UNDER PEER REVIEW

Exact solutions of the linear fractional diffusion and diffusion-convection equations via the Regular Perturbation Method (RPM)

Abstract

In this paper, we implement Regular Perturbation Method (RPM) for the Solving fractional diffusion and diffusion-convection equations, in order to determine the exact analytical solutions of some linear fractional diffusion and linear fractional diffusion-convection equations.

Keywords : Linear fractional diferential equation, Regular Perturbation Method, Mittag-Leffler, Caputo fractional derivative.

1 Introduction

Today, the notion of fractional calculus is essential in the resolution of partial differential equations. Indeed, several researchers have applied it to several methods such as the method of Adomian, the Homotopy Perturbation method (HPM), etc. However its application remains partial and fragmented. In this document, we will attempt to apply it using the Regular Perturbations method for solving linear fractional diffusion and linear fractional diffusion-convection equations.

Let be the following fractional equation (P):

$$(P)\begin{cases} \dfrac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = f_{\varepsilon}(t, u(x,t));\ 0 < \alpha \leq 1 \\ u(x,0) = g(x) \end{cases}$$

$(x;t) \in \Omega = \mathbb{R} \times [0, +\infty[$ and $u(x,t) \in L^2(\Omega)$

where g is any function dependent only on x, and f_{ε} is a continuous function. f_{ε} can be the right-hand member of a fractional diffusion- convection equation for example, and ε the perturbation coefficient.

2 Preliminaries

In this part, we will recall some very important notions that come into play imperatively in fractional calculus. These are the notions of gamma functions, Beta and Mitag Leffler function as well as some notions of convergence and of solution uniqueness.

2.1 Gamma, Beta and Mittag Leffler functions

2.1.1 Definition 1

The Gamma function is a function on]0;1[, defined by the following integral:

$$\Gamma(s) = \int_0^{+\infty} e^{-t} t^{s-1} dt; \quad s \in \mathbb{C} \ et \ Re(s) > 0.$$

Thus : $\Gamma(1) = 1$.

$\Gamma(s)$ is a monotonous and strictly decreasing function for $0 \leq s \leq 1$.

2.1.2 Property 1:

$$\bullet \forall s > 0, \ \Gamma(s+1) = s\Gamma(s); \ \Gamma(\frac{1}{2}) = \sqrt{\pi}.$$
$$\bullet \forall n \in \mathbb{N}, \ \Gamma(n+1) = n!; \ 0! = \ 1.$$

2.1.3 Definition 2:

The Beta function is the function defined by:

$$\beta(u, v) = \int_0^1 (1-z)^{u-1} z^{v-1} dz$$

where $Re(u) > 0 \ et \ Re(v) > 0$.

2.1.4 Property 2:

$\forall u \in \mathbb{C}$; $\forall v \in \mathbb{C}$ où $Re(u) > 0$ et $Re(v) > 0$,

Thus:

$$\beta(u, v) = \beta(v, u) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$$

2.1.5 Definition 3

For $s \in \mathbb{C}$, the Mittag-Leffler function denoted $E_a(s)$ is defined by:

$$E_a(s) = \sum_{k=0}^{+\infty} \frac{s^k}{\Gamma(k\alpha + 1)}$$

when it depends on a single parameter α.

2.1.6 Property 3:

$\forall s \in \mathbb{C}$ where $Re(s) > 0$, we have:
$E_1(s) = e^s, E_2(s) = ch(\sqrt{s})$, where ch denotes the hyberbolic cosine.

2.1.7 Definition 4

Let considere $f \in L^1([0,T]), T > 0$. The Riemann-Liouville fractional integral of the function f of order $\alpha \in \mathbb{C}$, $(Re(\alpha) > 0)$ noted I_α is defined by:

$$I_\alpha f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(x,\tau) d\tau$$

$t > 0$; $x \in \mathbb{R}$.

2.1.8 Definition 5

Let considere $f \in L^1([0,T]), T > 0$ a integrable function on $[0,T]$. The fractional derivative in the sens of Riemann-Liouville of the function f of order $\alpha \in \mathbb{C}, (Re(\alpha) > 0)$ noted $D_\alpha f$ is defined by:

$$D_\alpha f(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(x,\tau) d\tau; t > 0; x \in \mathbb{R}.$$

2.2 Concept of convergence and uniqueness of the solution of the problem (P)

Let's consider the problem (P) defined by:

$$(P) \begin{cases} \dfrac{\partial^\alpha u\,(x,t)}{\partial t^\alpha} = f_\varepsilon(t, u(x,t))\ ;\ 0 < \alpha \leq 1;\ 0 < \alpha \ll 1 \\ u(x,0) = g(x) \end{cases}$$

with its solution in the form:

$$u(x;t) = \sum_{n=0}^{+\infty} \varepsilon^n u_n(x,t)$$

,where $g(x) \in C(\Omega)$. we obtain:

$$\sum_{n=0}^{+\infty} \varepsilon^n u_n(x,t) = \sum_{n=0}^{+\infty} \varepsilon^n u_n(x,0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, \sum_{n=0}^{+\infty} \varepsilon^n u_n(x,\tau)) d\tau$$

As f is linear, we obtain:

$$\cos(\omega x) \sum_{k=0}^{+\infty} (-1)^k (2k+5)(2k+4)(2k+3)(2k+2) \frac{(\lambda \omega t^\alpha)^{2k+1}}{\Gamma((2k+5)\alpha+1)}]$$

$$\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,t) = \sum_{n=0}^{+\infty}\varepsilon^n u_n(x,0) + \frac{1}{\Gamma(\alpha)}\sum_{n=0}^{+\infty}\varepsilon^n \int_0^t (t-\tau)^{\alpha-1} f(\tau, u_n(x,\tau))d\tau.$$

