Original Research Article

Hybridization of Solitary Wave Solutions in (2+1)-dimentional Complex Ginzburg-Landau Equation

Abstract

The reflection carried out in this manuscript concerns the construction of prototypes of hybrid solitary waves, solutions of the (2+1)-dimensional complex Ginzburg-Landau equation. The principle of construction consists in injecting into the equation to be solved an ansatz that one would like solution, and that its analytical sequence results from a combination of the analytical sequences of the classical solitary waves. Then, the constraints imposed by the resolution allow to extract exact or approximate solution. As part of this work, the solution function to be constructed from the start is made up of a combination of four analytical sequences of solitary waves of the kink and pulse type. To this end, we have obtained, using a rigorous mathematical approach, important results whose graphic exploitations have made it possible to better characterize them.

Keywords: Hybridization of solitary wave solutions, (2+1)-dimensional complex Ginzburg-Landau equation, Bogning-Djeumen Tchaho-Kofané method, analytical sequences of the classical solitary waves.

1. Introduction

In everyday life, human beings as well as other living beings in the universe face the expression of many nonlinear phenomena [1-3], which at times, are detrimental to their well-being. To preserve themselves against this, many researchers in mathematical physics, in their commitments to understand and explain these nonlinear and dispersive phenomena, try to design nonlinear and dispersive evolution equations [2; 4-9] that can make it possible to describe the observed phenomena. One of the major particularities of the proposed mathematical models is that they admit an infinity of solutions, the most robust of which are solitary waves or solitons. These solitary wave solutions are either exact, approximate or forced; thus justifying the important role that these models play in soliton theory [10-12]. The formulation of many nonlinear and dispersive evolution equations makes solving them very complicated. Faced with these complications, a good number of effective direct methods[12] to obtain the exact solutions of these models have been proposed, in particular the homoclinic test technique[13-15], Exp-function method[16], the F-expansion method[17], the extended tanh-function method[18; 19], and so on. Hybrid solutions[20-22] formed by an assembly of analytical sequences of the classical solitary waves are very difficult to obtain. These last two decades have seen the birth of other techniques for solving some of the proposed models, which, apart from exact solutions, also offer approximate or forced solutions with several solitons, such as the Bogning-Djeumen Tchaho-Kofané method (BDKm)[23-28] extende to the implicit Bogning functions(iB-functions)[27-30]. Therefore, it is for adduce an adequate response to this lack of supply the literature with these other forms of multi-solitary wave solutions that this work is registers.

The aim of this work is to propose new exact, approximate or forced complex solitary wave solutions of the (2+1)-dimensional complex Ginzburg-Landau equation (2D CGLE) by the use of the BDKm extended to the iB-functions. This work is organized as follows: **Section 2** is responsible for presenting the studied model while **Section 3** briefly displays the implementation of the BDKm extended to iB-functions. **Section 4** is dedicated to the analytical and graphical results obtained. **Section 5** deals with the discussions while **Section 6** articulates the conclusion and the outlook.

2. The used mathematical model

A class of the Schrödinger equation with a nonlinear term is the well-known Ginzburg-Landau equation[31]. This equation is one of the most important nonlinear equations in physics[1]. Various forms of the Ginzburg-Landau equation are used to describe a wide variety of phenomena ranging from nonlinear waves to liquid in physics (case of hydrodynamic instabilities) through superconductivity, superfluidity and Bose-Einstein condensation. Strengthened by it all, the mathematical model on which we have set our sights is the (2+1)-dimensional complex Ginzburg-Landau equation which takes the following form[15; 32-36]

$$i\Phi_t + \frac{1}{2}\Phi_{xx} + \frac{1}{2}(\alpha - iG)\Phi_{yy} + (1 - i\lambda)|\Phi|^2\Phi = i\gamma\Phi. \tag{1}$$

Herein, Φ is a complex valued function and $\alpha, G, \lambda, \gamma$ are constants real parameters. Eq.(1) was used in[33] to construct new exact wave solutions including homoclinic wave, kink wave and soliton solutions by the aid of the auxiliary function method, generalized Hirota method and the ansatz function technique under the certain constraint conditions of coefficients in equation, respectively. Then, its was also used in[34] to obtain new exact periodic and blow up solutions via the homogeneous balance principle and general Jacobi elliptic-function method, and when $\alpha = \lambda = 0$, It takes, the name of real equation of Ginzburg-Landau. Recently in[35], an investigation was carried out in order to formulate new shape of the chirped soliton solutions for this equation as well as a study of the modulation instability gain spectrum under the effect of the power incident and the transverse wave number using the linear stability technic. Let us now, glance off the method to be used for the following.

3. Brief presentation of the used method

The BDKm [23-27; 37-47] extended to iB-functions [27-30] and used within the framework of this work applies to some partial differential equation types in which coexist the nonlinear terms and the dispersive terms (and others) under the form:

$$X(\Phi, \Phi_t, \Phi_{xy}, \Phi_{xzt}, \Phi_{ty}, \Phi_{yz}, \Phi_{ttz}, \Phi_{xxx}, ..., |\Phi|^2, (\Phi|\Phi|^2)_t, ...) = 0, \tag{2}$$

where $\Phi(x, y, z, t)$ is an unknown function to be determined, X is some function of Φ and its derivatives with respect to x, y, z, t and includes the highest order derivatives and the nonlinear terms. Most often, we use the change of variables $\Phi(x, t) = \Omega(\xi), \xi = \sum_{k=0}^{p} \alpha_k x_k$. In the case where Φ is a function of x, y, z and t, ξ becomes $\xi = x + y + z - \nu t$, where ν is the wave speed. In this context, eq.(2) gives rise to the ordinary differential equation(ODE) below:

$$X_{ODE}(\Omega, \Omega^{'}, \Omega^{''}, ..., \Omega^{'}|\Omega|^2, ...) = 0, \tag{3}$$

where $\Omega^{'}$, $\Omega^{''}$ represent respectively the first and second derivatives of the envelope Ω with respect to ξ. Then, the solution we are looking for can be expressed under contracted form

$$\Omega(\xi) = \sum_{ij} \mu_{ij} J_{j,i}(\eta\xi) \ , \tag{4}$$

where η is a real constant, μ_{ij} are the unknown constants to be determined and $J_{n,m}(\alpha x)$ is the iB-function whose explicit hyperbolic form is written as:

$$J_{n,m}(\sum_{i=0}^{p} \alpha_i x_i) = \frac{\sinh^m(\sum_{i=0}^{p} \alpha_i x_i)}{\cosh^n(\sum_{i=0}^{p}, \alpha_i x_i)}. \tag{5}$$

where $\alpha_i, (i = 0; 1; 2; ...; p)$ are the parameters associated to the independent variables $x_i, (i = 0; 1; 2; ...; p)$, m and n are powers of both terms of eq.(5). For more details, see [27-30]. Thus, inserting eq.(4) into (3) gives rise to the main equation of ranges

$$\begin{aligned} &\sum_{ijn} A_n(\mu_{ij}, \eta, \nu) J_{n,0}(\eta\xi) + \sum_{ijm} B_m(\mu_{ij}, \eta, \nu) J_{m,1}(\eta\xi) + \sum_{ijk} C_k(\mu_{ij}, \eta, \nu) J_{-k,0}(\eta\xi) + \sum_{ijl} D_l(\mu_{ij}, \eta, \nu) J_{-l,1}(\eta\xi) \\ &+ \sum_{ij} E(\mu_{ij}, \eta, \nu) J_{0,0}(\eta\xi) = 0, \end{aligned} \tag{6}$$

where i, j, k, l are positive natural integers and n, m the real numbers[27-30]. It can be noted here that eq.(6) is the one from which all the possible analyzes result. The identification of coefficients $A_n(\mu_{ij}, \eta, \nu), B_m(\mu_{ij}, \eta, \nu), C_k(\mu_{ij}, \eta, \nu)$, $D_l(\mu_{ij}, \eta, \nu), E(\mu_{ij}, \eta, \nu)$ at zero makes it possible to obtain the ranges of equations whose the resolutions could allow to obtain the expressions of the unknown coefficients μ_{ij}. It is important to point out here that, the resolution of this series of equations often leads to exact solutions[27; 40; 45; 47] for certain models and according to the form of the considered ansatz while, for other models and according to the form of the chosen ansatz , it (resolution) leads to approximate or forced solutions. In the case of approximate or forced solutions, the priority in the order of resolution is given to those from the highest clues of $J_{n,0}(\eta\xi)$, then to those of the highest clues of $J_{m,1}(\eta\xi)$. But, otherwise we go to those from the coefficients of lowest clues of $J_{-k,0}(\eta\xi)$ and $J_{-l,1}(\eta\xi)$. Here, the priority makes reference to the serie that permits to obtain good results or merely that tends more to the sought exact solution. Very often, the series of equations obtained by identify at zero the coefficient of $J_{n,0}(\eta\xi)$ gives satisfaction. For more understanding, one can refer to [23-28; 37-47]

4. Results

This part of the work deals with the construction of the solitary wave solutions of eq.(1) using the BDKm extended to the iB-function. The BDKm, by its implementation, made it possible to organize the obtained results in three large sections: the production of ranges of equations, the resolution of obtained ranges of equations and the graphical representations of some obtained solutions in order to better agree with theoretical predictions.

4.1. Production of the range equations

The range equations production is a tedious exercise which, in part, depends on a judicious choice of the analytical form of the solution to be constructed as well as a good mastery of the properties of iB-functions. So, consider the solution to be constructed in the compact form as being

$$\Phi(x,y,t)=\Psi[\xi(x,y,t)]e^{i\phi(x,y,t)}, \tag{7}$$

with $\xi(x,y,t)=x+y-\nu t$, $\phi(x,y,t)=-kx-sy+\omega t$ and where k, s ω are the real wave parameters, ν the wave speed. Substituting eq.(7) into (1) yields to the travelling wave equation which describes the dynamics of the amplitude Ψ below

$$[\omega+\frac{1}{2}k^2+\frac{1}{2}\alpha s^2+i(\gamma-\frac{1}{2}Gs^2)]\Psi+[Gs+i(\alpha s+k+\nu)]\Psi_\xi+[\frac{1}{2}iG-\frac{1}{2}(1+\alpha)]\Psi_{\xi\xi}+(i\lambda-1)|\Psi|^2\Psi=0. \tag{8}$$

The solution to be constructed is written in the form

$$\Psi(\xi)=A_1J_{2,1}(\eta\xi)+iB_1J_{1,1}(\eta\xi)+A_2J_{3,1}(\eta\xi)+iB_2J_{2,2}(\eta\xi), \tag{9}$$

where A_1, B_1, A_2 and B_2 are real constants to be determined, η, the inverse of the width at half-height, of each component of the choosen ansatz and i, an imaginary such that $i^2=-1$. In eq.(9), the respective coefficient terms A_1 and A_2 are hybrid solitary waves[39; 49] while the respective coefficient terms B_1 and B_2 are the well known classic solitons under the names: Kink and Dark. Thereafter, the consideration of eq.(9) into (8) provides the main equation of ranges in the following contracted form

$$\sum_{s=0}^{8}P_s(A_1,B_1,A_2,B_2,\eta,G,k,s,\lambda,\gamma,\alpha,\omega,\nu)J_{s,0}(\eta\xi)+\sum_{j=1}^{9}Q_j(A_1,B_1,A_2,B_2,\eta,G,k,s,\lambda,\gamma,\alpha,\omega,\nu)J_{j,1}(\eta\xi)$$
$$+i[\sum_{s'=0}^{8}P'_{s'}(A_1,B_1,A_2,B_2,\eta,G,k,s,\lambda,\gamma,\alpha,\omega,\nu)J_{s',0}(\eta\xi)+\sum_{j'=1}^{9}Q'_{j'}(A_1,B_1,A_2,B_2,\eta,G,k,s,\lambda,\gamma,\alpha,\omega,\nu)J_{j',1}(\eta\xi)]=0. \tag{10}$$

