Fixed Point Theorems For Mappings Satisfying Implicit Relation In Partial Metric Spaces

Abstract: In this manuscript, we shall introduce the implicit relation on ${\mathbb{R}_+}^4$. Using this implicit relation, we shall prove fixed point and common fixed point theorems on partial metric spaces.

Keywords: partial metric space, implicit relation, common fixed point.

1. Introduction

In fixed point theorem, the widely known Banach contraction principle is the most popular result. According to this principle, contraction map on complete metric space has a unique fixed point. Firstly Kannan [13] gave a new contractive condition in 1969. The generalization of the Banach contraction condition is done by Chatterjae [11] in 1972.

The widely-known Banach contraction principle is stated as follows.

Theorem 1.1 [8] Let (X, d) be a complete metric space and let $T: X \to X$ be a mapping satisfying the contractive condition

$$d(T(x), T(y)) \leq \alpha d(x, y)$$

for all $x, y \in X$, where $\alpha \in [0,1)$ is a constant. Then T has a unique fixed point in X and that point can be obtained as a limit of repeated iteration of the mapping at any point of X."

Matthews ([19, 20]) introduced a new space called partial metric space. The analogous of Banach contraction principle was proved by Matthews [20], which made the partial metric applicable in fixed point theory. By generalization of partial metric function named as weak partial metric function Heckmann [12] entrenched some outcomes.

Some generalizations are given by many authors of the outcomes of Matthews ([1-5, 14-17]).

The introduction of implicit relations in common fixed point theorems was established by V. Popa [22] in 1999. Further many authors extended common fixed point theorems using implicit relations ([7, 9, 10, 18, 21]).

Definition 1.2 [19] Let X be a nonempty set and let $p: X \times X \to \mathbb{R}^+$ be a function satisfying:

$(pm1)$ $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

$(pm2)$ $p(x, x) \leq p(x, y)$,

$(pm3)$ $p(x,y) = p(y,x)$,

$(pm4)$ $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$,

for all $x,y,z \in X$. Then p is called partial metric on X and the pair (X,p) is called partial metric space.

It is clear that if $p(x,y) = 0$, then from $(pm1)$ and $(pm2)$ we obtain $x = y$. But if $x = y$, $p(x,y)$ may not be zero.

Example 1.3 [6] Let $\tilde{X} = \mathbb{R}^+$ and $p: \tilde{X} \times \tilde{X} \to \mathbb{R}^+$ given by $p(x,y) = max\{x,y\}$ for all $x,y \in \mathbb{R}^+$. Then $(\mathbb{R}^+, p)$ is a partial metric space.

Example 1.4 [6] Let $\tilde{X} = \{[\tilde{a},\tilde{b}]: \tilde{a},\tilde{b} \in \mathbb{R}, \tilde{a} \leq \tilde{b}\}$. Then $p([\tilde{a},\tilde{b}],[\tilde{c},\tilde{d}] = max\{\tilde{b},\tilde{d}\} - min\{\tilde{a},\tilde{c}\}$ defines a partial metric p on $\tilde{X}$.

Lemma 1.5 ([19, 20])Let (X,p) be a partial metric space. Then:

$(c1)$ a sequence x_n in (X,p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X,d_p),

$(c2)$ (X,p) is complete if and only if the metric space (X,d_p) is complete ,

$(c3)$ a subset E of a partial metric space (X,p) is closed if a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to some $x \in X$, then $x \in E$.

Lemma 1.6 [2] Assume that $x_n \to z$ as $n \to \infty$ in a partial metric space (X,p) such that $p(z,z) = 0$. Then $lim_{n\to\infty} p(x_n, y) = p(z,y)$ for every $y \in X$.

2. Main Results

In this section, we shall prove fixed point theorems using implicit relation.

Definition 2.1 (Implicit Relation) Let $\boldsymbol{\Psi}$ be the family of all real valued continuous functions $\boldsymbol{\psi}: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ non decreasing in the first argument for three variables. For some $\boldsymbol{\alpha} \in [\mathbf{0}, \mathbf{1})$, we consider the following conditions.