2.2.1 Proposition 1

Let's suppose (P) is a linear diffusion equation, where $u(x,t) \in C^2(\Omega)$ and f is linear in u, with $\Omega = \mathbb{R}\times[0;t]$ and $0 \leq \tau \leq t < T < +\infty$. If $\exists M = \sup_{x,\tau\in\Omega}\left|\frac{\partial^2 u(x,\tau)}{\partial x^2}\right|$ and $m = \sup_{x\in\Omega}|u(x,0)|$ such as: $\forall x,\tau \in \Omega$, $\left|\frac{\partial^2 u(x,\tau)}{\partial x^2}\right| \leq M$, then the series $\left(\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,t)\right)$ is absolutely convergent.

2.2.2 Proof 1

We obtain progressively :

$$\begin{cases} \varepsilon^0 & |u_0(x,t)| \leq |g(x)| \leq m \\ \varepsilon^1 & |u_1(x,t)| = \left|I_\alpha\left(\frac{\partial^2 u_0(x,\tau)}{\partial x^2}\right)\right| \leq \left(M\frac{T^\alpha}{\Gamma(\alpha+1)}\right) \\ \cdots & \cdots \\ \varepsilon^n & |u_n(x,t)| = \left|I_\alpha\left(\frac{\partial^2 u_{n-1}(x,\tau)}{\partial x^2}\right)\right| \leq \left(M\frac{T^\alpha}{\Gamma(\alpha+1)}\right), n \geq 1 \end{cases}$$

We have: $\left|\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,t)\right| \leq m + \sum_{n=1}^{+\infty}\varepsilon^n\left(M\frac{T^\alpha}{\Gamma(\alpha+1)}\right)$

$$\leq m + \left(M\frac{T^\alpha}{\Gamma(\alpha+1)}\right)\sum_{n=1}^{+\infty}\varepsilon^n = m + \left(M\frac{T^\alpha}{\Gamma(\alpha+1)}\right)\left(-1+\sum_{n=0}^{+\infty}\varepsilon^n\right)$$

$$\Rightarrow \left|\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,t)\right| \leq m + \left(M\frac{T^\alpha}{\Gamma(\alpha+1)}\right)\left(-1+\frac{1}{1-\varepsilon}\right), \text{ because } 0<\varepsilon\ll 1$$

As a result, the series $\left(\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,t)\right)$ is absolutely convergent, therefore convergente.

2.2.3 Proposition 2

Let's suppose (P) a linear diffusion-convection equation, where $u(x,t) \in C(\Omega)$ and $f = \frac{\partial^2 u}{\partial x^2} + \lambda\frac{\partial u}{\partial x} \in C^2(\Omega)$, with $\Omega = \mathbb{R}\times[0;t]$ and $0 \leq \tau \leq t < T < +\infty$. If $\exists\ M = \sup_{x,\tau\in\Omega}\left|\frac{\partial^2 u(x,\tau)}{\partial x^2}\right|$, $N = \sup_{x,\tau\in\Omega}\left|\frac{\partial u(x,\tau)}{\partial x}\right|$ and $m = \sup_{x\in\Omega}|u(x,0)|$ such as: $\forall x,\tau\in\Omega$, $\left|\frac{\partial^2 u(x,\tau)}{\partial x^2}\right| \leq M$, $\left|\frac{\partial u(x,\tau)}{\partial x}\right| \leq N$ and $|u(x,0)| \leq m$, then the series $\left(\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,t)\right)$ is absolutely convergent.

2.2.4 Proof 2

We obtain progressively:

$$\begin{cases} \varepsilon^0 & |u_0(x,t)| = |u(x,0)| + \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(\lambda \frac{\partial u_0(x,\tau)}{\partial x} \right) d\tau \right| \\ \varepsilon^1 & |u_1(x,t)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(\frac{\partial^2 u_0(x,\tau)}{\partial x^2} + \lambda \frac{\partial u_1(x,\tau)}{\partial x} \right) d\tau \right| \\ \ldots & \ldots \\ \varepsilon^n & |u_n(x,t)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(\frac{\partial^2 u_{n-1}(x,\tau)}{\partial x^2} + \lambda \frac{\partial u_n(x,\tau)}{\partial x} \right) d\tau \right|, n \geq 1 \end{cases}$$

$$\Longrightarrow \begin{cases} \varepsilon^0 & |u_0(x,t)| \leq m + \left(\lambda N \frac{T^\alpha}{\Gamma(\alpha+1)} \right) \\ \varepsilon^1 & |u_1(x,t)| \leq \left((M+\lambda N) \frac{T^\alpha}{\Gamma(\alpha+1)} \right) \\ \ldots & \ldots \\ \varepsilon^n & |u_n(x,t)| \leq \left((M+\lambda N) \frac{T^\alpha}{\Gamma(\alpha+1)} \right), n \geq 1 \end{cases}$$

We have: $\left| \sum_{n=0}^{+\infty} \varepsilon^n u_n(x,t) \right| \leq m + \sum_{n=0}^{+\infty} \varepsilon^n \left((M+\lambda N) \frac{T^\alpha}{\Gamma(\alpha+1)} \right)$

$$\leq m + \left((M+\lambda N) \frac{T^\alpha}{\Gamma(\alpha+1)} \right) \sum_{n=0}^{+\infty} \varepsilon^n$$

$\left| \sum_{n=0}^{+\infty} \varepsilon^n u_n(x,t) \right| \leq m + \dfrac{\left((M+\lambda N) \frac{T^\alpha}{\Gamma(\alpha+1)} \right)}{1-\varepsilon}$, because $0 < \varepsilon \ll 1$

As a result, the series $\left(\sum_{n=0}^{+\infty} \varepsilon^n u_n(x,t) \right)$ is absolutely convergent, therefore convergent.

