Original Research Article

Fixed Point Theorems For Mappings Satisfying Implicit Relation In Partial Metric Spaces

Abstract: In this manuscript, we shall introduce the implicit relation on ${\mathbb{R}_+}^4$. Using this implicit relation, we shall prove fixed point and common fixed point theorems on partial metric spaces.

Keywords: partial metric space, implicit relation, common fixed point.

1 Introduction

In fixed point theorem, the widely known Banach contraction principle is the most popular result. According to this principle, contraction map on complete metric space has a unique fixed point. Firstly Kannan[13] gave a new contractive condition in 1969. The generalization of the Banach contraction condition done by Chatterjae[11] in 1972.

The widely-known Banach contraction principle is stated as follows.

Theorem 1.1 [8] Let (X, d) be a complete metric space and let $T: X \to X$ be a mapping satisfying the contractive condition

$$d(T(x), T(y)) \leq \alpha d(x, y)$$

for all $x, y \in X$, where $\alpha \in [0,1)$ is a constant. Then T has a unique fixed point in X and that point can be obtained as a limit of repeated iteration of the mapping at any point of X."

Matthews ([19, 20]) introduced a new space called partial metric space. The analogous of Banach contraction principle proved by Matthews[20], which made the partial metric applicable in fixed point theory. By generalization of partial metric function named as weak partial metric function Heckmann[12] entrenched some outcomes.

Some generalizations given by many authors of the outcomes of Matthews ([1, 2, 3, 4, 5, 14, 15, 16, 17]).

The introduction of implicit relations in common fixed point theorems was established by V. Popa[22] in 1999. Further many authors extended common fixed point theorems using implicit relations([7, 9, 10, 18]).

Definition 1.2 [19] Let X be a nonempty set and let $p: X \times X \to R^+$ be a function satisfy:

(pm1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

(pm2) $p(x,x) \leq p(x,y)$,

(pm3) $p(x,y) = p(y,x)$,

(pm4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$,

for all $x, y, z \in X$. Then p is called partial metric on X and the pair (X,p) is called partial metric space.

It is clear that if $p(x,y) = 0$, then from $(pm1)$ and $(pm2)$ we obtain $x = y$. But if $x = y$, $p(x,y)$ may not be zero.

Example 1.3 [6] Let $\widetilde{X} = \mathbb{R}^+$ and $p: \widetilde{X} \times \widetilde{X} \to \mathbb{R}^+$ given by $p(x,y) = \max\{x,y\}$ for all $x, y \in R^+$. Then (R^+, p) is a partial metric space.

Example 1.4 [6] Let $\widetilde{X} = \{[\tilde{a}, \tilde{b}]: \tilde{a}, \tilde{b} \in R, \tilde{a} \leq \tilde{b}\}$. Then $p([\tilde{a}, \tilde{b}], [\tilde{c}, \tilde{d}] = \max\{\tilde{b}, \tilde{d}\} - \min\{\tilde{a}, \tilde{c}\}$ defines a partial metric p on $\widetilde{X}$.

Lemma 1.5 ([19, 20])Let (X, p) be a partial metric space. Then:

$(c1)$ a sequence x_n in (X,p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X, d_p),

$(c2)$ (X,p) is complete if and only if the metric space (X, d_p) is complete ,

$(c3)$ a subset E of a partial metric space (X,p) is closed if a sequence x_n in E such that x_n converges to some $x \in X$, then $x \in E$.

Lemma 1.6 [2] Assume that $x_n \to z$ as $n \to \infty$ in a partial metric space (X, p) such that $p(z,z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z,y)$ for every $y \in X$.

2 Main Results

In this section, we shall prove fixed point theorems using implicit relation.

Definition 2.1 (Implicit Relation) Let Ψ be the family of all real valued continuous functions $\psi: R_+ \times R_+ \times R_+ \times R_+ \to R_+$ non decreasing in the first argument for three variables. For some $\alpha \in [0,1)$, we consider the following conditions.

(Imr1) For $x, y \in R_+$, if $y \leq \psi(x, x, y, \frac{x+y}{2})$, then $y \leq \alpha x$,

(Imr2) For $x \in R_+$, if $y \leq \psi(y, 0, 0, y)$, then $y = 0$ since $\alpha \in [0,1)$, (Imr3) For $x \in R_+$, if $y \leq \psi(0,0,y,\frac{y}{2})$, then $y = 0$.