$(Imr1)$ For $x,y \in \mathbb{R}_+$, if $y \leq \psi(x,x,y,\frac{x+y}{2})$, then $y \leq \alpha x$,

$(Imr2)$ For $x \in \mathbb{R}_+$, if $y \leq \psi(y,0,0,y)$, then $y = 0$ since $\alpha \in [0,1)$,

$(Imr3)$ For $x \in \mathbb{R}_+$, if $y \leq \psi(0,0,y,\frac{y}{2})$, then $y = 0$.

Theorem 2.2 Let $(\tilde{X},p)$ be a complete partial metric space and let $\tilde{T}: \tilde{X} \to \tilde{X}$ be a mapping, such that

$$p(\tilde{T}\tilde{u},\tilde{T}\tilde{v}) \leq \psi\{p(\tilde{u},\tilde{v}),p(\tilde{u},\tilde{T}\tilde{u}),p(\tilde{v},\tilde{T}\tilde{v}),\frac{p(\tilde{u},\tilde{T}\tilde{v})+p(\tilde{v},\tilde{T}\tilde{u})}{2}\} \tag{2.1}$$

for all $\tilde{u},\tilde{v} \in \tilde{X}$ and some $\tilde{\psi} \in \Psi$. If Ψ satisfies the condition $(Imr1),(Imr2)$ and $(Imr3)$, then $\tilde{T}$ has a unique fixed point in $\tilde{X}$.

Proof. For each $u_0 \in \tilde{X}$ and $n \in N$, put $u_{n+1} = \tilde{T}u_n$. It follows from equation (2.1) and $(pm4)$ that

$$\begin{aligned}
p(u_n,u_{n+1}) = \; & p(\tilde{T}u_{n-1},\tilde{T}u_n) \\
& \leq \psi\{p(u_{n-1},u_n),p(u_{n-1},\tilde{T}u_{n-1}),p(u_n,\tilde{T}u_n),\frac{p(u_{n-1},\tilde{T}u_n)+p(u_n,\tilde{T}u_{n-1})}{2}\} \\
& \leq \psi\{p(u_{n-1},u_n),p(u_{n-1},u_n),p(u_n,u_{n+1}),\frac{p(u_{n-1},u_{n+1})+p(u_n,u_n)}{2}\} \\
& \leq \psi\left\{\begin{matrix} p(u_{n-1},u_n),p(u_{n-1},u_n),p(u_n,u_{n+1}), \\ \dfrac{p(u_{n-1},u_n)+p(u_n,u_{n+1})-p(u_n,u_n)+p(u_n,u_n)}{2}\end{matrix}\right\} \\
& = \psi\{p(u_{n-1},u_n),p(u_{n-1},u_n),p(u_n,u_{n+1}),\frac{p(u_{n-1},u_n)+p(u_n,u_{n+1})}{2}\}
\end{aligned} \tag{2.2}$$

Since ψ satisfies the condition $(Imr1)$, there exists $\alpha \in [0,1)$, such that

$$p(u_n,u_{n+1}) \leq \alpha p(u_{n-1},u_n) \leq \ldots \leq \alpha^n p(x_0,x_1). \tag{2.3}$$

Set $S_n = p(u_n,u_{n+1})$ and $S_{n_1} = p(u_{n-1},u_n)$.

Then from equation (2.3), we obtain

$$S_n \leq \alpha S_{n-1} \leq \ldots \leq \alpha^n S_0.$$

Now we show that $\{u_n\}$ is a Cauchy sequence in $\tilde{X}$. Let $m,n > 0$ with $m > n$, then by using $(pm4)$ and equation (2.3), we have

$$\begin{aligned}
& p(u_n,u_m) \leq p(u_n,u_{n+1}) + p(u_{n+1},u_{n+2}) + \ldots + p(u_{n+m-1},u_m) \\
& -p(u_{n+1},u_{n+1}) - p(u_{n+2},u_{n+2}) - \ldots - p(u_{n+m-1},u_{n+m-1}) \\
& \leq \alpha^n p(u_0,u_1) + \alpha^{n+1}p(u_0,u_1) + \ldots + \alpha^{n+m-1}p(u_0,u_1) \\
& = \alpha^n[p(u_0,u_1) + \alpha p(u_0,u_1) + \ldots + \alpha^{m-1}p(u_0,u_1)] \\
& = \alpha^n[1 + \alpha + \ldots + \alpha^{m-1}]S_0 \\
& \leq \alpha(\frac{1-\alpha^{m-1}}{1-\alpha})S_0.
\end{aligned}$$