2.2.5 Proposition 3

The equation (P_1) and (P_2) each admit a unique solution which is written in the form:

$u(x,t) = u(x,0) + I_\alpha \left(f(\tau, u(x,\tau)) \right)$, where $I_\alpha(f)$ denotes the fractional integral of f defined by:

$I_\alpha \left(f(\tau, u(x,\tau)) \right) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u_0(x,\tau)) d\tau$

2.2.6 Proof 3

Now let's show the uniqueness of the solution.

Let's suppose there are two distinct solutions $u(x,t)$ and $v(x,t)$ such that:

there is $w(x,t) = u(x,t) - v(x,t) \neq 0$

Considering the following algorithms of u and v :

$$\begin{cases} u_0(x,t) = u(x,0) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau, u_0(x,\tau))d\tau \\ u_1(x,t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau, u_1(x,\tau))d\tau \\ ... \\ u_n(x,t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau, u_n(x,\tau))d\tau \end{cases}$$

and

$$\begin{cases} v_0(x,t) = v(x,0) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau, v_0(x,\tau))d\tau \\ v_1(x,t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau, v_1(x,\tau))d\tau \\ ... \\ v_n(x,t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau, v_n(x,\tau))d\tau \end{cases}$$

We have:

$$\begin{cases} w_0(x,t) = w(x,0) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau, w_0(x,\tau))d\tau = 0 \\ w_1(x,t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau, w_1(x,\tau))d\tau = 0 \\ ... \\ w_n(x,t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau, w_n(x,\tau))d\tau = 0 \end{cases}$$

Because from the conditions to the limits, it happens that
$u(x,0) = g(x) = v(x,0) \Rightarrow w(x,0) = 0$
In addition, $f(\tau, w_0(x,\tau))$ is a linear function of $w(x,0)$
Where from $w_0(x,0) = 0 \Rightarrow f(\tau, w_0(x,\tau)) = 0 \Rightarrow w_0(x,t) = 0$;absurd.
Similary,$f(\tau, w_n(x,\tau))$ is a linear function of $g(x), n \geq 1$
It follows that $w_n(x,t) = 0 = u_n(x,t) - v_n(x,t)$, $\forall n \geq 0$.
Thus $u(x,t) = v(x,t)$ hence the uniqueness of the solution.

3 APPLICATIONS

3.1 Fractional diffusion linear equation

Let's considere the following fonctional equation to fractional order :

$Lu = f + Ru + Nu$ and let (P_1) be the problem to be studied defined by:

$$(P_1)\begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \varepsilon \dfrac{\partial^2 u(x,t)}{\partial x^2} \ ; 0 < \alpha \leq 1; \\ u(x,0) = \sin \omega x \end{cases}$$

$(x;t) \in \Omega = \mathbb{R} \times [0, +\infty[$ et $u\,(x,t) \in L^2\,(\Omega)$

Let's find the solution of (P_1) in the form of:

$u(x,t) = \sum\limits_{n=0}^{+\infty} \varepsilon^n u_n(x,t)$, where $0 < \varepsilon \ll 1$

As

$$L(u(x,t)) = \frac{\partial^\alpha u\,(x,t)}{\partial t^\alpha}$$

and

$$Ru\,(x;t) = \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$L(u(x,t)) = Ru\,(x;t)$$

with

$$L^{-1}(u(x,t)) = I_\alpha(u(x,t))$$

By applying L^{-1} in equation (P_1), we obtain:

$$u(x,t) - u(x,0) = L^{-1}Ru(x,t)$$

which is equivalent to:

$$u_n(x,t) = u(x,0) + I_\alpha\left(\varepsilon \frac{\partial^2 u(x,t)}{\partial x^2}\right)$$

$$\Longrightarrow \sum_{n=0}^{+\infty} \varepsilon^n u_n(x,t) = u(x,0) + \sum_{n=0}^{+\infty} I_\alpha\left(\varepsilon^{n+1} \frac{\partial^2 u_n(x,t)}{\partial x^2}\right)$$

The following equations are obtained successively by identification of powers of ε:

$$\begin{cases} \varepsilon^0 : & u_0(x,t) = u(x,0) = g(x) = \sin \omega x \\ \varepsilon^1 : & u_1(x,t) = I_\alpha\left(\dfrac{\partial^2 u_0(x,t)}{\partial x^2}\right) \\ ... & ... \\ \varepsilon^n : & u_n(x,t) = I_\alpha\left(\dfrac{\partial^2 u_{n-1}(x,t)}{\partial x^2}\right); n \geq 1 \end{cases}$$

This leads us to solve the following equations:

$$\begin{cases} \varepsilon^0 : \begin{cases} \dfrac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} & = 0 \\ u_0(x,0) = \sin(\omega x) \end{cases} \\ \varepsilon^1 : \begin{cases} \dfrac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} & = \dfrac{\partial^2 u_0(x,t)}{\partial x^2} \\ u_1(x,0) = 0 \end{cases} \\ ... \\ \varepsilon^n : \begin{cases} \dfrac{\partial^\alpha u_n(x,t)}{\partial t^\alpha} & = \dfrac{\partial^2 u_{n-1}(x,t)}{\partial x^2}, \forall n \geq 1 \\ u_n(x,0) = 0, n \geq 1 \end{cases} \end{cases} \tag{1.2}$$