Theorem 2.2 Let $(\widetilde{X},p)$ be a complete partial metric space and let $\widetilde{T}:\widetilde{X} \to \widetilde{X}$ be a mapping, such that

$$p(\tilde{T}\tilde{u},\tilde{T}\tilde{v}) \leq \psi\{p(\tilde{u},\tilde{v}),p(\tilde{u},\tilde{T}\tilde{u}),p(\tilde{v},\tilde{T}\tilde{v}),\tfrac{p(\tilde{u},\tilde{T}\tilde{v})+p(\tilde{v},\tilde{T}\tilde{u})}{2}\} \tag{2.1}$$

for all $\tilde{u},\tilde{v} \in \tilde{X}$ and some $\tilde{\psi} \in \Psi$. If Ψ satisfies the condition $(Ir1),(Ir2)$ and $(Ir3)$, then $\tilde{T}$ has a unique fixed point in $\tilde{X}$.

Proof. For each $u_0 \in \tilde{X}$ and $n \in N$,

put $u_{n+1} = \tilde{T}u_n$. It follows from (2.1) and $(pm4)$ that

$$\begin{aligned}
p(u_n,u_{n+1}) = \quad & p(\tilde{T}u_{n-1},\tilde{T}u_n)\\
&\leq \psi\{p(u_{n-1},u_n),p(u_{n-1},\tilde{T}u_{n-1}),p(u_n,\tilde{T}u_n),\frac{p(u_{n-1},\tilde{T}u_n)+p(u_n,\tilde{T}u_{n-1})}{2}\}\\
&\leq \psi\{p(u_{n-1},u_n),p(u_{n-1},u_n),p(u_n,u_{n+1}),\frac{p(u_{n-1},u_{n+1})+p(u_n,u_n)}{2}\}\\
&\leq \psi\left\{\begin{matrix} p(u_{n-1},u_n),p(u_{n-1},u_n),p(u_n,u_{n+1}),\\ \frac{p(u_{n-1},u_n)+p(u_n,u_{n+1})-p(u_n,u_n)+p(u_n,u_n)}{2}\end{matrix}\right\}\\
&= \psi\{p(u_{n-1},u_n),p(u_{n-1},u_n),p(u_n,u_{n+1}),\tfrac{p(u_{n-1},u_n)+p(u_n,u_{n+1})}{2}\}
\end{aligned} \tag{2.2}$$

Since ψ satisfies the condition $(Imr1)$, there exists $\alpha \in [0,1)$, such that

$$p(u_n,u_{n+1}) \leq \alpha p(u_{n-1},u_n) \leq \ldots \leq \alpha^n p(x_0,x_1). \tag{2.3}$$

Set $S_n = p(u_n,u_{n+1})$

and $S_{n_1} = p(u_{n-1},u_n)$

then from (2.3), we obtain

$$S_n \leq \alpha S_{n-1} \leq \ldots \leq \alpha^n S_0.$$

Now we show that $\{u_n\}$ is a Cauchy sequence in $\tilde{X}$. Let $m,n > 0$ with $m > n$, then by using $(pm4)$ and equation (2.3), we have

$$\begin{aligned}
&p(u_n,u_m) \leq p(u_n,u_{n+1}) + p(u_{n+1},u_{n+2})+\ldots+p(u_{n+m-1},u_m)\\
&-p(u_{n+1},u_{n+1}) - p(u_{n+2},u_{n+2})-\ldots-p(u_{n+m-1},u_{n+m-1})\\
&\leq \alpha^n p(u_0,u_1) + \alpha^{n+1}p(u_0,u_1)+\ldots+\alpha^{n+m-1}p(u_0,u_1)\\
&= \alpha^n[p(u_0,u_1) + \alpha p(u_0,u_1)+\ldots+\alpha^{m-1}p(u_0,u_1)]\\
&= \alpha^n[1+\alpha+\ldots+\alpha^{m-1}]S_0\\
&\leq \alpha(\tfrac{1-\alpha^{m-1}}{1-\alpha})S_0.
\end{aligned}$$

In above inequality, taking the limits as $n, m \to \infty$, we have

$$p(u_n, u_m) \to 0 \text{ since } 0 < \alpha < 1,$$

hence, $\{u_n\}$ is a Cauchy sequence in $\tilde{X}$ and this sequence is also Cauchy in $(\tilde{X}, d_p)$.