In above inequality, taking the limits as $n,m \to \infty$, we have

$$p(u_n,u_m) \to 0 \text{ since } 0 < \alpha < 1,$$

hence, $\{u_n\}$ is a Cauchy sequence in $\tilde{X}$ and this sequence is also Cauchy in $(\tilde{X},d_p)$.

As $(\tilde{X},p)$ is complete, therefore $(\tilde{X},d_p)$ is also complete.

Thus by Lemma 1.5,

$$\lim_{n\to\infty} u_n \to w$$

$$p(w,w) = lim_{n\to\infty} p(w,u_n) = lim_{n\to\infty} p(u_n,u_m) = 0, \tag{2.4}$$

implies,

$$lim_{n\to\infty} d_p(w,u_n) = 0. \tag{2.5}$$

Now, we show that w is a fixed point of $\tilde{T}$.

From equation (2.4), we have

$$p(w,w) = 0.$$

By using inequality (2.1), we get

$$\begin{aligned} p(u_{n+1},\tilde{T}w) &= p(\tilde{T}u_n,\tilde{T}w) \\ &\leq \psi\{p(u_n,w),p(u_n,\tilde{T}u_n),p(w,\tilde{T}w),\tfrac{p(u_n,\tilde{T}w)+p(w,\tilde{T}u_n)}{2}\} \\ &= \psi\{p(u_n,w),p(u_n,u_{n+1}),p(w,\tilde{T}w),\tfrac{p(u_n,\tilde{T}w)+p(w,u_{n+1})}{2}\}. \end{aligned}$$

Note that $\psi \in \Psi$, then taking the limit as $n \to \infty$ and using equation (2.4), we get

$$p(u_{n+1},\tilde{T}w) \leq \psi\{0,0,p(w,\tilde{T}w),\tfrac{p(w,\tilde{T}w)}{2}\}.$$

Since ψ satisfies the condition $(Imr3)$, then

$$p(w,\tilde{T}w) = 0.$$

This implies $w = \tilde{T}w$.

Therefore, w is a fixed point of $\tilde{T}$.

Now we have to show that the fixed point of $\tilde{T}$ is a unique .

let w_1, w_2 be fixed points of $\tilde{T}$ with $w_1 \neq w_2$.

From equations (2.1) and (2.4)

$$\begin{aligned} p(w_1,w_2) &= p(\tilde{T}w_1,\tilde{T}w_2) \\ &\leq \psi\{p(w_1,w_2),p(w_1,\tilde{T}w_1),p(w_2,\tilde{T}w_2),\tfrac{p(w_1,\tilde{T}w_2),p(w_2,\tilde{T}w_1)}{2}\} \\ &= \psi\{p(w_1,w_2),p(w_1,w_1),p(w_2,w_2),\tfrac{p(w_1,w_2)+p(w_2,w_1)}{2}\} \\ &= \psi\{p(w_1,w_2),0,0,p(w_1,w_2)\}. \end{aligned}$$

Since ψ satisfies the condition $(Imr2)$, we have

this implies, that $p(w_1,w_2) = 0$, since $0 < \alpha < 1$.

This implies that $w_1 = w_2$.

Thus the fixed point of $\tilde{T}$ is unique.

Theorem 2.3 Let $(\tilde{X}, p)$ be a complete partial metric space and $\tilde{F}, \tilde{G}$ be two self maps on $\tilde{X}$ satisfying the following*:*

$$p(\tilde{F}u, \tilde{G}v) \leq \psi\{p(u,v), p(u,\tilde{F}u), p(v,\tilde{G}v), \frac{p(u,\tilde{G}v)+p(v,\tilde{F}u)}{2}\} \tag{2.6}$$

for all $u, v \in \tilde{X}$ and some $\psi \in \Psi$. Then $\tilde{F}$ and $\tilde{G}$ have a unique common fixed point in $\tilde{X}$.