Let's solve the system ε^0 :

$\dfrac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = 0$ with $u_0(x,0) = \sin(\omega x)$

In Caputo's sense, the solution to this equation is in the form:

$u_0\left(x;t\right) - u_0(x,0) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau, u_0(x,\tau))d\tau$

As $f(\tau, u_0(\tau)) = 0$,then,

$u_0\left(x;t\right) = u_0\left(x;0\right)$

where from $u_0\left(x;t\right) = u_0\left(x;0\right) = \sin(\omega x)$

The solution to this equation is:

$$u_0(x,t) = \sin(\omega x)$$

Let's solve the system ε^1 :

$\dfrac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 u_0(x,t)}{\partial x^2}$, $u_1(x,0) = 0$

In Caputo's sense, the solution to this equation is in the form:

$u_1\left(x;t\right) - u_1\left(x;0\right) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau, u_1(x,\tau))d\tau$

As $f(\tau, u_1(\tau)) = \dfrac{\partial^2 u_0(x,\tau)}{\partial x^2}$,

$\Rightarrow u_1\left(x;t\right) = \dfrac{-\omega^2 \sin(\omega x)}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1} d\tau$

Let's posing $\tau = tz$, $d\tau = tdz$; When $\tau \to t, alors\ z \to 1$

We obtain: $u_1\left(x;t\right) = \dfrac{-\omega^2 \sin(\omega x)}{\Gamma(\alpha)}\displaystyle\int_0^1 (t-tz)^{\alpha-1} tdz$

$\Rightarrow u_1\left(x;t\right) = \dfrac{-\omega^2 \sin(\omega x)}{\Gamma(\alpha)} t^\alpha \displaystyle\int_0^1 (1-z)^{\alpha-1} z^{1-1} dz$

$\Rightarrow u_1\left(x;t\right) = \dfrac{-\omega^2 \sin(\omega x)}{\Gamma\left(\alpha+1\right)} t^\alpha$

The solution to this equation is:

$$u_1(x;t) = \sin(\omega x)\frac{(-\omega^2 t^\alpha)}{\Gamma(\alpha+1)}$$

Let's solve the system ε^2 :

$\dfrac{\partial^\alpha u_2(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 u_1(x,t)}{\partial x^2}$ avec $u_2(x,0) = 0$

Using the same approach of resolution as in ε^0 and ε^1, we obtain as solution of the system ε^2 :

$$u_2(x;t) = \sin(\omega x)\frac{(-\omega^2 t^\alpha)^2}{\Gamma(2\alpha+1)}$$

In summary, we have:

$$\begin{cases} u_0(x,t) = \sin(\omega x) \\ u_1(x,t) = \sin(\omega x)\dfrac{(-\omega^2 t^\alpha)}{\Gamma(1\alpha+1)} \\ u_2(x,t) = \sin(\omega x)\dfrac{(-\omega^2 t^\alpha)^2}{\Gamma(2\alpha+1)} \\ \dots \\ u_n(x,t) = \sin(\omega x)\dfrac{(-\omega^2 t^\alpha)^n}{\Gamma(n\alpha+1)} \end{cases} \quad (1.3)$$

By posing,

$$\varphi_n(x,t) = \sum_{k=0}^{n} \varepsilon^k u_k(x,t); \quad (1.4)$$

The solution of this equation (P_1) is:

$$u(x,t) = \lim_{n\to+\infty} \varphi_n(x,t) = \sin(\omega x)\lim_{n\to+\infty}\sum_{k=0}^{n}\frac{(-\varepsilon\omega^2 t^\alpha)^k}{\Gamma(k\alpha+1)} = \sin(\omega x)\sum_{k=0}^{+\infty}\frac{(-\varepsilon\omega^2 t^\alpha)^k}{\Gamma(k\alpha+1)}$$

We thus obtain: $\quad u((x,t) = \sin(\omega x)E_\alpha(-\varepsilon\omega^2 t^\alpha) \quad (1.5)$

Note: When $\alpha = 1$, ,the solution of system is:$u(x,t) = \sin(\omega x)e^{-\varepsilon\omega^2 t}$

3.2 Fractional diffusion-convection linear equation

Let (P_2) be the problem to be studied defined by:

$$(P_2):\begin{cases}\dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\varepsilon\dfrac{\partial^2 u(x,t)}{\partial x^2}+\lambda\dfrac{\partial u(x,t)}{\partial x}\text{où } 0<\varepsilon\ll 1;\lambda>0 \text{ et } 0<\alpha\leq 1\\ u(x;0)=\sin(\omega x)\end{cases}$$

$(x;t)\in\Omega=\mathbb{R}\times[0,+\infty[$ et $u(x,t)\in L^2(\Omega)$

Let's posing $L(u(x,t))=\dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ et $Ru(x;t)=\varepsilon\dfrac{\partial^2 u(x,t)}{\partial x^2}+\lambda\dfrac{\partial u(x,t)}{\partial x}$
$\Longrightarrow L^{-1}(u(x,t))=I_\alpha(u(x,t))$ where $L(u(x,t))=Ru(x;t)$
By L^{-1} to the equation (P_2), we obtain: $u(x,t)-u(x,0)=L^{-1}Ru(x,t)$

Finding the solution in the form of: $$u(x,t)=\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,t) \qquad (2.1)$$

