bar{u} + t\,[v\,\bar{v} + u\,\bar{n}_t - \bar{n}\, u_t] - \bar{n}\, u, \end{cases} \tag{4.15}$$

Case 3: For the symmetry operator $X_3 = \partial_x$, we have $\xi^1 = 1$, $\xi^2 = \eta^1 = \eta^2 = \eta^3 = 0$ and $W^1 = -u_x$, $W^2 = -v_x$ and $W^3 = -n_x$. So, we obtain

$$\begin{cases} T^1 = \bar{u}(u_t - n\,v) + \bar{v}\,(v_t + n\,u) + \bar{n}\,n_{tt} + \frac{1}{2}(u_x\,\bar{v}_x + v_x\,\bar{u}_x) - \bar{n}_x(4\,u\,u_x + 4\,v\,v_x + n_x), \\ T^2 = \bar{n}_t\,n_x - \bar{n}\,n_{xt} - \bar{u}\,u_x - \bar{v}\,v_x, \end{cases} \tag{4.16}$$

Case 4: Using the symmetry operator $X_4 = \partial_t$, we have $\xi^2 = 1$, $\xi^1 = \eta^1 = \eta^2 = \eta^3 = 0$ with $W^1 = -u_t$, $W^2 = -v_t$ and $W^3 = -n_t$. so, we obtain the conserved vectors as

$$\begin{cases} T^1 = 4\,\bar{n}\,u_x u_t + u_{xt}\,(\frac{1}{2}\bar{v} + 4\,u\,\bar{n}) - v_{xt}\,(\frac{1}{2}\bar{u} - 4\,v\,\bar{n}) + \frac{1}{2}(v_t\bar{u}_x - u_t\bar{v}_x) + \bar{n}\,(4\,v_x\,v_t + n_{xt}), \\ -\bar{n}_x\,[\,4\,(u\,u_t + v\,v_t) + n_t], \\ T^2 = \bar{u}(\frac{1}{2}\,v_{xx} - n\,v) - \bar{v}(\frac{1}{2}\,u_{xx} - n\,u) - \bar{n}[n_{xx} + 4\,(u_x^2 + v_x^2 + u\,u_{xx} + v\,v_{xx})] + \bar{n}_t\,n_t, \end{cases} \tag{4.17}$$

Case 5: Using the symmetry operator $X_5 = v\,\partial_v - u\,\partial_n$, we have $\xi^1 = \xi^2 = \eta^1 = 0$, $\eta^2 = v$, $\eta^3 = -u$ and $W^1 = 0$, $W^2 = v$ and $W^3 = -u$. So, we obtain the conserved vectors as

$$\begin{cases} T^1 = \frac{1}{2}\,(\bar{u}\,v_x - v\,\bar{u}_x) + \bar{n}\,(u_x - 8\,v\,v_x) + \bar{n}_x\,(4\,v^2 - u), \\ T^2 = v\,\bar{v} + u\,\bar{n}_t - \bar{n}\,u_t. \end{cases} \tag{4.18}$$

5 Conclusion

In this paper, we considered the ISLWs model and we succeeded implementing the extended F-expansion method in the NLEEs for getting exact traveling wave solutions. As results, several kinds of solutions of the underlying model including periodic wave solutions with JEFs, hyperbolic function solutions dark and bright solutions have been obtained in the study, in which many are novel. The computer systems like as Maple is used to solve the complecated algebraic equations to get these solutions. The solutions (3.10), (3.11), (3.38) and (3.39) are the same as the results obtained in [24] and the solutions (3.39) and (3.40) are similar to the solutions given in [20, 25]. To the best of our knowledge, the obtained solutions of ISLWs model contain the known result in [20, 22, 24, 25] and other traveling wave solutions are new. The geometrical shape for some of the obtained results are plotted for various choices of the parameters that appear in the results which may help researchers to known some physical meaning of this model. Graphical simulation of some solutions in the form of two-dimentional and three-dimentional are helpfull to see the behaviour of these solutions. The 3D and 2D graphs indicated the properties of the periodic solutions (3.10), (3.13), (3.16) and (3.24) in Fig. 1, Fig. 2, Fig. 3 and Fig. 4, respectively. The bright solitary wave solution (3.38) are plotted in Fig. 5 and we have discussed the effect of the parameter Ω at different values on the intensity profile in Fig. (5.b). In addition, the conservation laws for the (1.1) are constructed. We hope that the obtained solutions are useful in the study of plasma physics and other important equations of mathematical physics.

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