Underscore herein that eq.(10) presents two ranges of equations, each consisting of two subranges coming from real and imaginary parts respectively: $P_s=0, P'_{s'}=0$ and $Q_j=0, Q'_{j'}=0$. By expliciting these equations, we obtain four subranges of equations of unknowns A_1, B_1, A_2, B_2 and apportioned as follows

4.1. 1. First range of equations

- From the real part:
the term in $J_{8,0}(\eta\xi)$,

$$-\lambda A_2^2B_2=0, \tag{11}$$

the term in $J_{7,0}(\eta\xi)$,

$$-2\lambda A_1A_2B_2=0, \tag{12}$$

the term in $J_{6,0}(\eta\xi)$,

$$B_2[\lambda(2A_2^2+B_2^2)-\lambda A_1^2]=0, \tag{13}$$

the term in $J_{5,0}(\eta\xi)$,

$$A_1B_2(4\lambda A_2-2B_1)=0, \tag{14}$$

the term in $J_{4,0}(\eta\xi)$,

$$B_2[2\lambda A_1^2+3A_2B_1-3\lambda(B_1^2+B_2^2))-\lambda A_2^2-3\eta^2G]+3\eta GsA_2=0, \tag{15}$$

the term in $J_{3,0}(\eta\xi)$,

$$A_1[B_2(4B_1-2\lambda A_2)+2\eta Gs]=0, \tag{16}$$

the term in $J_{2,0}(\eta\xi)$,

$$B_2[3\lambda(2B_1^2+B_2^2)-\lambda A_1^2-A_2B_1+2\eta^2G+\gamma-\frac{1}{2}Gs^2]-[2\eta GsA_2+\eta(\alpha s+k+\nu)B_1]=0, \tag{17}$$

the term in $J_{1,0}(\eta\xi)$,

$$-A_1(2B_1B_2 + \eta Gs) = 0, \tag{18}$$

the term in $J_{0,0}(\eta\xi)$,

$$B_2[-\lambda(3B_1^2 + B_2^2) - \gamma + \frac{1}{2}Gs^2] = 0, \tag{19}$$

- From the imaginary part:
the term in $J_{8,0}(\eta\xi)$,

$$-A_2^2B_2 = 0, \tag{20}$$

the term in $J_{7,0}(\eta\xi)$,

$$-2A_1A_2B_2 = 0, \tag{21}$$

the term in $J_{6,0}(\eta\xi)$,

$$B_2(2A_2^2 + B_2^2 - A_1^2] = 0, \tag{22}$$

the term in $J_{5,0}(\eta\xi)$,

$$A_1B_2(2\lambda B_1 + 4A_2) = 0, \tag{23}$$

the term in $J_{4,0}(\eta\xi)$,

$$B_2[2A_1^2 - 3\lambda A_2B_1 - (3B_1^2 + 3B_2^2 + A_2^2) - 3\eta^2(1+\alpha)] + 3\eta(\alpha s + k + \nu)A_2 = 0, \tag{24}$$

the term in $J_{3,0}(\eta\xi)$,

$$A_1[-B_2(2A_2 + 4\lambda B_1) + 2\eta(\alpha s + k + \nu)] = 0, \tag{25}$$

the term in $J_{2,0}(\eta\xi)$,

$$B_2[6B_1^2 + 3B_2^2 - A_1^2 + \lambda A_2B_1 + 2\eta^2(1+\alpha) - \omega - \frac{1}{2}k^2 - \frac{1}{2}\alpha s^2] + \eta GsB_1 - 2\eta(\alpha s + k + \nu)A_2 = 0, \tag{26}$$

the term in $J_{1,0}(\eta\xi)$,

$$A_1[2\lambda B_1B_2 - \eta(\alpha s + k + \nu)] = 0, \tag{27}$$

the term in $J_{0,0}(\eta\xi)$,

$$B_2(-3B_1^2 - B_2^2 + \omega + \frac{1}{2}k^2 + \frac{1}{2}\alpha s^2) = 0, \tag{28}$$

4.1. 2. Second range of equations

- From the real part:
the term in $J_{9,1}(\eta\xi)$,

$$A_2^3 = 0, \tag{29}$$

the term in $J_{8,1}(\eta\xi)$,

$$3A_2^2A_1 = 0, \tag{30}$$

the term in $J_{7,1}(\eta\xi)$,

$$A_2[3A_1^2 - (A_2^2 + B_2^2) + \lambda A_2B_2] = 0, \tag{31}$$

the term in $J_{6,1}(\eta\xi)$,

$$A_1[A_1^2 - (3A_2^2 + B_2^2) + 2\lambda A_2B_1] = 0, \tag{32}$$

the term in $J_{5,1}(\eta\xi)$,

$$\lambda B_1(A_1^2 - A_2^2 - 3B_2^2) + A_2[B_1^2 + B_2^2 - 3A_1^2 + 6\eta^2(1+\alpha)] = 0, \tag{33}$$

the term in $J_{4,1}(\eta\xi)$,

$$A_1[B_1^2 + B_2^2 - A_1^2 - 2\lambda A_2B_1 + 3\eta^2(1+\alpha)] = 0, \tag{34}$$

the term in $J_{3,1}(\eta\xi)$,

$$B_1[\lambda(B_1^2+6B_2^2)-\lambda A_1^2+\eta^2 G]-A_2(B_1^2+B_2^2)+A_2[\omega+\frac{1}{2}k^2+\frac{1}{2}\alpha s^2-2\eta^2(1+\alpha)]-2\eta(\alpha s+k+\nu)B_2=0, \quad (35)$$

the term in $J_{2,1}(\eta\xi)$,

$$A_1[-(B_1^2+B_2^2)+\omega++\frac{1}{2}k^2+\frac{1}{2}\alpha s^2-\frac{1}{2}\eta^2(1+\alpha)]=0, \quad (36)$$

the term in $J_{1,1}(\eta\xi)$,

$$B_1[-\lambda(B_1^2+2B_2^2)-\gamma+\frac{1}{2}Gs^2)=0, \quad (37)$$

- From the imaginary part:

the term in $J_{9,1}(\eta\xi)$,

$$-\lambda A_2^3=0, \quad (38)$$

the term in $J_{8,1}(\eta\xi)$,

$$-3\lambda A_2^2 A_1=0, \quad (39)$$

the term in $J_{7,1}(\eta\xi)$,

$$A_2[\lambda(A_2^2+B_2^2-3A_1^2)+A_2B_1]=0, \quad (40)$$

the term in $J_{6,1}(\eta\xi)$,

$$A_1[2A_2B_1+\lambda(3A_2^2+B_2^2-A_1^2)]=0, \quad (41)$$

the term in $J_{5,1}(\eta\xi)$,

$$A_2[3\lambda A_1^2-\lambda(B_1^2+B_2^2)-6\eta^2 G]+B_1[A_1^2-(A_2^2+3B_2^2)]=0, \quad (42)$$

the term in $J_{4,1}(\eta\xi)$,

$$A_1[\lambda A_1^2-\lambda(B_1^2+B_2^2)-2A_2B_1-3\eta^2 G]=0, \quad (43)$$

the term in $J_{3,1}(\eta\xi)$,

$$A_2[\lambda(B_1^2+B_2^2)+\gamma-\frac{1}{2}Gs^2+2\eta^2 G]+B_1[B_1^2+6B_2^2-A_1^2+\eta^2(1+\alpha)]+2\eta GsB_2=0, \quad (44)$$

the term in $J_{2,1}(\eta\xi)$,

$$A_1[\lambda(B_1^2+B_2^2)+\gamma-\frac{1}{2}Gs^2+\frac{1}{2}\eta^2 G]=0, \quad (45)$$

the term in $J_{1,1}(\eta\xi)$,

$$B_1[-(B_1^2+2B_2^2)+\omega+\frac{1}{2}k^2+\frac{1}{2}\alpha s^2]=0, \quad (46)$$

4.2. Resolution of the range equations after analysing

In view of the structure of the above ranges of equations , we realize that one can has: $A_1=0, B_1=0, A_2=0, B_2=0, \lambda=0, \gamma=0, s=0$ or $G=0$. Therefore, we are interested in cases which lead to non-trivial solutions. Thus, two types of solutions are resulting (**A** and **B**).

A: Solutions type I: case $A_1=0, B_1\neq 0, A_2\neq 0, B_2\neq 0, \lambda\neq 0$

In the case of these **Solutions type I**, only the equations of the first range suffice for the determination of the approximate solutions that we seek to construct. This being, eqs.(12), (14), (16), (18), (21), (23), (25) and (27) are vrified while eqs.(11), (13) , (20) and (22) suggest that we take $A_2=B_2=0$. Which supposes that, the contribution of these last four equations is negligible at these orders of clues of the corresponding iB-functions. Moreover, eqs. (15), (17) and (19) lead respectively to

$$B_2[3A_2B_1-3\lambda(B_1^2+B_2^2)-\lambda A_2^2-3\eta^2 G]+3\eta GsA_2=0, \quad (47)$$

$$B_2[3\lambda(2B_1^2+B_2^2)-A_2B_1+2\eta^2 G+\gamma-\frac{1}{2}Gs^2]-[2\eta GsA_2+\eta(\alpha s+k+\nu)B_1)]=0, \quad (48)$$

and

$$\frac{1}{2}Gs^2-\gamma-\lambda(3B_1^2+B_2^2)=0, \quad (49)$$

while eqs.(24), (26) and (28) successively give

$$B_2[-3\lambda A_2 B_1 - (3B_1^2 + 3B_2^2 + A_2^2) - 3\eta^2(1+\alpha)] + 3\eta(\alpha s + k + \nu)A_2 = 0, \tag{50}$$

$$B_2[6B_1^2 + 3B_2^2 + \lambda A_2 B_1 + 2\eta^2(1+\alpha) - \omega - \frac{1}{2}k^2 - \frac{1}{2}\alpha s^2] + \eta G s B_1 - 2\eta(\alpha s + k + \nu)A_2 = 0, \tag{51}$$

and

$$-3B_1^2 - B_2^2 + \omega + \frac{1}{2}k^2 + \frac{1}{2}\alpha s^2 = 0. \tag{52}$$

Then, combining eqs.(49) and (52) leads to the following constraint

$$\omega = \frac{1}{2}(\frac{G}{\lambda} - \alpha)s^2 - \frac{\gamma}{\lambda} - \frac{1}{2}k^2. \tag{53}$$

Firstly, combining eqs.(47) and (50), it comes

$$B_1 = \frac{m_1 B_2 + m_2 A_2}{m_3 B_2 A_2}, \tag{54}$$

where $m_1 = \eta^2[G + \lambda(1+\alpha)], m_2 = \eta[\lambda(\alpha s + k + \nu) - Gs], m_3 = 1 - \lambda^2$ with $\lambda \neq \pm 1$ and $m_3 B_2 A_2 \neq 0$. Secondly, taking into account eq.(48) in (51) provides another expression of coefficient B_1 in the form

$$B_1 = \frac{n_1 B_2 + n_2 A_2}{n_3 B_2 A_2 + n_4}, \tag{55}$$

where, considering eq.(53): $n_1 = 2\eta^2[G-(1+\alpha)\lambda]$ et $n_2 = 2\eta\lambda(\alpha s+k+\nu) - 2\eta G s, n_3 = 1+\lambda^2, n_4 = \eta\lambda G s + \eta(\alpha s + k + \nu)$ with $n_3 B_2 A_2 + n_4 \neq 0$. Since the coefficient B_1 is unique, the equality $B_1 = B_1$ highlights the quadratic equation with two unknowns B_2 and A_2 below

$$(n_1 m_3 - n_3 m_1)A_2 B_2^2 + (n_2 m_3 - n_3 m_2)B_2 A_2^2 - n_4 m_1 B_2 - n_4 m_2 A_2 = 0. \tag{56}$$

Since each of the two coefficients must be real, it is necessary to fix one or the other coefficient in R^* in order to reduce the eq.(56) to a quadratic equation with one unknown. Note also that this equation constitutes the fulcrum equation from which all the analyzes will be articulated with regard to these type I solutions for $\lambda \neq 0$. This Solution Type I gives rise to two large families of analytical solutions of eq.(1).