Which equals: $\displaystyle\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,t)=u(x,0)+\sum_{n=0}^{+\infty}\varepsilon^n I_\alpha R(u_n(x,t))$

or $$\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,t)=u(x,0)+\sum_{n=0}^{+\infty}\varepsilon^n I_\alpha\left(\varepsilon\frac{\partial^2 u_n(x,t)}{\partial x^2}+\lambda\frac{\partial u_n(x,t)}{\partial x}\right)$$
We successively obtain the following equations:

$$\begin{cases}\varepsilon^0: & u_0(x,t)=u(x,0)+\lambda I_\alpha\left(\dfrac{\partial u_0(x,t)}{\partial x}\right)\\ \varepsilon^1: & u_1(x,t)=I_\alpha\left(\dfrac{\partial^2 u_0(x,t)}{\partial x^2}+\lambda\dfrac{\partial u_1(x,t)}{\partial x}\right)\\ \varepsilon^2: & u_2(x,t)=I_\alpha\left(\dfrac{\partial^2 u_1(x,t)}{\partial x^2}+\lambda\dfrac{\partial u_2(x,t)}{\partial x}\right)\\ \dots & \dots\\ \varepsilon^n: & u_n(x,t)=I_\alpha\left(\dfrac{\partial^2 u_{n-1}(x,t)}{\partial x^2}+\lambda\dfrac{\partial u_n(x,t)}{\partial x}\right);n\geq 1\end{cases}$$

This leads us to solve the following equations:

$$\varepsilon^0:\begin{cases}\dfrac{\partial^\alpha u_0(x,t)}{\partial t^\alpha}=\lambda\dfrac{\partial u_0(x,t)}{\partial x}\\ u_0(x,0)=\sin(\omega x)\end{cases}$$
$$\varepsilon^1:\begin{cases}\dfrac{\partial^\alpha u_1(x,t)}{\partial t^\alpha}=\dfrac{\partial^2 u_0(x,t)}{\partial x^2}+\lambda\dfrac{\partial u_1(x,t)}{\partial x}\\ u_1(x,0)=0\end{cases}$$
$$\varepsilon^2:\begin{cases}\dfrac{\partial^\alpha u_2(x,t)}{\partial t^\alpha}=\dfrac{\partial^2 u_1(x,t)}{\partial x^2}+\lambda\dfrac{\partial u_2(x,t)}{\partial x}\\ u_2(x,0)=0\end{cases} \qquad (2.2)$$
$$\dots$$
$$\varepsilon^n:\begin{cases}\dfrac{\partial^\alpha u_n(x,t)}{\partial t^\alpha}=\dfrac{\partial^2 u_{n-1}(x,t)}{\partial x^2}+\lambda\dfrac{\partial u_n(x,t)}{\partial x};n>0\\ u_n(x,0)=0;n>0\end{cases}$$

Let's solve the system ε^0 :
$\dfrac{\partial^\alpha u_0(x,t)}{\partial t^\alpha}=\lambda\dfrac{\partial u_0(x,t)}{\partial x}$ avec $u_0(x,0)=\sin(\omega x)$

In Caputo's sense, the solution to this equation is in the form:

$$u_0\,(x;t) - u_0(x,0) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau, u_0(x,\tau))d\tau$$

As $f(\tau, u_0(x,\tau)) = \lambda \dfrac{\partial u_0(x,\tau)}{\partial x}$,then,

$$u_0\,(x;t) = u_0(x,0)\ + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\lambda\frac{\partial u_0(x,\tau)}{\partial x})d\tau$$

We have : $u_0(x,t) = \sum_{k=0}^{+\infty} u_{0_k}(x,t)$

Thus : $\begin{cases} u_{0_0}\,(x;t) = u_0\,(x;0)\ = \sin(\omega x) \\ u_{0_{n+1}}\,(x;t) = \dfrac{\lambda}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1}\frac{\partial u_{0_n}(x,\tau)}{\partial x}d\tau\ ;\ n \geq 0 \end{cases}$

we obtain after calculation :

$$\begin{cases} u_{0_{2p}}\,(x;t) = \sin(\omega x)\ (-1)^p\ \dfrac{(\lambda\omega t^\alpha)^{2p}}{\Gamma(2p\alpha+1)}; p \geq 0 \\ u_{0_{2p+1}}\,(x;t) = \cos(\omega x)(-1)^p\ \dfrac{(\lambda\omega t^\alpha)^{2p+1}}{\Gamma((2p+1)\alpha+1)}; p \geq 0 \end{cases}$$

We have:$u_0(x,t) = \sum_{k=0}^{+\infty} u_{0_k}(x,t)\ = \sum_{k=0}^{+\infty} u_{0_{2k}}(x,t)\ + \sum_{k=0}^{+\infty} u_{0_{2k+1}}(x,t)$

The solution to this equation is:

$$u_0(x,t) = \sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k\ \frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma(2k\alpha+1)} + \cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k\ \frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2k+1)\alpha+1)}$$

Let's solve the system ε^1 :

We have: $\dfrac{\partial^\alpha u_1(x,t)}{\partial t^\alpha}\ = \dfrac{\partial^2 u_0(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_1(x,t)}{\partial x}$, with $u_1(x,0) = 0$

In Caputo's sense, the solution to this equation is in the form:

$$u_1\,(x;t) = u_1(x,0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau, u_1(x,\tau))d\tau$$