As $(\tilde{X}, p)$ is complete, therefore $(\tilde{X}, d_p)$ is also complete.

Thus by Lemma 1.5,

$$\lim_{n\to\infty} u_n \to w$$

$$p(w,w) = lim_{n\to\infty} p(w, u_n) = lim_{n\to\infty} p(u_n, u_m) = 0, \tag{2.4}$$

implies,

$$lim_{n\to\infty} d_p(w, u_n) = 0. \tag{2.5}$$

Now, we show that w is a fixed point of $\tilde{T}$.

From equation (2.4), we have

$$p(w,w) = 0.$$

By using inequality (2.1), we get

$$\begin{aligned} p(u_{n+1}, \tilde{T}w) &= p(\tilde{T}u_n, \tilde{T}w) \\ &\leq \psi\{p(u_n, w), p(u_n, \tilde{T}u_n), p(w, \tilde{T}w), \tfrac{p(u_n, \tilde{T}w) + p(w, \tilde{T}u_n)}{2}\} \\ &= \psi\{p(u_n, w), p(u_n, u_{n+1}), p(w, \tilde{T}w), \tfrac{p(u_n, \tilde{T}w) + p(w, u_{n+1})}{2}\}. \end{aligned}$$

Note that $\psi \in \Psi$, then taking the limit as $n \to \infty$ and using equation (2.4), we get

$$p(u_{n+1}, \tilde{T}w) \leq \psi\{0,0, p(w, \tilde{T}w), \tfrac{p(w, \tilde{T}w)}{2}\}.$$

Since ψ satisfies the condition $(Imr3)$, then

$$p(w, \tilde{T}w) = 0.$$

This implies $w = \tilde{T}w$.

Therefore, w is a fixed point of $\tilde{T}$.

Now to show that the fixed point of $\tilde{T}$ is a unique .

let w_1, w_2 be fixed points of $\tilde{T}$ with $w_1 \neq w_2$.

From equations (2.1) and (2.4)

$$\begin{aligned} p(w_1, w_2) &= p(\tilde{T}w_1, \tilde{T}w_2) \\ &\leq \psi\{p(w_1, w_2), p(w_1, \tilde{T}w_1), p(w_2, \tilde{T}w_2), \tfrac{p(w_1, \tilde{T}w_2), p(w_2, \tilde{T}w_1)}{2}\} \\ &= \psi\{p(w_1, w_2), p(w_1, w_1), p(w_2, w_2), \tfrac{p(w_1, w_2) + p(w_2, w_1)}{2}\} \\ &= \psi\{p(w_1, w_2), 0, 0, p(w_1, w_2)\}. \end{aligned}$$

Since ψ satisfies the condition $(Imr2)$, we have

this implies, that $p(w_1, w_2) = 0$, since $0 < \alpha < 1$.

This implies that $w_1 = w_2$.

Thus the fixed point of $\tilde{T}$ is unique.

Theorem 2.3 Let $(\tilde{X}, p)$ be a complete partial metric space and $\tilde{F}, \tilde{G}$ be two self maps on $\tilde{X}$ satisfying the following:

$$p(\tilde{F}u, \tilde{G}v) \le \psi\{p(u,v), p(u,\tilde{F}u), p(v,\tilde{G}v), \frac{p(u,\tilde{G}v)+p(v,\tilde{F}u)}{2}\} \tag{2.6}$$

for all $u, v \in \tilde{X}$ and some $\psi \in \Psi$. Then $\tilde{F}$ and $\tilde{G}$ have a unique common fixed point in $\tilde{X}$.

Proof. For each $u_0 \in \tilde{X}$ and $n \in \mathbb{N}$.