Proof. For each $u_0 \in \tilde{X}$ and $n \in \mathbb{N}$.

Put $u_{n+1} = \tilde{F}u_n$ and

$u_{n+2} = \tilde{G}u_{n+1}$ for $n = 0,1,2...$

It follows from equation (2.5), Lemma 1.5 and $(pm4)$ that

$$\begin{aligned} p(u_n, u_{n+1}) &= p(\tilde{F}u_{n-1}, \tilde{G}u_n) \\ &\leq \psi\{p(u_{n-1}, u_n), p(u_{n-1}, \tilde{F}u_{n-1}), p(u_n, \tilde{G}u_n), \frac{p(u_{n-1},\tilde{G}u_n)+p(u_n,\tilde{F}u_{n-1})}{2}\} \\ &\leq \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u-n+1), \frac{p(u_{n-1},u_{n+1})+p(u_n,u_n)}{2}\} \\ &\leq \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u_{n+1}), \frac{p(u_{n-1},u_n)p(u_n,u_{n+1})-p(u_n,u_n)+p(u_n,u_n)}{2}\} \\ &= \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u_{n+1}), \frac{p(u_{n-1},u_n)p(u_n,u_{n+1})}{2}\} \end{aligned}$$

$$p(u_n, u_{n+1}) = \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u_{n+1}), \frac{p(u_{n-1},u_n)p(u_n,u_{n+1})}{2}\}. \tag{2.7}$$

Since ψ satisfies the condition $(Imr1)$, therefore there exists $\alpha \in [0,1)$ such that

$$p(u_n, u_{n+1}) \leq \alpha p(u_{n-1}, u_n) \leq \ldots \leq \alpha^n p(x_0, x_1) \tag{2.8}$$

Now we show that $\{u_n\}$ is a Cauchy sequence

Let $m, n > 0$ with $m > n$, then by using $(pm4)$ and equation (2.3), we have

$$\begin{aligned} p(u_n, u_m) &\leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \ldots + p(u_{n+m-1}, u_m) \\ &\quad - p(u_{n+1}, u_{n+1}) - p(u_{n+2}, u_{n+2}) - \ldots - p(u_{n+m-1}, u_{n+m-1}) \\ &\leq \alpha^n p(u_0, u_1) + \alpha^{n+1} p(u_0, u_1) + \ldots + \alpha^{n+m-1} p(u_0, u_1) \\ &= \alpha^n [p(u_0, u_1) + \alpha p(u_0, u_1) + \ldots + \alpha^{m-1} p(u_0, u_1)] \\ &= \alpha^n [1 + \alpha + \ldots + \alpha^{m-1}] S_0 \\ &\leq \alpha (\frac{1-\alpha^{m-1}}{1-\alpha}) S_0. \end{aligned}$$

Assume that, $lim_{n\to\infty} u_n \to r$

by Lemma 1.5,

$$p(r,r) = lim_{n\to\infty} p(r, u_n) = lim_{n,m\to\infty} p(u_n, u_m) = 0. \tag{2.9}$$

This implies,

$$lim_{n\to\infty} d_p(r,u_n) = 0. \quad (2.10)$$

Now we have to prove that r is a common fixed point of $\tilde{F}$ and $\tilde{G}$.

$$\begin{aligned} p(u_{n+1},\tilde{F}r) &= p(\tilde{F}u_n,\tilde{F}r) \\ &\le \psi\{p(u_n,r),p(u_n,\tilde{F}u_n),p(r,\tilde{F}r),\tfrac{p(u_n,\tilde{F}r)+p(r,\tilde{F}u_n)}{2}\} \\ &= \psi\{p(u_n,r),p(u_n,u_{n+1}),p(r,\tilde{F}r),\tfrac{p(u_n,\tilde{F}r)+p(r,u_{n+1})}{2}\} \end{aligned}$$

Note that $\psi \in \Psi$, using (2.8), Lemma 1.6 and taking the limit as $n \to \infty$,

$$p(r,\tilde{F}r) \le \psi\{0,0,p(r,\tilde{F}r),\frac{p(r,\tilde{F}r)}{2}\}$$

Since ψ satisfies the condition $(Imr3)$, then

$p(r,\tilde{F}r) = 0$.