As $f(\tau, u_1(\tau)) = \dfrac{\partial^2 u_0(x,\tau)}{\partial x^2} + \lambda\dfrac{\partial u_0(x,\tau)}{\partial x}$,then,

$$u_1\,(x;t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(\frac{\partial^2 u_0(x,\tau)}{\partial x^2} + \lambda\frac{\partial u_1(x,\tau)}{\partial x})d\tau$$

$$u_1(x,t) = \sum_{k=0}^{+\infty} u_{1_k}(x,t)$$

Thus: $\begin{cases} u_{1_0}\,(x;t) = u_1\,(x;0)\ = 0 \\ u_{1_{n+1}}\,(x;t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1}(\frac{\partial^2 u_{0_n}(x,\tau)}{\partial x^2} + \lambda\frac{\partial u_{1_n}(x,\tau)}{\partial x})d\tau \end{cases}$

Using the values of u_{2_k} obtained in ε^2,we obtain after calculation:

$$\begin{cases} u_{1_{2k}}\,(x;t) = (-1)^k 2k\ \dfrac{(\lambda\omega t^\alpha)^{2k-1}}{\Gamma(2k\alpha+1)}(\omega^2 t^\alpha)\cos(\omega x); k \geq 0 \\ u_{1_{2k+1}}\,(x;t) = -(-1)^k(2k+1)\ \dfrac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2k+1)\alpha+1)}(-\omega^2 t^\alpha)\sin(\omega x); k \geq 0 \end{cases}$$

We have: $u_1(x,t) = \sum_{k=0}^{+\infty} u_{1_{2k}}(x,t) + \sum_{k=0}^{+\infty} u_{1_{2k+1}}(x,t)$

The solution to this equation is:

$$u_1(x,t) = (\omega^2 t^\alpha)\cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k 2k\ \frac{(\lambda\omega t^\alpha)^{2k-1}}{\Gamma(2k\alpha+1)}$$
$$-(\omega^2 t^\alpha)\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+1)\ \frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2k+1)\alpha+1)}$$

Let's solve the system ε^2 :

We have: $\frac{\partial^\alpha u_2(x,t)}{\partial t^\alpha} = \frac{\partial^2 u_1(x,t)}{\partial x^2} + \lambda\frac{\partial u_2(x,t)}{\partial x}$, with $u_2(x,0) = 0$

Using the same approach of resolution as in ε^0 and $\varepsilon^1$and using the values of u_{1_k} obtained in ε^1, we obtain after calculation:

$$\begin{cases} u_{2_{2k}}(x,t) = -\frac{1}{2}(-1)^k 2k(2k-1)(\omega^4 t^{2\alpha})\sin(\omega x)\frac{(\lambda\omega t^\alpha)^{2k-2}}{\Gamma((2k)\alpha+1)}; k \geq 0 \\ u_{2_{2k+1}}(x,t) = -\frac{1}{2}(-1)^k(2k+1)(2k)(\omega^4 t^{2\alpha})\cos(\omega x)\frac{(\lambda\omega t^\alpha)^{2k-1}}{\Gamma((2k+1)\alpha+1)}; k \geq 0 \end{cases}$$

We have: $u_2(x,t) = \sum_{k=0}^{+\infty} u_{2_{2k}}(x,t) + \sum_{k=0}^{+\infty} u_{2_{2k+1}}(x,t)$

The solution to this equation is:

$$u_2(x,t) = -\frac{1}{2}(\omega^2 t^\alpha)^2\sin(\omega x)\sum_{k=1}^{+\infty}(-1)^k 2k(2k-1)\frac{(\lambda\omega t^\alpha)^{2k-2}}{\Gamma((2k)\alpha+1)}$$
$$-\frac{1}{2}(\omega^2 t^\alpha)^2\cos(\omega x)\sum_{k=1}^{+\infty}(-1)^k(2k+1)(2k)\frac{(\lambda\omega t^\alpha)^{2k-1}}{\Gamma((2k+1)\alpha+1)}$$

Let's solve the system ε^3 :

We have: $\frac{\partial^\alpha u_3(x,t)}{\partial t^\alpha} = \frac{\partial^2 u_2(x,t)}{\partial x^2} + \lambda\frac{\partial u_3(x,t)}{\partial x}$, with $u_3(x,0) = 0$

Using the same approach of resolution as in ε^0 and $\varepsilon^1$and using the values of u_{2_k} obtained in ε^2, we obtain after calculation:

$$\begin{cases} u_{3_{2k}}(x,t) = -(-1)^k\frac{1}{3!}2k(2k-1)(2k-2)(\omega^6 t^{3\alpha})\cos(\omega x)\frac{(\lambda\omega\ t^\alpha)^{2k-3}}{\Gamma(2k\alpha+1)}; k \geq 0 \\ u_{3_{2k+1}}(x,t) = (-1)^k\frac{1}{3!}(2k+1)(2k)(2k-1)(\omega^6 t^{3\alpha})\sin(\omega x)\frac{(\lambda\omega\ t^\alpha)^{2k-2}}{\Gamma((2k+1)\alpha+1)}; k \geq 0 \end{cases}$$

We have : $u_3(x,t) = \sum_{k=0}^{+\infty} u_{3_{2k}}(x,t) + \sum_{k=0}^{+\infty} u_{3_{2k+1}}(x,t)$

The solution to this equation is:

$$u_3(x,t) = -\frac{1}{3!}(\omega^2 t^\alpha)^3 \cos(\omega x) \sum_{k=0}^{+\infty} (-1)^k 2k(2k-1)(2k-2) \frac{(\lambda\omega\ t^\alpha)^{2k-3}}{\Gamma(2k\alpha+1)}$$
$$+\frac{1}{3!}(\omega^2 t^\alpha)^3 \sin(\omega x) \sum_{k=0}^{+\infty} (-1)^k (2k+1)(2k)(2k-1) \frac{(\lambda\omega\ t^\alpha)^{2k-2}}{\Gamma((2k+1)\alpha+1)}$$