4.2.1. First large family of analytical solutions: Case: $A_1 = 0, B_1 \neq 0, A_2 = \beta, \beta \in R^*, B_2 \neq 0, \lambda \neq 0$

Under these conditions, eq.(56) becomes

$$\theta B_2^2 + \mu B_2 + \delta = 0, \tag{57}$$

where $\theta = (n_1 m_3 - n_3 m_1)\beta, \mu = (n_2 m_3 - n_3 m_2)\beta^2 - n_4 m_1$ and $\delta = -n_4 m_2 \beta$. Eq.(57) is a quadratic equation with one unknown B_2 which gives rise to four families, three of which come from special cases.

4.2.1.1. Family I of solutions: Particular case I: $\theta \neq 0, \mu \neq 0, \delta = 0$

For $\delta = 0 \iff n_4 = 0$ or $m_2 = 0$, we obtain two subfamilies of solutions.

4.2.1.1.1. Subfamily I of family I of solutions: case: $A_2 = \beta, \beta \in R^*, n_4 = 0, m_2 \neq 0$

When $n_4 = 0 \iff \nu = -\lambda G s - \alpha s - k$, only the quantities n_2 and m_2 of the amplitudes B_1 and B_2 are modified and become respectively: $n_2' = -2\eta G s(1+\lambda^2)$ and $m_2' = -3\eta G s(1+\lambda^2)$. Thus, equations (55) and (57) yield respectively

$$B_1 = \frac{n_2'}{n_3}\frac{1}{B_2} + \frac{n_1}{n_3\beta}, \tag{58}$$

and

$$B_2 = \frac{n_3 m_2' - n_2' m_3}{n_1 m_3 - n_3 m_1}\beta. \tag{59}$$

This being so, we obtain **subfamily I** of ***Family I*** of **Solutions type I** in the form

$$\begin{aligned}\Phi(x,y,t) = \{&\beta J_{3,1}[\eta(x+y-\nu t)] + i[\frac{n_2'}{n_3}\frac{1}{B_2} + \frac{n_1}{n_3\beta}J_{1,1}[\eta(x+y-\nu t)] \\ &+ \frac{n_3 m_2' - n_2' m_3}{n_1 m_3 - n_3 m_1}\beta J_{2,2}[\eta(x+y-\nu t)]\}e^{i(-kx-sy+\omega t)},\end{aligned} \tag{60}$$

where B_2 is given by eq.(59) with $n_3 \neq 0$ and $n_1 m_3 \neq n_3 m_1$. Eq.(60) indicates that, for given values of $n_3, m_2', n_2', m_3, n_1$ and m_1, the amplitudes B_2 is a linear fonction of $A_2 = \beta, \beta \in R^*$.

4.2.1.1.2. Subfamily II of family I of solutions: case: $A_2 = \beta, \beta \in R^*, n_4 \neq 0, m_2 = 0$

When $m_2 = 0 \Longleftrightarrow \nu = \frac{Gs}{\lambda} - \alpha s - k$, only the coefficients n_2 and n_4, considering ν, are modified and correspond to, respectively : $n_2 = 0$ and $n_4' = \frac{Gs}{\lambda}(1+\lambda^2)$. Thus, eqs.(55) and (57) successively give

$$B_1 = \frac{n_1 B_2}{n_3 \beta B_2 + n_4'}, \tag{61}$$

and

$$B_2 = \frac{n_4' m_1}{(n_1 m_3 - n_3 m_1)} \frac{1}{\beta}, \tag{62}$$

where $(n_1 m_3 - n_3 m_1)\beta \neq 0, n_3 \beta B_2 + n_4' \neq 0$ with B_2 given by eq.(62). So, we obtain the **subfamily II** of ***Family I*** of **Solutions type I** as

$$\Phi(x,y,t) = \{\beta J_{3,1}[\eta(x+y-\nu t)] + i[\frac{n_1 B_2}{n_3 \beta B_2 + n_4'} J_{1,1}[\eta(x+y-\nu t)] + \frac{n_4' m_1}{(n_1 m_3 - n_3 m_1)} \frac{1}{\beta}] J_{2,2}[\eta(x+y-\nu t)]\} e^{i(-kx-sy+\omega t)}, \tag{63}$$

where $(n_1 m_3 - n_3 m_1)\beta \neq 0, n_3 \beta B_2 + n_4' \neq 0$ with B_2 given by eq.(62). Eq.(62) indicates that, for this subfamily of solutions, the coefficients B_2 and $A_2 = \beta, \beta \in R^*$, have antagonistic actions: when B_2 is high A_2 is low vice versa.

4.2.1.2. Family II of solutions: Particular case II: $\theta \neq 0, \mu = 0, \delta \neq 0$

For $\mu = 0$, we obtain the expression of coefficient $A_2 = \beta, \beta \in R^*$ under the form

$$\beta = \pm \sqrt{\frac{n_4 m_1}{n_2 m_3 - n_3 m_2}}, \tag{64}$$

where $(n_2 m_3 - n_3 m_2) n_4 m_1 \succ 0$. So, B_1 is given by eq.(55), and eq.(57) gives

$$B_2 = \pm \sqrt{\frac{n_4 m_2}{(n_1 m_3 - n_3 m_1)}}, \tag{65}$$

where $n_4 m_2 (n_1 m_3 - n_3 m_1) \succ 0$ with B_2. Then, we obtain the ***Family II*** of **Solutions Type I** below

$$\begin{aligned}\Phi(x,y,t) = \{ & \pm \sqrt{\frac{n_4 m_1}{n_2 m_3 - n_3 m_2}} J_{3,1}[\eta(x+y-\nu t)] + i[\frac{n_1 B_2 + n_2 \beta}{n_3 \beta B_2 + n_4} J_{1,1}[\eta(x+y-\nu t)] \\ & \pm \sqrt{\frac{n_4 m_2}{n_1 m_3 - n_3 m_1}} J_{2,2}[\eta(x+y-\nu t)]]\} e^{i(-kx-sy+\omega t)},\end{aligned} \tag{66}$$

where B_2 is given by eq.(65) with constraint of eq.(53), $n_4 m_2 (n_1 m_3 - n_3 m_1) \succ 0$ and $n_4 m_1 (n_2 m_3 - n_3 m_2) \succ 0$. Notice herein that, coefficients B_2 and $A_2 = \beta, \beta \in R^*$ are independent of each other unlike the two previous cases.

4.2.1.3. Family III of solutions: Particular case III: $\theta = 0, \mu \neq 0, \delta \neq 0$

When $\theta = 0 \Longleftrightarrow n_1 m_3 = n_3 m_1$, we obtain from this equivalency that

$$G = \frac{\lambda^3 - 3\lambda}{3\lambda^2 - 1}(1+\alpha), \tag{67}$$

with $\lambda \neq \pm \frac{\sqrt{3}}{3}$. Thus, B_1 is given by eq.(55) with $A_2 = \beta, \beta \in R^*$, and then, eq.(57) delivers

$$B_2 = \frac{n_4 m_2 \beta}{(n_2 m_3 - n_3 m_2)\beta^2 - n_4 m_1}. \tag{68}$$

We obtain the ***Family III*** of **Solutions Type I** as below

$$\begin{aligned}\Phi(x,y,t) = \{ & \beta J_{3,1}[\eta(x+y-\nu t)] + i[\frac{n_1 B_2 + n_2 \beta}{n_3 \beta B_2 + n_4} J_{1,1}[\eta(x+y-\nu t)] \\ & + \frac{n_4 m_2 \beta}{(n_2 m_3 - n_3 m_2)\beta^2 - n_4 m_1} J_{2,2}[\eta(x+y-\nu t)]]\} e^{i(-kx-sy+\omega t)},\end{aligned} \tag{69}$$

where B_2 is given by eq.(68) with $n_3 \beta B_2 + n_4 \neq 0, (n_2 m_3 - n_3 m_2)\beta^2 - n_4 m_1 \neq 0$. In the case of this ***family*** III of the **Solutions Type I** , the coefficient B_2 is a rational fraction of the function $n_4 m_2 \beta$ by the function $(n_2 m_3 - n_3 m_2)\beta^2 - n_4 m_1$ of variable $A_2 = \beta, \beta \in R^*$.

4.2.1.4. Family IV of solutions: general case : $\theta \neq 0, \mu \neq 0, \delta \neq 0$

Under these conditions, eq.(57) has for discriminant $\triangle = \mu^2 - 4\theta\delta = (n_2m_3 - n_3m_2)^2\beta^4 + [4n_4m_2(n_1m_3 - n_3m_1) - 2n_4m_1(n_2m_3 - n_3m_2)]\beta^2 + n_4^2m_1^2$. Thus, for $\triangle \geq 0$, eq.(57) admits two distinct solutions below

$$B_2 = \frac{(n_3m_2 - n_2m_3)\beta^2 + n_4m_1 \pm \sqrt{\triangle}}{2(n_1m_3 - n_3m_1)\beta}. \tag{70}$$

So, we obtain the expression of the ***Family IV*** of **Solutions Type I** as follows

$$\begin{aligned}\Phi(x,y,t) = \{&\beta J_{3,1}[\eta(x+y-\nu t)] + i[\frac{n_1B_2 + n_2\beta}{n_3\beta B_2 + n_4}J_{1,1}[\eta(x+y-\nu t)] \\ &+ \frac{(n_3m_2 - n_2m_3)\beta^2 + n_4m_1 \pm \sqrt{\triangle}}{2(n_1m_3 - n_3m_1)\beta}J_{2,2}[\eta(x+y-\nu t)]]\}e^{i(-kx-sy+\omega t)},\end{aligned} \tag{71}$$

with $n_1m_3 \neq n_3m_1$ and $n_3\beta B_2 + n_4 \neq 0$. $\triangle$ is a bisquare polynomial in β which must frame the choice of acceptable values of $A_2 = \beta, \beta \in R^*$ during the propagation tests in approved laboratories.