This shows that $r = \tilde{F}r$ for all $r \in \tilde{X}$.

Similarly, we can show $r = \tilde{G}r$.

Thus r is a common fixed point of $\tilde{F}$ and $\tilde{G}$.

Now we will prove the uniqueness of fixed point of $\tilde{F}$ and $\tilde{G}$.

Let r' be another common fixed point of $\tilde{F}$ and $\tilde{G}$, that is, $\tilde{F}r' = \tilde{G}r' = r'$ with $r' \ne r$.

Thus we have to show to show that $r = r'$.

From equations (2.5) and (2.8)

$$\begin{aligned} p(r,r') = \; & p(\tilde{F}r,\tilde{G}r') \\ &\le \psi\{p(r,r'),p(r,\tilde{F}r),p(r',\tilde{G}r'),\tfrac{p(r,\tilde{G}r')+p(r',\tilde{F}r)}{2}\} \\ &= \psi\{p(r,r'),p(r,r),p(r',r'),\tfrac{p(r,r')+p(r',r)}{2}\} \\ &= \psi\{p(r,r'),0,0,p(r,r')\}. \end{aligned}$$

Since ψ satisfies the condition $(Imr2)$, then we get

$p(r,r') = 0$, since $0 < \alpha < 1$.

Thus, we get $r = r'$.

This shows that r is the unique common fixed point of $\tilde{F}$ and $\tilde{G}$.

Theorem 2.4 Let $(\tilde{X},p)$ be a complete partial metric space and $\tilde{F}_1$, $\tilde{F}_2$ be continuous mappings on $\tilde{X}$ satisfying

$$p(\tilde{F}_1{}^m u,\tilde{F}_1{}^n v) \le \psi\{p(u,v),p(u,\tilde{F}_1{}^m u),p(v,\tilde{F}_2{}^n v),\frac{p(u,\tilde{F}_2{}^n v)+p(v,\tilde{F}_1{}^m u)}{2} \quad (2.11)$$

for all $u, v \in \tilde{X}$, where m and n are some positive integers and some $\psi \in \Psi$. Then $\tilde{F}_1$ and $\tilde{F}_2$ have a unique common fixed point in X.

Proof. Since ${\tilde{F}_1}^m$ and ${\tilde{F}_2}^n$ satisfy the condition of Theorem 2.2. So ${\tilde{F}_1}^m$ and ${\tilde{F}_2}^n$ have a unique common fixed point.

Then, we have

$$
\begin{gathered}
{\tilde{F}_1}^m \tilde{z} = \tilde{z} \\
\Rightarrow \tilde{F}_1({\tilde{F}_1}^m \tilde{z}) = \tilde{F}_1 \tilde{z} \\
{\tilde{F}_1}^m (\tilde{F}_1 \tilde{z}) = \tilde{F}_1 \tilde{z}.
\end{gathered}
$$

If $\tilde{F}_1 \tilde{z} = \tilde{r}_0$, then ${\tilde{F}_1}^m \tilde{r}_0 = \tilde{r}_0$.

So, $\tilde{F}_1 \tilde{z}$ is a point of ${\tilde{F}_1}^m$.

Similarly, $\tilde{F}_2({\tilde{F}_2}^n \tilde{z}) = \tilde{F}_2 \tilde{z}$.

Now using equation (2.11) and Lemma 1.5, we obtain

$$
\begin{aligned}
p(\tilde{z}, \tilde{F}_1 \tilde{z}) &= p({\tilde{F}_1}^m \tilde{z}, {\tilde{F}_1}^m (\tilde{F}_1 \tilde{z})) \\
&\leq \{p(\tilde{z}, \tilde{F}_1 z), p(\tilde{F}_1 z, {\tilde{F}_1}^m (\tilde{F}_1 \tilde{z})), p(\tilde{z}, {\tilde{F}_1}^m \tilde{z}), \frac{p(\tilde{z}, {\tilde{F}_1}^m (\tilde{F}_1 \tilde{z})) + p(\tilde{F}_1 \tilde{z}, {\tilde{F}_1}^m \tilde{z})}{2}\} \\
&= \psi\{p(\tilde{z}, \tilde{F}_1 \tilde{z}), p(\tilde{z}, \tilde{z}), p(\tilde{F}_1 z, \tilde{F}_1 z), \frac{p(\tilde{z}, \tilde{F}_1 \tilde{z}) + p(\tilde{F}_1 \tilde{z}, \tilde{z})}{2}\} \\
&= \psi\{p(\tilde{z}, \tilde{F}_1 \tilde{z}), 0, 0, p(\tilde{z}, \tilde{F}_1 \tilde{z})\}
\end{aligned}
$$