Let's solve the system ε^4 :

We have: $\dfrac{\partial^\alpha u_4(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 u_3(x,t)}{\partial x^2} + \lambda \dfrac{\partial u_4(x,t)}{\partial x}$, with $u_4(x,0) = 0$

Using the same approach of resolution as in ε^0 and $\varepsilon^1$and using the values of u_{3_k} obtained in ε^3, we obtain after calculation:

$$\begin{cases} u_{4_{2k}}(x,t) = (-1)^k \frac{1}{4!} 2k(2k-1)(2k-2)(2k-3)(\omega^8 t^{4\alpha}) \sin(\omega x) \frac{(\lambda\omega t^\alpha)^{2k-4}}{\Gamma(2k\alpha+1)}; k \geq 0 \\ u_{4_{2k+1}}(x,t) = (-1)^k \frac{1}{4!}(2k+1)(2k)(2k-1)(2k-2)(\omega^8 t^{4\alpha}) \cos(\omega x) \frac{(\lambda\omega t^\alpha)^{2k-3}}{\Gamma((2k+1)\alpha+1)}; k \geq 0 \end{cases}$$

We have: $\quad u_4(x,t) = \sum_{k=0}^{+\infty} u_{4_{2k}}(x,t) + \sum_{k=0}^{+\infty} u_{4_{2k+1}}(x,t)$

The solution of this equation is:

$$u_4(x,t) = \frac{(\omega^2 t^\alpha)^4}{4!} \sin(\omega x) \sum_{k=0}^{+\infty} (-1)^k 2k(2k-1)(2k-2)(2k-3) \frac{(\lambda\omega t^\alpha)^{2k-4}}{\Gamma(2k\alpha+1)}$$
$$+\frac{(\omega^2 t^\alpha)^4}{4!} \cos(\omega x) \sum_{k=0}^{+\infty} (-1)^k (2k+1)(2k)(2k-1)(2k-2) \frac{(\lambda\omega t^\alpha)^{2k-3}}{\Gamma((2k+1)\alpha+1)}$$

From close to close, we obtain:

$$
\begin{cases}
u_0(x,t) = \sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k \dfrac{(\lambda\omega t^\alpha)^{2k}}{\Gamma(2k\alpha+1)} \\
\qquad + \cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k \dfrac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2k+1)\alpha+1)} \\
u_1(x,t) = (\omega^2 t^\alpha)\cos(\omega x)\sum_{k=1}^{+\infty}(-1)^k 2k \dfrac{(\lambda\omega t^\alpha)^{2k-1}}{\Gamma(2k\alpha+1)} \\
\qquad -(\omega^2 t^\alpha)\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k (2k+1) \dfrac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2k+1)\alpha+1)} \\
u_2(x,t) = \dfrac{(\omega^2 t^\alpha)^2}{2!}\sin(\omega x)\sum_{k=1}^{+\infty}(-1)^k 2k(2k-1)\dfrac{(\lambda\omega\ t^\alpha)^{2k-2}}{\Gamma(2k\alpha+1)} \\
\qquad -\dfrac{(\omega^2 t^\alpha)^2}{2!}\cos(\omega x)\sum_{k=1}^{+\infty}(-1)^k (2k+1)(2k)\dfrac{(\lambda\omega\ t^\alpha)^{2k-1}}{\Gamma((2k+1)\alpha+1)} \\
u_3(x,t) = -\dfrac{(\omega^2 t^\alpha)^3}{3!}\cos(\omega x)\sum_{k=2}^{+\infty}(-1)^k 2k(2k-1)(2k-2)\dfrac{(\lambda\omega\ t^\alpha)^{2k-3}}{\Gamma(2k\alpha+1)} \\
\qquad +\dfrac{(\omega^2 t^\alpha)^3}{3!}\sin(\omega x)\sum_{k=1}^{+\infty}(-1)^k (2k+1)(2k)(2k-1)\dfrac{(\lambda\omega\ t^\alpha)^{2k-2}}{\Gamma((2k+1)\alpha+1)} \\
u_4(x,t) = \dfrac{(\omega^2 t^\alpha)^4}{4!}\sin(\omega x)\sum_{k=2}^{+\infty}(-1)^k 2k(2k-1)(2k-2)(2k-3)\dfrac{(\lambda\omega t^\alpha)^{2k-4}}{\Gamma(2k\alpha+1)} \\
\qquad +\dfrac{(\omega^2 t^\alpha)^4}{4!}\cos(\omega x)\sum_{k=2}^{+\infty}(-1)^k (2k+1)(2k)(2k-1)(2k-2)\dfrac{(\lambda\omega t^\alpha)^{2k-3}}{\Gamma((2k+1)\alpha+1)} \\
...
\end{cases} : \quad (2.3)
$$

Let's posing:

$$\varphi_n(x,t) = \sum_{k=0}^{n}\varepsilon^k u_k(x,t)$$

(2.4)

with

$$u(x,t) = \lim_{n\to+\infty} \varphi_n(x,t)$$

The solution of this equation (P_2) is:

$$u(x,t) = \varepsilon^0 u_0(x,t) + \varepsilon^1 u_1(x,t) + \varepsilon^2 u_2(x,t) + \varepsilon^3 u_3(x,t) + \varepsilon^4 u_4(x,t) + ...$$