Put $u_{n+1} = \tilde{F}u_n$ and

$u_{n+2} = \tilde{G}u_{n+1}$ for $n = 0,1,2...$

It follows from equation (2.5), Lemma 1.5 and $(pm4)$ that

$$\begin{aligned}
p(u_n, u_{n+1}) &= p(\tilde{F}u_{n-1}, \tilde{G}u_n)\\
&\le \psi\{p(u_{n-1},u_n), p(u_{n-1},\tilde{F}u_{n-1}), p(u_n,\tilde{G}u_n), \tfrac{p(u_{n-1},\tilde{G}u_n)+p(u_n,\tilde{F}u_{n-1})}{2}\}\\
&\le \psi\{p(u_{n-1},u_n), p(u_{n-1},u_n), p(u_n, u-n+1), \tfrac{p(u_{n-1},u_{n+1})+p(u_n,u_n)}{2}\}\\
&\le \psi\{p(u_{n-1},u_n), p(u_{n-1},u_n), p(u_n,u_{n+1}), \tfrac{p(u_{n-1},u_n)p(u_n,u_{n+1})-p(u_n,u_n)+p(u_n,u_n)}{2}\}\\
&= \psi\{p(u_{n-1},u_n), p(u_{n-1},u_n), p(u_n,u_{n+1}), \tfrac{p(u_{n-1},u_n)p(u_n,u_{n+1})}{2}\}
\end{aligned}$$

$$p(u_n,u_{n+1}) = \psi\{p(u_{n-1},u_n), p(u_{n-1},u_n), p(u_n,u_{n+1}), \frac{p(u_{n-1},u_n)p(u_n,u_{n+1})}{2}\}. \tag{2.7}$$

Since ψ satisfies the condition $(Imr1)$, therefore there exists $\alpha \in [0,1)$ such that

$$p(u_n,u_{n+1}) \le \alpha p(u_{n-1},u_n) \le \ldots \le \alpha^n p(x_0,x_1) \tag{2.8}$$

Now we show that $\{u_n\}$ is a Cauchy sequence

Let $m, n > 0$ with $m > n$, then by using $(pm4)$ and equation (2.3), we have

$$\begin{aligned}
&p(u_n,u_m) \le p(u_n,u_{n+1}) + p(u_{n+1},u_{n+2}) + \ldots + p(u_{n+m-1},u_m)\\
&-p(u_{n+1},u_{n+1}) - p(u_{n+2},u_{n+2}) - \ldots - p(u_{n+m-1},u_{n+m-1})\\
&\le \alpha^n p(u_0,u_1) + \alpha^{n+1}p(u_0,u_1) + \ldots + \alpha^{n+m-1}p(u_0,u_1)\\
&= \alpha^n[p(u_0,u_1) + \alpha p(u_0,u_1) + \ldots + \alpha^{m-1}p(u_0,u_1)]\\
&= \alpha^n[1 + \alpha + \ldots + \alpha^{m-1}]S_0\\
&\le \alpha(\tfrac{1-\alpha^{m-1}}{1-\alpha})S_0.
\end{aligned}$$

Assume that, $lim_{n\to\infty} u_n \to r$

by Lemma 1.5,

$$p(r,r) = lim_{n\to\infty} p(r,u_n) = lim_{n,m\to\infty} p(u_n,u_m) = 0. \quad (2.9)$$

This implies,

$$lim_{n\to\infty} d_p(r,u_n) = 0. \quad (2.10)$$

Now we have to prove that r is a common fixed point of $\tilde{F}$ and $\tilde{G}$.

$$\begin{aligned} p(u_{n+1},\tilde{F}r) &= p(\tilde{F}u_n,\tilde{F}r) \\ &\leq \psi\{p(u_n,r),p(u_n,\tilde{F}u_n),p(r,\tilde{F}r),\tfrac{p(u_n,\tilde{F}r)+p(r,\tilde{F}u_n)}{2}\} \\ &= \psi\{p(u_n,r),p(u_n,u_{n+1}),p(r,\tilde{F}r),\tfrac{p(u_n,\tilde{F}r)+p(r,u_{n+1})}{2}\} \end{aligned}$$

Note that $\psi \in \Psi$, using (2.8), Lemma 1.6 and taking the limit as $n \to \infty$,

$$p(r,\tilde{F}r) \leq \psi\{0,0,p(r,\tilde{F}r),\frac{p(r,\tilde{F}r)}{2}\}$$

Since ψ satisfies the condition $(Imr3)$, then

$p(r,\tilde{F}r) = 0$.