Since ψ satisfies the condition $(Imr2)$, then we get

$$p(\tilde{z}, \tilde{F}_1 \tilde{z}), \text{ since } 0 < \alpha < 1.$$

Thus, we have $\tilde{z} = \tilde{F}_1 \tilde{z}$ for all $\tilde{z} \in \tilde{X}$.

Similarly, we can show that $\tilde{z} = \tilde{F}_2 \tilde{z}$.

This shows that $\tilde{z}$ is a common fixed point of $\tilde{F}_1$ and $\tilde{F}_2$.

For uniqueness of $\tilde{z}$, let $\tilde{z}' \neq \tilde{z}$ be another common fixed point of $\tilde{F}_1$ and $\tilde{F}_2$. Then clearly $\tilde{z}'$ is also a common fixed point of ${\tilde{F}_1}^m$ and ${\tilde{F}_2}^n$ which implies $\tilde{z}' = \tilde{z}$.

Hence $\tilde{F}_1$ and $\tilde{F}_2$ have a unique common fixed point.

Theorem 2.5 Let $\tilde{u}_\gamma$ be a family of continuous mappings on a complete partial metric space $(\tilde{X}, p)$ satisfying

$$p(\tilde{u}_\gamma u, \tilde{u}_\beta v) \leq \psi\{p(u, v), p(u, \tilde{u}_\gamma u), p(v, \tilde{u}_\beta v), \frac{p(u, \tilde{u}_\beta v) + p(v, \tilde{u}_\gamma u)}{2}\} \tag{2.12}$$

for all $\gamma, \beta \in \Psi$ with $\gamma \neq \beta$ and $u, v \in \tilde{X}$. Then there exists a unique $\tilde{z} \in \tilde{X}$ satisfying $\tilde{u}_\gamma \tilde{z} = \tilde{z}$ for all $\gamma \in \Psi$.

Proof. For $u_0 \in \tilde{X}$, we define a sequence as follows:-

$u_{n+1} = \tilde{u}_\gamma u_n, \quad u_{n+2} = \tilde{u}_\beta u_{n+1}, \quad$ n=0,1,2...

It follows from $(2.12), (pm4)$ and Lemma 1.5 that

$$\begin{aligned} p(u_n, u_{n+1}) = \; & p(\tilde{u}_\gamma u_{n-1}, \tilde{u}_\beta u_n) \\ & \leq \psi\{p(u_{n-1}, u_n), p(u_{n-1}, \tilde{u}_(\gamma) u_{n-1}), p(u_n, \tilde{u}_\beta u_n), \tfrac{p(u_{n-1}, \tilde{u}_\beta u_n) + p(u_n, \tilde{u}_\gamma u_{n-1})}{2}\} \\ & = \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u_{n+1}), \tfrac{p(u_{n-1}, u_{n+1}) + p(u_n, u_n)}{2}\} \\ & \leq \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u_{n+1}), \tfrac{p(u_{n-1}, u_n) + p(u_n, u_{n+1}) - p(u_n, u_n) + p(u_n, u_n)}{2}\} \\ & = \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u_{n+1}), \tfrac{p(u_{n-1}, u_n) + p(u_n, u_{n+1})}{2}\} \end{aligned}$$

$$p(u_n, u_{n+1}) \leq \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u_{n+1}), \tfrac{p(u_{n-1}, u_n) + p(u_n, u_{n+1})}{2}\}. \quad (2.13)$$