4.2.2. Second large family of analytical solutions: Case:$A_1 = 0, A_2 \neq 0, B_1 \neq 0, B_2 = \chi, \chi \in R^*, \lambda \neq 0$

For $B_2 = \chi, \chi \in R^*$, eq.(56) produces a quadratic equation with an unknown coefficient A_2 below

$$\theta' A_2^2 + \mu' A_2 + \delta' = 0. \tag{72}$$

where $\theta' = (n_2m_3 - n_3m_2)\chi$, $\mu' = (n_1m_3 - n_3m_1)\chi^2 - n_4m_2$ and $\delta' = -n_4m_1\chi$. Eq.(72) gives rise to four families, three of which come from special cases and which taking into account eq.(53).

4.2.2.1. Family I of the Second large family of analytical solutions: Particular case 1: $\delta' = 0, A_1 = 0, A_2 \neq 0, B_1 \neq 0, B_2 = \chi, \chi \in R^*, \lambda \neq 0$

For $\delta' = 0$, we obtain $n_4 = 0$ or $m_1 = 0 \Longleftrightarrow \nu = -\lambda Gs - \alpha s - k$ or $G = -\lambda(1+\alpha)$. Thus, we discover for this first family of solutions, three subfamilies of solutions: cases $n_4 = 0 \Longleftrightarrow \nu = -\lambda Gs - \alpha s - k, m_1 \neq 0$; $n_4 \neq 0, m_1 = 0 \Longleftrightarrow G = -\lambda(1+\alpha)$ and $n_4 = 0, m_1 = 0 \Longleftrightarrow \nu = -\lambda Gs - \alpha s - k$ and $G = -\lambda(1+\alpha) \Rightarrow \nu = [\lambda^2(1+\alpha) - \alpha]s - k$.

4.2.2.1.1. First subfamily of Family I of the Second large family of analytical solutions: sub-case 1: $\delta' = 0 \Longleftrightarrow n_4 = 0 \Rightarrow \nu = -\lambda Gs - \alpha s - k, m_1 \neq 0$

Under these conditions, only coefficients n_2 and m_2 are impacted and become, respectively: $n_2'' = -2\eta Gs(\lambda^2 + 1)$ and $m_2'' = -3\eta Gs(\lambda^2 + 1)$. Thenceforward, we obtain from eq.(72)

$$A_2 = \frac{(n_3m_1 - n_1m_3)\chi}{n_2''m_3 - n_3m_2''}. \tag{73}$$

Given expressions of n_2'' and m_2'' above, eqs.(7), (9), (55) and (73) lead to the first sought subfamily as being

$$\begin{aligned}\Phi(x,y,t) = \{&\frac{(n_3m_1 - n_1m_3)\chi}{n_2''m_3 - n_3m_2''}J_{3,1}[\eta(x+y-\nu t)] + i[(\frac{n_1}{n_3}\frac{1}{A_2} + \frac{n_2''}{n_3\chi})J_{1,1}[\eta(x+y-\nu t)] \\ &+ \chi J_{2,2}[\eta(x+y-\nu t)]]\}e^{i(-kx-sy+\omega t)},\end{aligned} \tag{74}$$

where ω and A_2 are given by eqs.(53) and (73), respectively, with $n_2''m_3 \neq n_3m_2''$ and $n_3 \neq 0$. Eq.(74) shows that, for given values of $n_3, m_1, n_1, m_3, n_2''$ and m_2'', the amplitude A_2 is a linear fonction of $B_2 = \chi, \chi \in R^*$.

4.2.2.1.2. Second subfamily of Family I of the Second large family of analytical solutions: sub-case 2: $\delta' = 0 \Longleftrightarrow n_4 \neq 0, m_1 = 0 \Rightarrow G = -\lambda(1+\alpha)$

For $\delta' = m_1 = 0 \Rightarrow G = -\lambda(1+\alpha)$, eq.(53) and the expressions of n_1, n_2, n_4, m_2 undergo modifications and become, respectively: $\omega' = -\frac{1}{2}(1+2\alpha)s^2 - \frac{\gamma}{\lambda} - \frac{1}{2}k^2, n_1' = -4\lambda\eta^2(1+\alpha), n_2''' = 2\eta\lambda[s(1+\alpha)+k+\nu], n_4'' = \eta s[\alpha - \lambda^2(1+\alpha)] + \eta(k+\nu)$ and $m_2''' = 3\eta\lambda[(1+\alpha)s + k + \nu]$. In this context, eq.(72) leads to

$$A_2 = \frac{n_1'm_3\chi^2 - n_4''m_2'''}{n_3m_2''' - n_2'''m_3}. \tag{75}$$

The previous new expressions of $\omega', n_1', n_2''', n_4''$ and m_2''', associated with eqs.(7), (9), (55) and (75), produce this second subfamily of solutions in the form

$$\begin{aligned}\Phi(x,y,t) = \{&\frac{n_1'm_3\chi^2 - n_4''m_2'''}{(n_3m_2''' - n_2'''m_3)\chi}J_{3,1}[\eta(x+y-\nu t)] + i[\frac{n_1'\chi + n_2'''A_2}{n_3\chi A_2 + n_4''}J_{1,1}[\eta(x+y-\nu t)] \\ &+ \chi J_{2,2}[\eta(x+y-\nu t)]]\}e^{i(-kx-sy+\omega' t)},\end{aligned} \tag{76}$$

where A_2 is given by eq.(75) with $n_3m_2''' \neq n_2'''m_3$ and $n_3\chi A_2 + n_4'' \neq 0$. For this subfamily of solutions, the coefficient A_2 is a parabolic function of the coefficient $B_2 = \chi, \chi \in R^*$.

4.2.2.1.3. Third subfamily of Family I of the Second large family of analytical solutions: sub-case 3: $\delta' = 0 \Rightarrow n_4 = 0 \Longleftrightarrow \nu = -\lambda Gs - \alpha s - k, m_1 = 0 \Longleftrightarrow G = -\lambda(1+\alpha)$

For $m_1 = n_4 = 0$, we obtain successively: $G = -\lambda(1+\alpha)$ and $\nu' = [\lambda^2(1+\alpha) - \alpha]s - k$. Eq.(53) leads to ω' previously obtained, while n_1, n_2 and m_2 undergo a modification and become, respectively: $n_1' = -4\lambda\eta^2(1+\alpha), n_2'''' = 2\lambda\eta s[\lambda^2(1+\alpha)+1]$ and $m_2'''' = \lambda\eta s(1+\alpha)(\lambda^2+1)$. Thus, eq.(72) provides

$$A_2 = \frac{n_1' m_3 \chi}{n_3 m_2'''' - n_2'''' m_3}. \tag{77}$$

Therewith, we obtain this third subfamily, considering the expressions of $\nu', n_1', n_2'''', m_2''''$ as well as eqs.(7), (9), (55) and (77), in the form

$$\begin{aligned}\Phi(x,y,t) = \{&\frac{n_1' m_3 \chi}{n_3 m_2'''' - n_2'''' m_3} J_{3,1}[\eta(x+y-\nu' t)] + i[(\frac{n_1'}{n_3}\frac{1}{A_2} + \frac{n_2''''}{n_3\chi})J_{1,1}[\eta(x+y-\nu' t)] \\ &+ \chi J_{2,2}[\eta(x+y-\nu' t)]]\}e^{i(-kx-sy+\omega' t)},\end{aligned} \tag{78}$$

where A_2 is given by eq.(77) with $n_3 m_2'''' \neq n_2'''' m_3$ and $n_3 \neq 0$. Herein, coefficients A_2 is a linear function of coefficient $B_2 = \chi, \chi \in R^*$ as in eq.(74).

4.2.2.2. Family II of the Second large family of analytical solutions: Particular case 2: $\mu' = 0, \delta' \neq 0, \theta' \neq 0, A_1 = 0, A_2 \neq 0, B_1 \neq 0, B_2 = \chi, \chi \in R^*, \lambda \neq 0$

For $\mu' = 0$, we obtain

$$\chi = \pm\sqrt{\frac{n_4 m_2}{n_1 m_3 - n_3 m_1}}. \tag{79}$$

Then, eq.(72) delivers

$$A_2 = \pm\sqrt{\frac{n_4 m_1}{n_2 m_3 - n_3 m_2}}. \tag{80}$$

Subsequently, taking into account eq.(79) in (55), as well as eqs.(9), (79) and (80) in (7) produce this second family of solutions under the form

$$\begin{aligned}\Phi(x,y,t) = \{&\pm\sqrt{\frac{n_4 m_1}{n_2 m_3 - n_3 m_2}} J_{3,1}[\eta(x+y-\nu t)] + i[\frac{n_1\chi + n_2 A_2}{n_3\chi A_2 + n_4} J_{1,1}[\eta(x+y-\nu t)] \\ &\pm\sqrt{\frac{n_4 m_2}{n_1 m_3 - n_3 m_1}} J_{2,2}[\eta(x+y-\nu t)]]\}e^{i(-kx-sy+\omega t)},\end{aligned} \tag{81}$$

where χ and A_2 are given by eqs.(79) and (80) with constraints $n_4 m_1(n_2 m_3 - n_3 m_2) \succ 0, n_4 m_2(n_1 m_3 - n_3 m_1) \succ 0$ and eq.(53). Notice also herein that, coefficients A_2 and $B_2 = \chi, \chi \in R^*$ are independent of each other.

4.2.2.3. Family III of the Second large family of analytical solutions: Particular case 3: $\theta' = 0, \mu' \neq 0, \delta' \neq 0, A_1 = 0, A_2 \neq 0, B_1 \neq 0, B_2 = \chi, \chi \in R^*, \lambda \neq 0, s \neq 0$

For $\theta' = 0$, we obtain $\lambda = \pm\frac{\sqrt{3}}{3}$ or $G = (\alpha s + k + \nu)\frac{\lambda}{s}$ with $s \neq 0$. This gives rise to three subfamilies of solutions

4.2.2.3.1. Subfamily 1 of Family III of the Second large family of analytical solutions: Sub-case 1: $\lambda = \pm\frac{\sqrt{3}}{3}; G \neq (\alpha s + k + \nu)\frac{\lambda}{s}; B_2 = \chi, \chi \in R^*$

For $\lambda = \pm\frac{\sqrt{3}}{3}, G \neq (\alpha s + k + \nu)\frac{\lambda}{s}$, Eq.(72) reveals that

$$A_2 = \frac{n_4 m_1 \chi}{(n_1 m_3 - n_3 m_1)\chi^2 - n_4 m_2}, \tag{82}$$

where $\chi \neq \pm\sqrt{\frac{n_4 m_2}{n_1 m_3 - n_3 m_1}}$. Consideration of $B_2 = \chi, \chi \in R^*$ in eq.(55) and insertion of eqs.(9), (55) and (82) in (7) provide this first subfamily as being

$$\begin{aligned}\Phi(x,y,t) = \{&\frac{n_4 m_1 \chi}{(n_1 m_3 - n_3 m_1)\chi^2 - n_4 m_2} J_{3,1}[\eta(x+y-\nu t)] + i[\frac{n_1\chi + n_2 A_2}{n_3\chi A_2 + n_4} J_{1,1}[\eta(x+y-\nu t)] \\ &+ \chi J_{2,2}[\eta(x+y-\nu t)]]\}e^{i(-kx-sy+\omega t)},\end{aligned} \tag{83}$$

with $\chi \neq \pm\sqrt{\frac{n_4 m_2}{n_1 m_3 - n_3 m_1}}, (n_1 m_3 - n_3 m_1) n_4 m_2 \succ 0, \chi \neq -\frac{n_4}{n_3 A_2}$ and where ω and A_2 are given by eqs.(53) and (82), respectively. In the case of this **Subfamily 1** of ***Family*** III of the **Second large family**, the coefficient A_2 is a rational fraction of the function $n_4 m_1 \chi$ by the function $(n_1 m_3 - n_3 m_1)\chi^2 - n_4 m_2$ of variable $B_2 = \chi, \chi \in R^*$.