This shows that $r = \tilde{F}r$ for all $r \in \tilde{X}$.

Similarly, we can show $r = \tilde{G}r$.

Thus r is a common fixed point of $\tilde{F}$ and $\tilde{G}$.

Now we will prove the uniqueness of fixed point of $\tilde{F}$ and $\tilde{G}$.

Let r' be another common fixed point of $\tilde{F}$ and $\tilde{G}$, that is, $\tilde{F}r' = \tilde{G}r' = r'$ with $r' \neq r$.

Thus we have to show to show that $r = r'$.

From equations (2.5) and (2.8)

$$\begin{aligned} p(r,r') = \; & p(\tilde{F}r,\tilde{G}r') \\ &\leq \psi\{p(r,r'),p(r,\tilde{F}r),p(r',\tilde{G}r'),\tfrac{p(r,\tilde{G}r')+p(r',\tilde{F}r)}{2}\} \\ &= \psi\{p(r,r'),p(r,r),p(r',r'),\tfrac{p(r,r')+p(r',r)}{2}\} \\ &= \psi\{p(r,r'),0,0,p(r,r')\}. \end{aligned}$$

Since ψ satisfies the condition $(Imr2)$, then we get

$p(r,r') = 0$, since $0 < \alpha < 1$.

Thus, we get $r = r'$.

This shows that r is the unique common fixed point of $\tilde{F}$ and $\tilde{G}$.

Theorem 2.4 Let $(\widetilde{X}, p)$ be a complete partial metric space and $\tilde{F}_1$, $\tilde{F}_2$ be continuous mappings on $\widetilde{X}$ satisfying

$$p(\tilde{F}_1{}^m u, \tilde{F}_1{}^n v) \leq \psi\{p(u,v), p(u, \tilde{F}_1{}^m u), p(v, \tilde{F}_2{}^n v), \frac{p(u,\tilde{F}_2{}^n v)+p(v,\tilde{F}_1{}^m u)}{2} \qquad (2.11)$$

for all $u, v \in \tilde{X}$, where m and n are some positive integers and some $\psi \in \Psi$. Then $\tilde{F}_1$ and $\tilde{F}_2$ have a unique common fixed point in X.

Proof. Since $\tilde{F}_1{}^m$ and $\tilde{F}_2{}^n$ satisfy the condition of Theorem 2.2. So $\tilde{F}_1{}^m$ and $\tilde{F}_2{}^n$ have a unique common fixed point.

Then, we have

$$\tilde{F}_1{}^m \tilde{z} = \tilde{z}$$
$$\Rightarrow \tilde{F}_1(\tilde{F}_1{}^m \tilde{z}) = \tilde{F}_1 \tilde{z}$$
$$\tilde{F}_1{}^m(\tilde{F}_1 \tilde{z}) = \tilde{F}_1 \tilde{z}.$$

If $\tilde{F}_1 \tilde{z} = \tilde{r}_0$, then $\tilde{F}_1{}^m \tilde{r}_0 = \tilde{r}_0$.

So, $\tilde{F}_1 \tilde{z}$ is a point of $\tilde{F}_1{}^m$. Similarly,$\tilde{F}_2(\tilde{F}_2{}^n \tilde{z}) = \tilde{F}_2 \tilde{z}$.