Since ψ satisfies the condition $(Imr1)$, there exists $\alpha \in (0,1)$ such that

$$p(u_n, u_{n+1}) \leq \alpha p(u_{n-1}, u_n) \leq \ldots \leq \alpha^n p(x_0, x_1) \quad (2.14)$$

Now we show that $\{u_n\}$ is a Cauchy sequence in $\tilde{X}$. Let $m, n > 0$ with $m > n$, then by using (pm4) and equation (2.14), we have

$$\begin{aligned} & p(u_n, u_m) \leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \ldots + p(u_{n+m-1}, u_m) \\ & -p(u_{n+1}, u_{n+1}) - p(u_{n+2}, u_{n+2}) - \ldots - p(u_{n+m-1}, u_{n+m-1}) \\ & \leq \alpha^n p(u_0, u_1) + \alpha^{n+1} p(u_0, u_1) + \ldots + \alpha^{n+m-1} p(u_0, u_1) \\ & = \alpha^n [p(u_0, u_1) + \alpha p(u_0, u_1) + \ldots + \alpha^{m-1} p(u_0, u_1)] \\ & = \alpha^n [1 + \alpha + \ldots + \alpha^{m-1}] S_0 \\ & \leq \alpha (\tfrac{1-\alpha^{m-1}}{1-\alpha}) S_0. \end{aligned}$$

Taking $\lim_{n,m\to\infty}$in the above inequality, we get

$p(u_n, u_m) \to 0$, since $0 < \alpha < 1$.

$u_n \to \tilde{r}$ as $n \to \infty$.

Moreover by Lemma 1.5,

$$p(\tilde{r}, \tilde{r}) = \lim_{n\to\infty} p(\tilde{r}, u_{2n}) = \lim_{n,m\to\infty} p(\tilde{u}_n, \tilde{u}_m) = 0 \quad (2.15)$$

implies,

$$\lim_{n\to\infty} d_p(\tilde{r}, u_n) = 0 \tag{2.16}$$

By the continuity of $\tilde{u}_\alpha$ and $\tilde{u}_\beta$, it is clear that

$$\tilde{u}_\gamma \tilde{r} = \tilde{u}_\beta \tilde{r} = \tilde{r}.$$

Therefore $\tilde{r}$ is a common fixed point of $\tilde{u}_\gamma$ for all $\gamma \in \Psi$.

To prove the uniqueness, let us consider the another common fixed point $\tilde{r}'$ of $\tilde{u}_\gamma$ and $\tilde{u}_\beta$ where

$$\tilde{r} \neq \tilde{r}'$$

from equations (2.12) and (2.15), we obtain

$$\begin{aligned} p(\tilde{r}, \tilde{r}') &= p(\tilde{u}_\gamma \tilde{r}, \tilde{u}_\beta \tilde{r}') \\ &\leq \psi\{p(\tilde{r}, \tilde{r}'), p(\tilde{r}, \widetilde{U}_\gamma \tilde{r}), p(\tilde{r}', \widetilde{U}_\beta), \tfrac{p(\tilde{r}, \tilde{u}_\beta \tilde{r}') + p(\tilde{r}', \tilde{u}_\gamma \tilde{r})}{2}\} \\ &= \psi\{p(\tilde{r}, \tilde{r}'), p(\tilde{r}, \tilde{r}), p(\tilde{r}', \tilde{r}'), \tfrac{p(\tilde{r}, \tilde{r}') + p(\tilde{r}', \tilde{r})}{2}\} \\ &= \psi\{p(\tilde{r}, \tilde{r}'), 0, 0, p(\tilde{r}, \tilde{r}')\} \end{aligned}$$

Since ψ satisfies the condition $(Imr2)$, then we get

$$p(\tilde{r}, \tilde{r}') = 0, \text{ since } 0 < \alpha < 1.$$

Thus, we get $\tilde{r} = \tilde{r}'$ for all $\tilde{r} \in \tilde{X}$.

This shows that $\tilde{r}$ is a unique common fixed point of $\tilde{u}_\gamma$ for all $\gamma \in \Psi$.

3. Conclusion

We have introduced the implicit relation on ${\mathbb{R}_+}^4$. Using this implicit relation, we have proved fixed point and common fixed point theorems on partial metric spaces.

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