4.2.2.3.2. Subfamily 2 of Family III of the Second large family of analytical solutions: Sub-case 2: $\lambda \neq 0; \lambda \neq \pm\frac{\sqrt{3}}{3}; G = (\alpha s + k + \nu)\frac{\lambda}{s}; B_2 = \chi, \chi \in R^*$

For $\lambda \neq \pm\frac{\sqrt{3}}{3}; G = (\alpha s + k + \nu)\frac{\lambda}{s}$, we obtain $n_2 = m_2 = 0$ and coefficients n_1, n_4 and m_1 undergo some adjustments and become, respectively: $n_{01} = \frac{2\eta^2\lambda}{s}(k + \nu - s), n_{04} = \eta(\alpha s + k + \nu)n_3$ and $m_{01} = \frac{\eta^2\lambda}{s}[(1 + 2\alpha)s + k + \nu]$. However, n_3 and m_3 remain unchanged, while eq.(72) gives the expression of the coefficient A_2 below

$$A_2 = \frac{n_{04}m_{01}}{(n_{01}m_3 - n_3m_{01})\chi}, \tag{84}$$

with $n_{01}m_3 \neq n_3m_{01}$. In this context, taking into account $B_2 = \chi, \chi \in R^*$ and $n_2 = 0$ in eq.(55) as well as insertion of eqs.(9), (55) and (84) in (7) give this second subfamily of solutions as being

$$\begin{aligned}\Phi(x, y, t) = \{&\frac{n_{04}m_{01}}{(n_{01}m_3 - n_3m_{01})\chi}J_{3,1}[\eta(x + y - \nu t)] + i[\frac{n_{01}\chi}{n_3\chi A_2 + n_{04}}J_{1,1}[\eta(x + y - \nu t)] \\ &+ \chi J_{2,2}[\eta(x + y - \nu t)]]\}e^{i(-kx-sy+\omega t)},\end{aligned} \tag{85}$$

with $n_{01}m_3 \neq n_3m_{01}, \lambda \neq \pm\frac{\sqrt{3}}{3}, \chi \neq -\frac{n_{04}}{n_3A_2}, G = (\alpha s + k + \nu)\frac{\lambda}{s}$ and where ω and A_2 are given by eqs.(53) and (84), respectively. Eq.(85) indicates that, for this subfamily of solutions, the coefficients A_2 and $B_2 = \chi, \chi \in R^*$, have antagonistic actions: when A_2 is high B_2 is low vice versa.

4.2.2.3.3. Subfamily 3 of Family III of the Second large family of analytical solutions: Sub-case 3: $\lambda = \pm\frac{\sqrt{3}}{3}; G = (\alpha s + k + \nu)\frac{\lambda}{s}; B_2 = \chi, \chi \in R^*$

Under these conditions, we arrive at $n_2 = m_2 = 0$, while coefficients $n_{01}, n_3, n_{04}, m_{01}$ and m_3 become, respectively: $n'_{01} = \pm\frac{2\eta^2\sqrt{3}}{3s}(k+\nu-s), n_{03} = \frac{4}{3}, n'_{04} = \frac{4}{3}\eta(\alpha s+k+\nu), m'_{01} = \pm\frac{\eta^2\sqrt{3}}{3s}[(1+2\alpha)s+k+\nu]$ and $m_{03} = \frac{2}{3}$. Thus, we obtain the coefficient A_2 in the form

$$A_2 = \frac{n'_{04}m'_{01}}{(n'_{01}m_{03} - n_{03}m'_{01})\chi}, \tag{86}$$

with $n'_{01}m_{03} \neq n_{03}m'_{01}, \lambda = \pm\frac{\sqrt{3}}{3}, \chi \neq -\frac{n'_{04}}{n_{03}A_2}, G = (\alpha s + k + \nu)\frac{\lambda}{s}$. In this context, the consideration of $B_2 = \chi, \chi \in R^*$ and $n_2 = 0$ in eq.(55) as well as insertion of eqs .(9), (55) and (84) in (7) lead to this third subfamily of solutions as follows

$$\begin{aligned}\Phi(x, y, t) = \{&\frac{n'_{04}m'_{01}}{(n'_{01}m_{03} - n_{03}m'_{01})\chi}J_{3,1}[\eta(x + y - \nu t)] + i[\frac{n'_{01}\chi}{n_{03}\chi A_2 + n'_{04}}J_{1,1}[\eta(x + y - \nu t)] \\ &+ \chi J_{2,2}[\eta(x + y - \nu t)]]\}e^{i(-kx-sy+\omega t)},\end{aligned} \tag{87}$$

with $n'_{01}m_{03} \neq n_{03}m'_{01}, \lambda = \pm\frac{\sqrt{3}}{3}, \chi \neq -\frac{n'_{04}}{n_{03}A_2}, G = (\alpha s + k + \nu)\frac{\lambda}{s}$ and where ω and A_2 are given by eqs.(53) and (86), respectively. Eq.(87) also indicates that, for this subfamily of solutions, the coefficients A_2 and $B_2 = \chi, \chi \in R^*$, have antagonistic actions: when A_2 is high B_2 is low vice versa.

4.2.2.4. Family IV of the Second large family of analytical solutions: General case: $\theta' \neq 0, \mu' \neq 0, \delta' \neq 0, A_1 = 0, A_2 \neq 0, B_1 \neq 0, B_2 = \chi, \chi \in R^*, \lambda \neq 0$

Considering these conditions, eq.(72) yields as a discriminant a bisquare polynomial in χ: $\triangle' = (n_1m_3 - n_3m_1)^2\chi^4 + [4n_4m_1(n_2m_3 - n_3m_2) - 2n_4m_2(n_1m_3 - n_3m_1)]\chi^2 + n_4^2m_2^2$. For $\triangle' \geq 0$, eq.(72) produces two distinct expressions for the coefficient A_2 as follows

$$A_2 = \frac{-\mu' \pm \sqrt{\triangle'}}{2\theta'}. \tag{88}$$

Thus, the inclusion of $B_2 = \chi, \chi \in R^*$ in eq.(55) as well as insertion of eqs.(9), (55) and (88) in (7) lead to this fourth family of solutions such that

$$\Phi(x, y, t) = \{\frac{-\mu' \pm \sqrt{\triangle'}}{2\theta'}J_{3,1}[\eta(x + y - \nu t)] + i[\frac{n_1\chi + n_2A_2}{n_3\chi A_2 + n_4}J_{1,1}[\eta(x + y - \nu t)] + \chi J_{2,2}[\eta(x + y - \nu t)]]\}e^{i(-kx-sy+\omega t)}, \tag{89}$$

with $\chi \neq -\frac{n_4}{n_3A_2}$ and where ω and A_2 are given by eqs.(53) and (88), respectively. $\triangle'$ is a bisquare polynomial in χ which must frame the choice of acceptable values of $B_2 = \chi, \chi \in R^*$ during the propagation tests in approved laboratories.

However, it should globally be noted in this case of **Solutions type I** that all the families of solutions obtained are from the outset hybrid and new, since they each form a mixed wave packet [27-30]. This is justified by the fact that the real part is a hybrid solitaire wave of the types Dark-Bright, Bright-Dark or Kink-Dark-Bright solitaire wave; and, the imaginary part as for it, is made up of solitons Kink and Dark, respectively. Thus, the results of the interactions between these different components are wave structures which depend above all on the nature of the relations which exist between the coefficients A_2 and B_2: this is how we can see emerge in their propagation media solitary wave molecules of the Antikink-Bright-Dark, Kink-Bright-Dark, Double-Bright Dark or Kink-Dark-Bright solitary wave types (see figures 2 and 3), and so on. Let us now address those of the families of **Solutions type II**.

B: Solutions type II: case $A_2 = 0, \lambda \neq 0$

For $A_2 = 0$, eqs.(11), (12), (20) and (21) are verified while eqs.(14) and (23) supplied $A_1 = 0$ or $B_2 = 0$ or $B_1 = 0$. Therefore, with the intention of getting nontrivial solutions, we only hold from this type II of analytical solutions, four families of solutions:$A_2 = B_1 = 0, A_1 \neq 0, B_2 \neq 0$; $A_2 = A_1 = 0, B_1 \neq 0, B_2 \neq 0$; $A_2 = B_2 = 0, A_1 \neq 0, B_1 \neq 0$ and $A_2 = A_1 = B_2 = 0, B_1 \neq 0$.

4.2.3. Family I of the Solutions type II: Case: $A_2 = B_1 = 0, A_1 \neq 0, B_2 \neq 0, s \neq 0, \lambda \neq 0$

In this case, eqs.(11), (12), (14), (20), (21) and (23) are verified. Thenceforth, eqs.(13) and (22) lead to $B_2 = \pm A_1$, while eqs.(16) and (18) impose to take $s = 0$(the case G =0 being not interesting). As a consequence, equations (25) and (27) provide the same constraint $\nu = -k$. Continuing our analysis:

- eqs(15), (17) and (19) successively reduce to

$$-\lambda B_2^2 - 3\eta^2 G = 0, \tag{90}$$

$$2\lambda B_2^2 + 2\eta^2 G + \gamma = 0, \tag{91}$$

and

$$-\gamma - \lambda B_2^2 = 0, \tag{92}$$

- eqs.(24), (26) and (28) also reduce and become respectively

$$-B_2^2 - 3\eta^2(1+\alpha) = 0, \tag{93}$$

$$2B_2^2 + 2\eta^2(1+\alpha) - \omega - \frac{1}{2}k^2 = 0, \tag{94}$$

and

$$-B_2^2 + \omega + \frac{1}{2}k^2 = 0. \tag{95}$$

On the one hand, combining eqs.(90), (91) and (92), and, on the other hand, combining equations (93), (94) and (95) also, and given the fact that B_2 is unique, we get some relations between parameters: $\eta = \sqrt{\frac{\gamma}{3G}}, G = \lambda(1+\alpha), \omega = -\frac{\gamma}{\lambda} - \frac{1}{2}\nu^2$ as well as the expression of the coefficient B_2 as being

$$B_2 = \pm A_1 = \pm\eta\sqrt{-\frac{5}{2}(1+\alpha)}, \tag{96}$$

where $\eta = \sqrt{\frac{\gamma}{3G}}$ with constraints $\gamma G \succ 0, \alpha \prec -1, \lambda\gamma \prec 0$ and $sign(G) = sign(\gamma)$. We thus obtain, the ***Family*** I of **solutions type II** in the form

$$\Phi(x,y,t) = \{\pm\eta\sqrt{-\frac{5}{2}(1+\alpha)}J_{2,1}[\eta(x+y-\nu t)] \pm \eta i\sqrt{-\frac{5}{2}(1+\alpha)}J_{2,2}[\eta(x+y-\nu t)]\}e^{i[-kx-sy+(-\frac{\gamma}{\lambda}-\frac{1}{2}k^2)t]}, \tag{97}$$

where $\eta = \sqrt{\frac{\gamma}{3G}}, G = \lambda(1+\alpha)$ with constraints $\gamma G \succ 0, \alpha \prec -1, \lambda\gamma \prec 0$ and $sign(G) = sign(\gamma)$. This family of solutions is a complex family which, from the base, is hybrid, since the first term is a hybrid wave of the Dark-Bright or Bright-Dark type, and the second is a Kink wave. This mixture, during the interactions in the propagation media, will produce complex intermediate structures which will highlight each of the characteristics of the basic waves.