Now using equation (2.11) and Lemma 1.5, we obtain

$$\begin{aligned} p(\tilde{z}, \tilde{F}_1 \tilde{z}) &= p(\tilde{F}_1{}^m \tilde{z}, \tilde{F}_1{}^m(\tilde{F}_1 \tilde{z})) \\ &\leq \{p(\tilde{z}, \tilde{F}_1 z), p(\tilde{F}_1 z, \tilde{F}_1{}^m(\tilde{F}_1 \tilde{z})), p(\tilde{z}, \tilde{F}_1{}^m \tilde{z}), \frac{p(\tilde{z}, \tilde{F}_1{}^m(\tilde{F}_1 \tilde{z})) + p(\tilde{F}_1 \tilde{z}, \tilde{F}_1{}^m \tilde{z})}{2}\} \\ &= \psi\{p(\tilde{z}, \tilde{F}_1 \tilde{z}), p(\tilde{z}, \tilde{z}), p(\tilde{F}_1 z, \tilde{F}_1 z), \frac{p(\tilde{z}, \tilde{F}_1 \tilde{z}) + p(\tilde{F}_1 \tilde{z}, \tilde{z})}{2}\} \\ &= \psi\{p(\tilde{z}, \tilde{F}_1 \tilde{z}), 0, 0, p(\tilde{z}, \tilde{F}_1 \tilde{z})\} \end{aligned}$$

Since ψ satisfies the condition $(Imr2)$, then we get

$$p(\tilde{z}, \tilde{F}_1 \tilde{z}), \text{ since } 0 < \alpha < 1.$$

Thus, we have $\tilde{z} = \tilde{F}_1 \tilde{z}$ for all $\tilde{z} \in \tilde{X}$.

Similarly, we can show that $\tilde{z} = \tilde{F}_2 \tilde{z}$.

This shows that $\tilde{z}$ is a common fixed point of $\tilde{F}_1$ and $\tilde{F}_2$.

For uniqueness of $\tilde{z}$, let $\tilde{z}' \neq \tilde{z}$ be another common fixed point of $\tilde{F}_1$ and $\tilde{F}_2$. Then clearly $\tilde{z}'$ is also a common fixed point of $\tilde{F}_1{}^m$ and $\tilde{F}_2{}^n$ which implies $\tilde{z}' = \tilde{z}$.

Hence $\tilde{F}_1$ and $\tilde{F}_2$ have a unique common fixed point.

Theorem 2.5 Let $\tilde{u}_{(}\gamma)$ be a family of continuous mappings on a complete partial metric space

$(\widetilde{X}, p)$ satisfying

$$p(\tilde{u}_(\gamma)u, \tilde{u}_(\beta)v) \leq \psi\{p(u,v), p(u, \tilde{u}_(\gamma)u), p(v, \tilde{u}_(\beta)v), \frac{p(u,\tilde{u}_(\beta)v)+p(v,\tilde{u}_(\gamma)u)}{2}\} \qquad (2.12)$$

for all $\gamma, \beta \in \Psi$ with $\gamma \neq \beta$ and $u, v \in \tilde{X}$. Then there exists a unique $\tilde{z} \in \tilde{X}$ satisfying $\tilde{u}_(\gamma)\tilde{z} = \tilde{z}$ for all $\gamma \in \Psi$.

Proof. For $u_0 \in \tilde{X}$, we define a sequence as follows:-

$u_{n+1} = \tilde{u}_(\gamma)u_n$, $u_{n+2} = \tilde{u}_(\beta)u_{n+1}$, n=0,1,2...

It follows from $(2.12), (pm4)$ and Lemma 1.5 that

$$\begin{aligned} p(u_n, u_{n+1}) = \ & p(\tilde{u}_(\gamma)u_{n-1}, \tilde{u}_(\beta)u_n) \\ & \leq \psi\{p(u_{n-1}, u_n), p(u_{n-1}, \tilde{u}_(\gamma)u_{n-1}), p(u_n, \tilde{u}_(\beta)u_n), \frac{p(u_{n-1},\tilde{u}_(\beta)u_n)+p(u_n,\tilde{u}_(\gamma)u_{n-1})}{2}\} \\ & = \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u_{n+1}), \frac{p(u_{n-1},u_{n+1})+p(u_n,u_n)}{2}\} \\ & \leq \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u_{n+1}), \frac{p(u_{n-1},u_n)+p(u_n,u_{n+1})-p(u_n,u_n)+p(u_n,u_n)}{2}\} \\ & = \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u_{n+1}), \frac{p(u_{n-1},u_n)+p(u_n,u_{n+1})}{2}\} \end{aligned}$$

$$p(u_n, u_{n+1}) \leq \psi\{p(u_{n-1}, u_n), p(u_{n-1}, u_n), p(u_n, u_{n+1}), \frac{p(u_{n-1},u_n)+p(u_n,u_{n+1})}{2}\}. \qquad (2.13)$$