Which allows us to get by reducing the smallest value from k to 0, we obtain:

$$
\begin{cases}
u(x,t) = & [\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k \ \frac{(\lambda\omega t^{\alpha})^{2k}}{\Gamma(2k\alpha+1)} + \cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k \ \frac{(\lambda\omega t^{\alpha})^{2k+1}}{\Gamma((2k+1)\alpha+1)}]+ \\
& (-\varepsilon\omega^2 t^{\alpha})[\cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k 2(k+1) \ \frac{(\lambda\omega t^{\alpha})^{2k+1}}{\Gamma(2(k+1)\alpha+1)}+ \\
& \sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+1) \ \frac{(\lambda\omega t^{\alpha})^{2k}}{\Gamma((2k+1)\alpha+1)}]+ \\
& \frac{(-\varepsilon\omega^2 t^{\alpha})^2}{2!}[\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k 2(k+1)(2(k+1)-1)\frac{(\lambda\omega t^{\alpha})^{2k}}{\Gamma(2(k+1)\alpha+1)}+ \\
& \cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2(k+1)+1)(2(k+1))\frac{(\lambda\omega t^{\alpha})^{2k+1}}{\Gamma((2(k+1)+1)\alpha+1)}]+ \\
& \frac{(-\varepsilon\omega^2 t^{\alpha})^3}{3!}[\cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k 2(k+2)(2(k+2)-1)(2(k+2)-2)\frac{(\lambda\omega t^{\alpha})^{2k+1}}{\Gamma(2(k+2)\alpha+1)}+ \\
& \sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2(k+1)+1)(2(k+1))(2(k+1)-1)\frac{(\lambda\omega t^{\alpha})^{2k}}{\Gamma((2(k+1)+1)\alpha+1)}]+ \\
& \frac{(-\varepsilon\omega^2 t^{\alpha})^4}{4!}[\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k 2(k+2)(2(k+2)-1)(2(k+2)-2)(2(k+2)-3)\frac{(\lambda\omega t^{\alpha})^{2k}}{\Gamma(2(k+2)\alpha+1)}+ \\
& \cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2(k+2)+1)(2(k+2))(2(k+2)-1)(2(k+2)-2)\frac{(\lambda\omega t^{\alpha})^{2k+1}}{\Gamma((2(k+2)+1)\alpha+1)}]+...
\end{cases}
$$

Which still gives:

$$\left\{\begin{array}{ll} u(x,t) = & [\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k \ \frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma(2k\alpha+1)} + \cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k \ \frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2k+1)\alpha+1)}]+ \\ & (-\varepsilon\omega^2 t^\alpha)[\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+1) \ \frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2k+1)\alpha+1)}+ \\ & \cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+2) \ \frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2k+2)\alpha+1)}]+ \\ & \frac{(-\varepsilon\omega^2 t^\alpha)^2}{2!}[\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+2)(2k+1)\frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2k+2)\alpha+1)}+ \\ & \cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+3)(2k+2)\frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2k+3)\alpha+1)}]+ \\ & \frac{(-\varepsilon\omega^2 t^\alpha)^3}{3!}[\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+3)(2k+2)(2k+1)\frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2k+3)\alpha+1)}+ \\ & \cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+4)(2k+3)(2k+2)\frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2k+4)\alpha+1)}]+ \\ & \frac{(-\varepsilon\omega^2 t^\alpha)^4}{4!}[\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+4)(2k+3)(2k+2)(2k+1)\frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2k+4)\alpha+1)}+ \\ & \cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k(2k+5)(2k+4)(2k+3)(2k+2)\frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2k+5)\alpha+1)} + ... \end{array}\right.$$

In the end, we obtain:

$$\begin{array}{ll} u(x,t) = & \sum_{p=0}^{+\infty}\frac{(-\varepsilon\omega^2 t^\alpha)^p}{p!}[\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k A_{2k+p}^p\frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2k+p)\alpha+1)}+ \\ & \cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k A_{2k+p+1}^p\frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2k+p+1)\alpha+1)}] \end{array}$$

where A_m^r avec $m \geq r$ is the arrangement of r in m

Note: For $\alpha = 1$, we have:

$$\sum_{k=0}^{+\infty}(-1)^k A_{2k+p}^p\frac{(\lambda\omega t^\alpha)^{2k}}{\Gamma((2k+p)\alpha+1)} = \sum_{k=0}^{+\infty}(-1)^k\frac{(\lambda\omega t)^{2k}}{(2k)!} = \cos(\lambda\omega t)$$

et

$$\sum_{k=0}^{+\infty}(-1)^k A_{2k+p+1}^p\frac{(\lambda\omega t^\alpha)^{2k+1}}{\Gamma((2k+p+1)\alpha+1)} = \sum_{k=0}^{+\infty}(-1)^k\frac{(\lambda\omega t)^{2k+1}}{(2k+1)!} = \sin(\lambda\omega t)$$

$\Longrightarrow$

$$u(x,t) = [\sin(\omega x)\cos(\lambda\omega t) + \cos(\omega x)\sin(\lambda\omega t)]\sum_{p=0}^{+\infty}\frac{(-\varepsilon\omega^2 t)^p}{p!}$$

$$u(x,t) \ = \sin(\omega x + \lambda\omega t)e^{-\varepsilon\omega^2 t}.$$

3.3 Conclusion

The resolution of fractional equations by the Perturbation method regular is very tedious because it requires much more vigilance in calculations. The use of the Mittag-Leffler function is essential. In the event that $\alpha = 1$, the solutions become simpler because they appeal to the exponential function.

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