4.2.4. Family II of the Solutions type II: Case: $A_1 = A_2 = B_2 = 0, B_1 \neq 0, \lambda \succ 0, s = 0, \gamma \prec 0, G \prec 0$

Under these conditions, eqs.(26) and (17) respectively lead to: $s = 0$ and $\nu = -k$, while, all the other equations of the first range are verified. As a result, equations of the first range provided only the relations between the parameters of the system and those of the solitary wave that we are looking for. Thus, we have to refer to equations of the second range. In this context, eqs.(29) to (34) and (36) as well as eqs.(38) to (43) and (45) are verified. Then, Combining eqs.(35), (37), (44) and (46), and taking into account the fact that B_1 is unique, we successively obtain constraints between the parameters of the system and those of the solitary wave: $G = (1+\alpha)\lambda, \gamma = \eta^2(1+\alpha)\lambda, \omega = -\eta^2(1+\alpha) - \frac{1}{2}\nu^2 = -\frac{\gamma}{\lambda} - \frac{1}{2}\nu^2$ as well as the expression of the coefficient B_1 under the form

$$B_1 = \pm\eta\sqrt{-(1+\alpha)} = \pm\sqrt{-\frac{\gamma}{\lambda}}, \tag{98}$$

with $\alpha \prec -1, \gamma\lambda \prec 0, G\lambda \prec 0, \eta^2(1+\alpha) \prec -\frac{1}{2}\nu^2 \Rightarrow \frac{\gamma}{\lambda} \prec -\frac{1}{2}\nu^2$. Thus, we obtain the ***Family II*** of **Solutions type II** under the pure imaginary exacte solution as being

$$\Phi(x,y,t) = \{\pm\eta i\sqrt{-(1+\alpha)}J_{1,1}[\eta(x+y-\nu t)]\}e^{-i[kx+(\frac{\gamma}{\lambda}+\frac{1}{2}\nu^2)t]}, \tag{99}$$

with $\alpha \prec -1, \gamma\lambda \prec 0, G\lambda \prec 0, \nu = -k, \frac{\gamma}{\lambda} \prec -\frac{1}{2}\nu^2$ and $sign(G) = sign(\gamma)$. Eq.(99) shows that eq.(1) admits the kink wave as a pure imaginary solution. Let us point out herein that this kink wave had been proposed in[33] as a pure real solution and qualified as a homoclinic wave solution which tends to periodic wave solution $\pm\sqrt{-\frac{\gamma}{d}}e^{i(px+\omega t)}$ when we match $d \to \lambda$ and $p \to -k$ when $t \to \pm\infty$.

4.2.5. Family III of the Solutions type II: Case: $A_2 = A_1 = 0, B_1 \neq 0, B_2 \neq 0, s \neq 0, \lambda \neq 0$

In the case $A_1 = A_2 = 0$, eqs.(11), (12), (14), (16), (18), (20), (21), (23) , (25) and (27) are verified. On the other side, eqs.(13) and (22) give $B_2 = 0$, thus, leading to the result obtained in paragraph 4.2.3., which is not interesting in this case. This supposes that we neglect the contribution of each of eqs.(13) and (22) to the order of the indices of $J_{6,0}$ (or to this order of the powers of sech). Continuing our investigations, it comes from the combination of eqs.(15) and (24), the constraint below

$$G = (1 + \alpha)\lambda. \tag{100}$$

Combining, on the one hand eqs.(19) and (28), then, on the other hand, eqs.(17) and (26), and, taking into account the constraint given by eq.(100), we obtain expressions of B_1 and B_2 under the respective forms

$$B_1 = \pm\frac{1}{2}\sqrt{\frac{(2\omega + k^2 - 1)\lambda + 2\gamma}{\lambda}}, \tag{101}$$

and

$$B_2 = \frac{2\eta[\alpha s(1 + \lambda^2) + \lambda^2 s + k + \nu]}{(2\omega + k^2 - s^2)\lambda + 2\gamma} B_1. \tag{102}$$

Thus, we obtain the ***Family III*** of **Solutions type II** under the forced solution form

$$\begin{aligned}\Phi(x, y, t) = \{&\pm\frac{1}{2}i\sqrt{\frac{(2\omega + k^2 - 1)\lambda + 2\gamma}{\lambda}} J_{1,1}[\eta(x + y - \nu t)] \\ &+ i\frac{2\eta[\alpha s(1 + \lambda^2) + \lambda^2 s + k + \nu]}{(2\omega + k^2 - s^2)\lambda + 2\gamma} B_1 J_{2,2}[\eta(x + y - \nu t)]\}e^{i(-kx-sy+\omega t)},\end{aligned} \tag{103}$$

with $(2\omega + k^2 - 1)\lambda^2 + 2\gamma\lambda \succ 0, (2\omega + k^2 - s^2)\lambda + 2\gamma \neq 0$ with the constraints given by eqs.(53) and (100). Eq.(103) is a new prototype of pure imaginary solitary wave solution of eq.(1). It is a combination of a classical Kink wave and a classical dark wave, both belonging to the large Dark soliton family. The interactions between them generate hybrid structures which are intermediate forms which can for example display a Kink-Dark-Bright solitary wave which brings out the three characters, namely: Kink wave, Dark wave and Bright wave solutions (see figure 1 (C) and (D)).

4.2.6. Family IV of the Solutions type II: Case: $A_2 = B_2 = 0, A_1 \neq 0, B_1 \neq 0, \lambda \neq 0$

In this case, eqs.(11) to (15) and (19), as well as the equations (20) to (24) and (28), of the first range are verified. On the other hand, eqs.(16), (17), (18), (25), (26) and (27) of this first range are reduced to, respectively

First range of equations

- From the real part:

the term in $J_{3,0}(\eta\xi)$,

$$2\eta G s A_1 = 0, \tag{104}$$

the term in $J_{2,0}(\eta\xi)$,

$$-\eta(\alpha s + k + \nu)B_1 = 0, \tag{105}$$

the term in $J_{1,0}(\eta\xi)$,

$$-\eta G s A_1 = 0, \tag{106}$$

- From the imaginary part:

the term in $J_{3,0}(\eta\xi)$,

$$2\eta(\alpha s + k + \nu)A_1 = 0, \tag{107}$$

the term in $J_{2,0}(\eta\xi)$,

$$\eta G s B_1 = 0, \tag{108}$$

the term in $J_{1,0}(\eta\xi)$,

$$-\eta(\alpha s + k + \nu)A_1 = 0. \tag{109}$$

In the continuity, with regard to the second range of equations, we note that eqs.(29), (30), (31), (38), (39) and (40) are verified while eqs.(32) to (37) as well as (41) to (46) lead to, respectively

Second range of equations

- From the real part:

the term in $J_{6,1}(\eta\xi)$,

$$A_1^3 = 0, \tag{110}$$

the term in $J_{5,1}(\eta\xi)$,

$$\lambda B_1 A_1^2 = 0, \tag{111}$$

the term in $J_{4,1}(\eta\xi)$,

$$[B_1^2 - A_1^2 + 3\eta^2(1+\alpha)]A_1 = 0, \tag{112}$$

the term in $J_{3,1}(\eta\xi)$,

$$[\lambda(B_1^2 - A_1^2) + \eta^2 G]B_1 = 0, \tag{113}$$

the term in $J_{2,1}(\eta\xi)$,

$$[-B_1^2 + \omega + \frac{1}{2}k^2 + \frac{1}{2}\alpha s^2 - \frac{1}{2}\eta^2(1+\alpha)]A_1 = 0, \tag{114}$$

the term in $J_{1,1}(\eta\xi)$,

$$(-\lambda B_1^2 - \gamma + \frac{1}{2}Gs^2)B_1 = 0, \tag{115}$$

- From the imaginary part:

the term in $J_{6,1}(\eta\xi)$,

$$-\lambda A_1^3 = 0, \tag{116}$$

the term in $J_{5,1}(\eta\xi)$,

$$B_1 A_1^2 = 0, \tag{117}$$

the term in $J_{4,1}(\eta\xi)$,

$$[\lambda(A_1^2 - B_1^2) - 3\eta^2 G]A_1 = 0, \tag{118}$$

the term in $J_{3,1}(\eta\xi)$,

$$[B_1^2 - A_1^2 + \eta^2(1+\alpha)]B_1 = 0, \tag{119}$$

the term in $J_{2,1}(\eta\xi)$,

$$[\lambda B_1^2 + \gamma + \frac{1}{2}G(\eta^2 - s^2)]A_1 = 0, \tag{120}$$

the term in $J_{1,1}(\eta\xi)$,

$$(-B_1^2 + \omega + \frac{1}{2}k^2 + \frac{1}{2}\alpha s^2)B_1 = 0, \tag{121}$$

Under these conditions, and, following the order of priority in the resolution of the range equations, it emerges that eqs.(104), (106) and (108) propose $G = 0$ or $s = 0$ while eqs.(105), (107) and (109) lead to the only constraint: $\nu = -\alpha s - k$. In view of all these observations, it looms that equations of the first range only inform about the relations which must exist between certain parameters of the system and those of the solitary wave which one builds. In such a context, the equations of the second range become priorities in the order of resolutions. Thus, by continuing in our investigations, it clearly appears that the equations (110), (111), (116) and (117) propose $A_1 = B_1 = 0$, leading therefore to a trivial solution, which is not important. This supposes, concerning this family of solutions, that the contribution of each eqs.(110), (111), (116) and (117) remains negligible at these orders of the clues of $J_{n,m}(\eta\xi)$ with $n \in \{5; 6\}, m = 1$. In this context, this family gives rise to two subfamilies of approximate solutions.