Since ψ satisfies the condition $(Imr1)$, there exists $\alpha \in (0,1)$ such that

$$p(u_n, u_{n+1}) \leq \alpha p(u_{n-1}, u_n) \leq \ldots \leq \alpha^n p(x_0, x_1) \qquad (2.14)$$

Now we show that $\{u_n\}$ is a Cauchy sequence in $\tilde{X}$. Let $m, n > 0$ with $m > n$, then by using (pm4) and equation (2.14), we have

$$\begin{aligned} & p(u_n, u_m) \leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \ldots + p(u_{n+m-1}, u_m) \\ & -p(u_{n+1}, u_{n+1}) - p(u_{n+2}, u_{n+2}) - \ldots - p(u_{n+m-1}, u_{n+m-1}) \\ & \leq \alpha^n p(u_0, u_1) + \alpha^{n+1} p(u_0, u_1) + \ldots + \alpha^{n+m-1} p(u_0, u_1) \\ & = \alpha^n [p(u_0, u_1) + \alpha p(u_0, u_1) + \ldots + \alpha^{m-1} p(u_0, u_1)] \\ & = \alpha^n [1 + \alpha + \ldots + \alpha^{m-1}] S_0 \\ & \leq \alpha(\frac{1-\alpha^{m-1}}{1-\alpha}) S_0. \end{aligned}$$

Taking $\lim_{n,m\to\infty}$ in the above inequality, we get

$p(u_n, u_m) \to 0$, since $0 < \alpha < 1$.

$u_n \to \tilde{r}$ as $n \to \infty$.

Moreover by Lemma 1.5,

$$p(\tilde{r},\tilde{r}) = \lim_{n\to\infty} p(\tilde{r},u_{2n}) = \lim_{n,m\to\infty} p(\tilde{u}_n,\tilde{u}_m) = 0 \tag{2.15}$$

implies,

$$\lim_{n\to\infty} d_p(\tilde{r},u_n) = 0 \tag{2.16}$$

By the continuity of $\tilde{u}_\alpha$ and $\tilde{u}_\beta$, it is clear that

$$\tilde{u}_\gamma \tilde{r} = \tilde{u}_\beta \tilde{r} = \tilde{r}.$$

Therefore $\tilde{r}$ is a common fixed point of $\tilde{u}_\gamma$ for all $\gamma \in \Psi$.

To prove the uniqueness, let us consider the another common fixed point $\tilde{r}'$ of $\widetilde{U}_\gamma$ and $\widetilde{U}_\beta$ where

$$\tilde{r} \neq \tilde{r}'$$

from (2.12) and (2.15), we obtain

$$\begin{aligned} p(\tilde{r},\tilde{r}') &= p(\widetilde{U}_\gamma \tilde{r}, \widetilde{U}_\beta \tilde{r}') \\ &\leq \psi\{p(\tilde{r},\tilde{r}'), p(\tilde{r},\widetilde{U}_\gamma \tilde{r}), p(\tilde{r}',\widetilde{U}_\beta), \tfrac{p(\tilde{r},\widetilde{U}_\beta \tilde{r}')+p(\tilde{r}',\widetilde{U}_\gamma \tilde{r})}{2}\} \\ &= \psi\{p(\tilde{r},\tilde{r}'), p(\tilde{r},\tilde{r}), p(\tilde{r}',\tilde{r}'), \tfrac{p(\tilde{r},\tilde{r}')+p(\tilde{r}',\tilde{r})}{2}\} \\ &= \psi\{p(\tilde{r},\tilde{r}'),0,0,p(\tilde{r},\tilde{r}')\} \end{aligned}$$

Since ψ satisfies the condition $(Imr2)$, then we get

$$p(\tilde{r},\tilde{r}') = 0, \text{ since } 0 < \alpha < 1.$$

Thus, we get $\tilde{r} = \tilde{r}'$ for all $\tilde{r} \in \tilde{X}$.

This shows that $\tilde{r}$ is a unique common fixed point of $\widetilde{U}_\gamma$ for all $\gamma \in \Psi$.