4.2.6.1. Subfamily I of the Family IV of the Solutions type II: Case: $A_2 = B_2 = 0, A_1 \neq 0, B_1 \neq 0, \lambda \prec 0, G = 0, s \neq 0, \nu = s - k$

For $A_2 = B_2 = G = 0$ and, taking into account the previous analyses, eqs.(113) and (118) give: $A_1 = \pm B_1$ while eqs.(112) and (119) lead to: $\alpha = -1$. As a consequence, eqs.(105), (107) and (109) yield $\nu = s - k$. On the other hand, the combination of eqs.(115) and (120) as well as eqs.(114) and (121), and, given the fact that each of the coefficients A_1 or B_1 is unique, it comes respectively

$$B_1 = \pm\sqrt{-\frac{\gamma}{\lambda}}, \tag{122}$$

and

$$\omega = \frac{1}{2}\nu(k+s) - \frac{\gamma}{\lambda}, \tag{123}$$

with $\frac{\gamma}{\lambda} \prec \frac{1}{2}\nu(k+s), k \geq s, \gamma\lambda \prec 0$. Thus, we obtain **Subfamily I** of ***family IV*** of **solutions Type II** in the form

$$\Phi(x,y,t) = \{\pm\sqrt{-\frac{\gamma}{\lambda}}J_{2,1}[\eta(x+y-\nu t)] \pm i\sqrt{-\frac{\gamma}{\lambda}}J_{1,1}[\eta(x+y-\nu t)]\}e^{-i\{kx+sy+[-\frac{1}{2}\nu(k+s)+\frac{\gamma}{\lambda}]t\}}, \tag{124}$$

with $\alpha = -1, \gamma\lambda \prec 0, \frac{\gamma}{\lambda} \prec \frac{1}{2}\nu(k+s), \nu = s-k, k \geq s$. Eq.(123) shows that, for fixed values of the parameters s_0, λ_0 and γ_0 of the system whose dynamics is described by eq.(1), the angular frequency ω is a parabolic function of $k \in]0; \sqrt{s_0^2 - \frac{2\gamma_0}{\lambda_0}}[$ of the solitary wave, since $\nu = s_0 - k$. According to this observation, the angular frequency reaches its maximum $\omega_0 = \frac{1}{2}s_0^2 - \frac{\gamma_0}{\lambda_0}, \gamma_0\lambda_0 \prec 0$; this for the value of $k = 0$. Then, the angular frequency ω tends to its maximum value ω_0 for low values of k, and low for large values of k. Thenceforward, it appears that this angular frequency ω decreases for $k \in]0; \sqrt{s_0^2 - \frac{2\gamma_0}{\lambda_0}}[$. From this result, it follows that we can control the deployment of the solitary wave, in brief, the control of the energy of the considered system through eq.(123).

4.2.6.2. Subfamily II of the Family IV of the Solutions type II: Case: $A_2 = B_2 = 0, A_1 \neq 0, B_1 \neq 0, \lambda \neq 0, G = s = 0, \alpha = -1, \nu = -k$

For $A_2 = B_2 = s = G = 0$, eqs.(104), (106) and (108) are verified while eqs.(105), (107) and (109) gives rise to the constraint $\nu = -k$. As before, the equations of the first range again anew give constraints between some parameters of the solitary wave. Thus, eqs.(110), (111), (116) and (117) suggest: $A_1 = B_1 = 0$, thus giving rise to a trivial solution which is not important. This supposes that we neglect the contribution of each of these eqs.(110), (111), (116) and (117) to these orders of clues of $J_{n,m}(\eta\xi)$ with $n \in \{5;6\}, m = 1$. In this context, eqs.(113) and (118) give: $B_1 = \pm A_1$. As a consequence, eqs.(112) and (119) lead to: $\alpha = -1$. On the other hand, eqs.(114) and (121) are identical, as well as eqs.(115) and (120). Given the fact that the coefficients A_1 and B_1 are each unique, we successively obtain

$$\omega = -\frac{1}{2}\nu^2 - \frac{\gamma}{\lambda}, \tag{125}$$

and

$$B_1 = \pm A_1 = \pm\sqrt{-\frac{\gamma}{\lambda}}, \tag{126}$$

with $\nu = -k, \gamma\lambda \prec 0$ and $\alpha = -1$. We obtain this second **Subfamily II** of ***Family IV*** of **solutions Type II** in the form

$$\Phi(x,y,t) = \{\pm\sqrt{-\frac{\gamma}{\lambda}}J_{2,1}[\eta(x+y-\nu t)] \pm i\sqrt{-\frac{\gamma}{\lambda}}J_{1,1}[\eta(x+y-\nu t)]\}e^{i[\nu x-(\frac{1}{2}\nu^2+\frac{\gamma}{\lambda})t]}, \tag{127}$$

with $\nu = -k, \gamma\lambda \prec 0$ and $\alpha = -1$. Eq.(125) indicates that, for fixed values of the parameters λ_0 and γ_0 of the system whose dynamics is described by eq.(1), the angular frequency ω is a parabolic function of the velocity $\nu \in]-\sqrt{-\frac{2\gamma_0}{\lambda_0}}; 0[$ or $k \in]0; \sqrt{-\frac{2\gamma_0}{\lambda_0}}[$ of the solitary wave, since $\nu = -k$. According to this observation, the angular frequency reaches its maximum $\omega_0 = -\frac{\gamma_0}{\lambda_0}, \gamma_0\lambda_0 \prec 0$; this for the value of the speed $\nu = 0$ (which can be considered as a state of pseudo equilibrium). In this context, it appears that this angular frequency ω increases for $\nu \in]-\sqrt{-\frac{2\gamma_0}{\lambda_0}}; 0[$ and decreases for $k \in]0; \sqrt{-\frac{2\gamma_0}{\lambda_0}}[$. Then, the angular frequency ω tends to its maximum value ω_0 for great values of ν, and for low values of k. From this result, it follows that we can control the deployment of the solitary wave, in brief, the control of the energy of the considered system through eq.(125). These reflections can also be carried out in the cases of Families I and II of Solutions Type II

4.3. Plot of profiles of some obtained solutions

We present in this part of work, the profiles of certain solitary wave structures that we tracked down from packages constituted by the eqs. (85), (99) and (103), this thanks to the graphical tool MAPLE. This exercise therefore gave rise to three figures, each containing four profiles called (A), (B), (C) and (D), respectively.

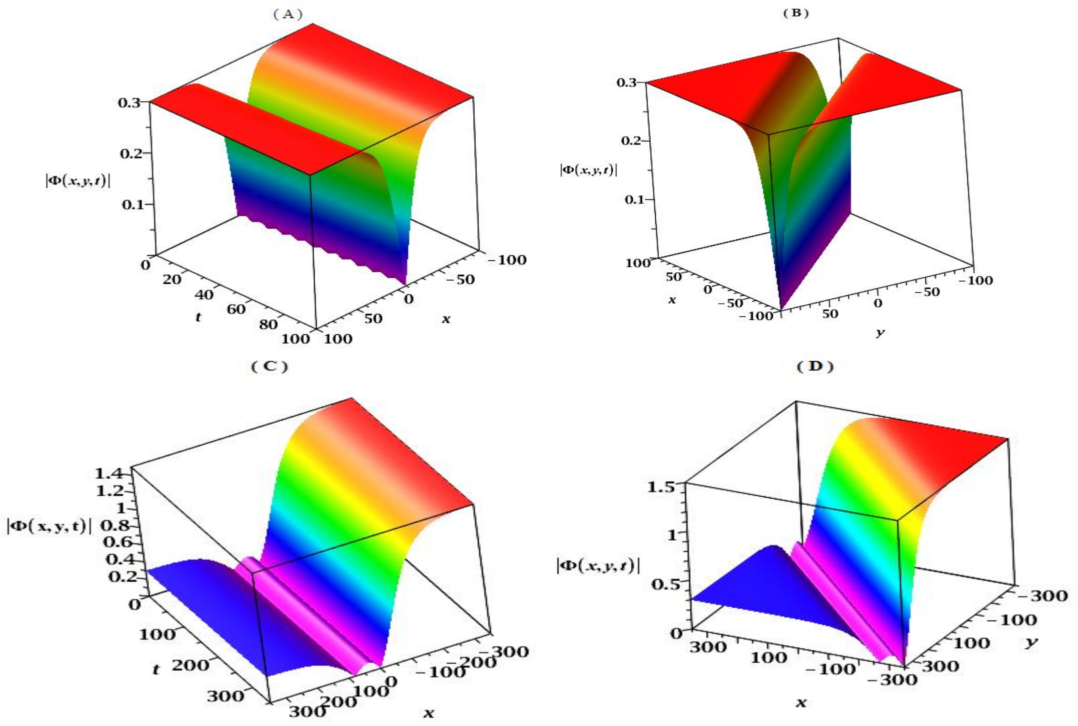

Figure 1: Graphical representations given by eq.(99) and (103). Eq.(99): for $\eta = 0.05; B_1 = 0.3; \gamma = 0.03; k = 0.1; s = 0; \nu = -0.1; \alpha = -37; \lambda = -0.3333333333; G = 12; \omega = 0.08500000001$: Dark solitons with a sawtooth bottom: **(A)**: $y = 0$ **(B)**: $t = 0$. Eq.(103): for $\eta = 0.01; \lambda = 0.02; \alpha = -2.5; k = 0.1; s = 0.2; \gamma = -0.0015; \nu = 0.00026; B_1 = 0.6; B_2 = -1.5B_1 = -0.9; G = 0.30966; \omega = 0.42966$: Kink-Dark-Bright solitary waves **C)** $y = 0$; **D)** $t = 0$.

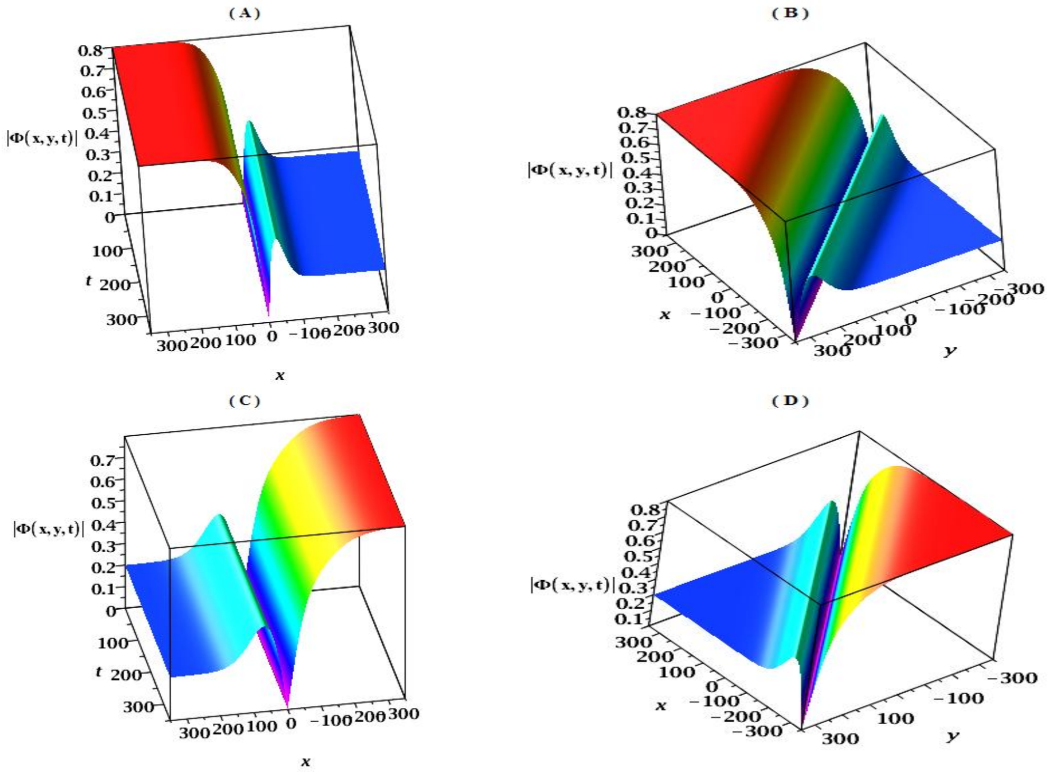

Figure 2: Graphical representation given by eq.(85). For $\eta = 0.01; \gamma = 1; \nu = 0.0013; \alpha = -0.67; k = 0.1; s = 0.2; B_2 = \chi = 0.3; A_2 = 3B_2 = 3; B_1 = \frac{5}{3}B_2; \lambda = -1.720977806; G = 0.2813798713; \omega = 0.5861950181$: Antikink-Bright-Dark solitary waves: **A)** $y = 0$; **B)** $t = 0$. For $\eta = 0.01; \gamma = 1; \nu = 0.0013; \alpha = -0.67; k = 0.1; s = 0.2; B_2 = \chi = 0.3; A_2 = 3B_2; B_1 = -\frac{5}{3}B_2; \lambda = -1.720977806; G = 0.2813798713; \omega = 0.5861950181$: Kink-Bright-Dark solitary waves **C)** $y = 0$; **D)** $t = 0$.

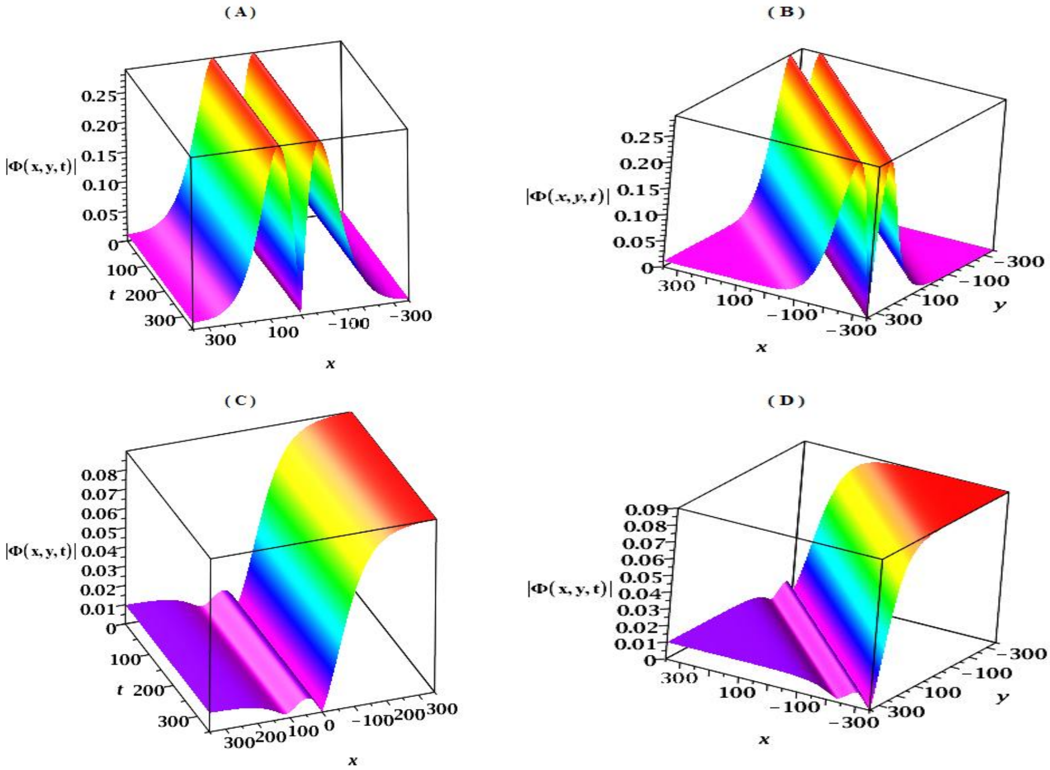

Figure 3: Graphical representation given by eq.(85). For $\eta = 0.01; \gamma = 1; \nu = 0.0013; \alpha = -0.67; k = 0.1; s = 0.2; B_2 = \chi = 0.006; A_2 = 125B_2; B_1 = \frac{5}{6}B_2; \lambda = -0.9586430515; G = 0.1567381389; \omega = 1.048271134$: Double-Bright Dark solitary waves: **A)** $y = 0$; **B)** $t = 0$. For $\eta = 0.01; \gamma = 1; \nu = 0.0013; \alpha = -0.67; k = 0.1222; s = 0.2; B_2 = \chi = 0.05; A_2 = 0.16B_2; B_1 = -0.8B_2; \lambda = -0.5211797094; G = 0.02736193474; \omega = 1.923607548$: Kink-Dark-Bright solitary waves: **C)** $y = 0$; **D)** $t = 0$.

Withal, it is appropriate herein to highlight the process which made it possible to produce the different profiles of certain obtained solitary wave structures . For example, in the case of figures 2 (A) or (B), we set, $\eta = 0.01; \gamma = 1; \nu = 0.0013; \alpha = -0.67; k = 0.1; s = 0.2; B_2 = \chi = 0.3; A_2 = 3B_2; B_1 = \frac{5}{3}B_2$. Then, one deduced from the equality (arbitrarily fixed): $A_2 = 3B_2$ and from one of the constraints that accompany eq.(85) as well as eq.(53): $\lambda = -1.720977806; G = 0.2813798713$ and $\omega = 0.5861950181$.

5. Discussions

In the rest of our analyses, it is important in this section to place particular emphasis on two aspects essential to understanding the obtained results: the analytical aspect and the graphical aspect.
Eq.(9) is a complex analytical multi-soliton. This is justified by the fact that it groups together two hybrid solitary waves represented by terms with respective coefficients A_1 and A_2 as well as two classical solitons represented by terms with respective coefficients B_1 and B_2. These two classic solitons belong to the great family of Dark solitons. In this context and, by operating the choice of the ansatz of the eq.(9), we opted to construct new prototypes of solitary waves of multiple forms which are more robust and able to resist to a number of constraints impose their propagation media which are more generally non-linear and dispersive. These more robust and new prototypes result from the interactions between the basic forms of the different constituent terms of eq.(9). This state of affairs is favored by the values taken by the characteristic parameters of the solitary wave and those of the studied system. Subsequently, it turned out that the series of ordinary equations with four unknowns resulting from the main eq.(10) are very complicated to solve. And so it was necessary to neglect the contribution of the coefficient term A_1 by setting $A_1 = 0$ so that the three remaining terms and their coefficients A_2, B_1 and B_2 give rise to several families of approximate **Solutions Type I**.
Graphical results as for them and following the construction plan, have produced three Figures.

Figure 1 comes from eqs.(99) and (103) and each containing two profiles for the same values taken by the parameters of the solitary wave and those of the considered system. Thus, we notice that ***figure 1 (A) or (B)*** comes from the Kink type analytical form, but, because of its modulus, has provided a sawtooth bottom Dark soliton given by eq.(99), while, ***figure 1 (C) or (D)*** presents a mixed and new structure that we have called Kink-Dark-Bright solitary wave. We estimate that this obtained mixed figure is the fruit of the interactions induced by the combinations of the two classical solitons of the Kink and Dark types that constitutes eq.(103).

Figures 2 and 3 both come from eq.(85). A deep observation of these figures reveals that:

- **Figure 2** displays four profiles which contain the mixed forms that are new intermediate forms that are more robust. These come from the interactions between the Kink, Hybrid and Dark solitons with respective coefficients B_1, A_2 and B_2 of eq.(9) and which we have called Antikink-Bright-Dark and Kink-Bright-Dark, respectively;

- **Figure 3** on the other hand, in its version ***(A) or (B)*** shows a mixed solitary wave structure which is a new intermediate form between the Bright and the Dark soliton with a strong Bright soliton tendency which we have

qualified as Double-Bright-Dark solitary wave. However, version ***(C) or (D)*** offers the Kink-Dark-Bright structure that we obtained to the ***figure 1 (C) or (D)***. This observation made at the level of version ***(C) or (D)*** of **Figure 3** sufficiently indicates that the multi-soliton of eq.(85) is a wide package [27-30] of solitary waves which put together within it, structures contained in the less wide packets of eqs.(99) and (103) respectively.
Consequently, all these graphical results thus confirm the theoretical or analytical predictions considered when adopting the ansatz given to the eq.(9). In comparison with previous works contained in [13; 15; 20-22; 31-36; 39; 40; 47], there appears a clear difference, at least under is analytical and graphical forms. To be a little more precise, almost all of the proposed results in those work are solutions of the types homoclinic waves, periodic and blow up solutions, singular function solutions, periodic kink wave, double kink wave, periodic soliton, double periodic wave solutions, dark periodic, chirped bright wave, chirped soliton solutions, kink waves and single soliton solutions.

6. Conclusion

At the end of this work in which we used the BDKm extended to iB-functions, new packets [27-30] of mixed and complex solitary waves are proposed. It is established that some of the obtained wave structures owe their existence in their propagation media thanks to the nature of the relations which exist mainly between the coefficients A_2, B_2 of eq.(56), other parameters of the wave and those of the system whose dynamics are described by eq.(1): this is the case of **Solutions Type I** (see ***figures 2 and 3***). Then, in the case of certain families of **Solutions Type II**, the angular frequency ω is a parabolic function of the speed ν or of the wave number k of the solitary wave (where the parabola has a concavity facing down) and that ω reaches a maximum value ω_0 for a value ν_0 or k_0 of the speed or the wave number. It is also established that ω tends to ω_0 for large values of ν or small values of k. We have also graphically established that the families of **Solutions Type I** contain within them a large part of the undulatory structures from certain families of **Solutions Type II** (see ***figure 1 (C) or (D)*** and ***figure 3 (C) or (D)***). Graphical results corroborate at best with the theoretical predictions. During laboratory propagation tests, these results obtained will be very useful in the simple choice of new wave structures that one would like to inject into the propagation media by taking into account the ratios that exist among the coefficients A_2 and B_2 of the pivot equation (56) and thus limiting the loss of time. In addition, with regard to certain **Solutions Type II**, these results must allow the control of the deployment of the solitary wave through the relation which gives ω according to ν or k. The authors are all convinced that these new packages of proposed wave structures will find applications in various fields of science and engineering. They will be used in these fields to describe a wide variety of nonlinear wave phenomena, in particular, from superconductivity to liquids in physics, including superfluidity and Bose-Einstein condensation. Beyond the results obtained with satisfaction, this work also had the merit of developing a technic of construction of solitary wave solutions. The principle has consisted of injecting into the nonlinear partial differential equation the solitary wave ansatz previously adopted and as far as possible, proceeded by elimination of certain coefficients in order to determine with accuracy new prototypes of suitable hybrid solutions.
However, it would be appropriate in future work to use an appropriate bifurcation technique to track down all of these new wave structures that will allow in the short, medium or long term to understand and explain new phenomena that remain unexplainable so far.

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