References

[1] M. Abbas, T. Nazir and S. Ramaguera, Fixed point results for generalized cyclic con-tractive mappings in partial metric spaces, *Revista de la Real Academia de CienciasExactas, F´Ãŋsicas y Naturales, Serie A – Matemáticas (RACSAM)*, **106** (1) (2012), pp. 287–297.

[2] T. Abdeljawad, E. Karapinar and K. Tas, Existence and uniqueness of common fixedpoint partial metric spaces, *Applied Mathematics Letters*, **24** (11) (2011), pp. 1900–1904.

[3] O. Acar, V. Berinde and I. Altun, Fixed point theorems for Ciric-type strong almost contractions on partial metric spaces, *Journal of Fixed Point Theory and*

Applica-tions, **12** (2012), pp. 247–259.

[4] I. Altun, F. Sola and H. Simsek, Generalized contractions on partial metric spaces, *Topology and its Applications*, **157** (2010), pp. 2778–2785.

[5] I. Altun and A. Erduran, Fixed point theorems for monotone mappings on partialmetric spaces, *Fixed Point Theory and Applications*, 2011 (2011), ArticleID 508730, 10 pages.

[6] H. Aydi, M. Abbas and C. Vetro, Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces, *Topology and its Applications*, **159** (14) (2012), pp. 3234–3242.

[7] G. V. R. Babu and P. D. Sailaja, Coupled fixed point theorems with new implicit relations and an application, *Journal of Operators*, 2014 (2014), ArticleID 389646,16 pages.

[8] S. Banach, Surles operation dans les ensembles abstraits et leur application aux equa-tion integrals, *Fundamenta Mathematicae*, **3** (1922), pp. 133–181.

[9] V. Berinde, and F. Vetro, Common fixed points of mappings satisfying implicit con-tractive conditions, *Fixed Point Theory and Applications*, 2012 (2012), ArticleID2012:105, 12 pages.

[10] S. Beloul and A. Tomar, A coincidence and common fixed point theorem for subse-quentially continuous hybrid pairs of maps satisfying an implicit relation, *Mathemat-ica Moravica*, **21** (2) (2017), pp. 15–25.

[11] S. K. Chatterjae, Fixed point theorems compactes, *Comptes rendus de I'Academie bulgare des Sciences,* **25** (1972), pp. 727-730.

[12] R. Heckmann, Approximation of metric spaces by partial metric spaces, *Applied Cat-egorical Structures*, **7** (1) (1999), pp. 71–83.

[13] R. Kannan, Some results on fixed point theorems, *Bulletin of Calcutta Mathematical Society,* **60** (1969), pp. 71-78.

[14] E. Karapinar, Generalization of Caristi-Kirk's theorem on partial metric spaces, *Fixed Point Theory and Applications*, 2011 (4) (2011).

[15] E. Karapinar, Weak φ-contraction on partial metric spaces, *Journal of Computational Analysis and Applications*, **14** (2) (2012), pp. 206–210.

[16] E. Karapinar, W. Shatanawi and K. Tas, Fixed point theorems on partial met-ric

spaces involving rational expressions, *Miskolc Mathematical Notes*, **14** (2013), pp. 135–142.

[17] E. Karapinar and Inci M. Erhan, Fixed point theorems for operators on partial metricspaces, *Applied Mathematics Letters*, 24 (2011), pp. 1894–1899.

[18] S. Manro, *A common fixed point theorem in Fuzzy metric space using implicit relation*,International Journal of Mathematics and Statistics, **16** (2) (2015), 35–42.

[19] S. G. Matthews, Partial metric topology, *Research report 2012*, *Deptartment of Com-puter Science*, University of Warwick, 1992.

[20] S. G. Matthews, Partial metric topology, Proceedings of the 8th summer conferenceon topology and its applications, *Annals of the New York Academy of Sciences*, 728 (1994), pp. 183–197.

[21] H. K. Nashine, Common fixed point theorems under implicit relations on ordered partial metric spaces and application to integral equation, *Bulletin of Mathematical Sciences*, **3** (2013), 18–204.

[22] V. Popa, Some fixed point theorems for compatible mappings satisfying an implicit relation, *Demonstratio Mathematica*, **32** (1) (1999), pp. 157–164.