

# Application of the Some Blaise Abbo (SBA) method to solving the time-fractional Schrödinger equation and comparison with the homotopy perturbation method

## **Abstract**

We have solved the Schrödinger equation with the HPM method and the SBA method. We have noticed that with these two methods we find the same result.

**Key words :** Fractional equation, Some Blaise Abbo (SBA) method, the homotopy perturbation method (HPM) ; Schrödinger equation ; fractional Partial Differential Equations (PDEs).

**Mathematics Subject Classification :** 49J20 ; 58J45 ; 35L86 ; 81T13.

## **1 Introduction**

For more than three decades, mathematicians and research authors have paid special attention to fractional calculus because a fractional derivative is not a derivative at the local point, it considers history and non-local distributed effects.

Therefore, fractional mathematical models are more realistic and practical than other classical models. Thus, while the latter do not include information on mechanisms related to memory and learning, Fractional models take into account past and distributed effects in the system under study. Any dynamic process modeled by fractional equations then has a memory effect. This is why applications of fractional calculus can be found in several scientific fields such as : quantum mechanics, electrical signal propagation, biomechanics, signal processing and bio-engineering.

In the field of quantum mechanics, the resolution of the Schrödinger equation, which allows to determine the different energies or eigenvalues of the energy of an atomic, nuclear or solid state system, has been the subject of several studies, among others, Wazwaz [1] for the variational iteration method (VIM), Ravi Kanth [2] for the differential transformation method (DTM), and Golmankhaneh et al [3] for the homotopy perturbation method (HPM).

For the time-dependent Schrödinger equation, many efforts are made to try to find robust numerical and analytical methods, e.g. Rams Wroop et al using the homotopy analysis transformation method (HATM), Khan et al [4] the homotopy analysis method (HAM). Twenty years ago, the Some Blaise Abbo (SBA) method was proposed by an African team to solve linear and nonlinear ODEs and PDEs.

In this paper, we propose to extend the applications of the SBA method to the solution of the fractional time dependent Schrödinger equation.

This paper is organized as follows : in section 2, the fractional time-dependent Schrödinger equation is described and we give some basic definitions and properties. In sections 3 and 4, we present some properties of the SBA method and the HPM method respectively. In section 5, we present and compare the numerical solutions of the time-fractional Schrödinger equation in time using these two methods. Finally, conclusions are given in section 6.

## 2 Preliminaries and description of the fractional Schrödinger equation

The Schrödinger Equation, first obtained in 1926 by Erwin Schrödinger, was an extension of de Broglie's hypotheses, proposed two years earlier, that each material particule has associated with it a wave-length  $\lambda$  related to the linear manentum  $p$  of the partide by the equation :  $\lambda = \frac{h}{p}$  where  $h$  is planck's constant.

If the time-independent of the Schrödinger Equation is given by

$$H \left( -ih \frac{\partial}{\partial q_i}, q_i \right) u = -ih \frac{\partial u}{\partial t}$$

$u$  is a wave function,  $q_i$  is a dynamical variable and  $H$  the Hamiltonian operateur. In the peper, the time-fractional Schrödinger Equation (FSE) has the following form :

$$\begin{cases} i^c D_t^\alpha u(x, t) = \gamma \nabla^2 (u(x, t)) + V_d(x)u(x, t) + q|u(x, t)|^2 u(x, t), & 0 < \alpha \leq 1, \quad t > 0 \\ u(x, 0) = f(x), & x \in \mathbb{R}^d \end{cases} \quad (1)$$

where  $V_d(x)$  is the trapping potential and  $\gamma, q$  is a reals constants. The physical model (1) and its generalized forms occur in various areas of physics, including nonlinear optics, plasma physics, superconductivity, and quantum mechanics.

We give some basic definitions, notations, and properties of the fractional calculus theory, which will be used later in this work.

### 2.1 Definition

The Euler Gamma function is defined by [5;6;7].

$$\Gamma(z) = \int_0^\infty t^{z+1} e^{-t} dt \quad (2)$$

with  $z$  is any complex, number such that  $Re(z) > 0$ . The function is strictly decreasing on  $]0; 1]$ .

## 2.2 Definition

The Beta function is defined by [5;6;7]

$$\beta(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt \quad (3)$$

with  $Re(p) > 0$ ,  $Re(q) > 0$ . The relation between the beta function and the gamma function is given by :

$$\beta(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad (4)$$

## 2.3 Definition

The Mittag-Leffler function is defined by [5;6;7]

$$E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha + 1)} \quad (5)$$

where  $z$  is a complex  $\alpha$  is a strictly positive real.

## 2.4 Definition

Let  $f \in C([a, b])$ . The operator  $I_a^\alpha$  defined on  $[a, b]$  by :

$$(I_a^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha > 0 \quad (6)$$

is called Riemann-Liouville fractional integral of order  $\alpha$ .

## 2.5 Propriety

Let  $\alpha$  and  $\beta$  two complex numbers,  $f \in C([a, b])$

- i)  $I_a^\alpha(I_a^\beta f) = I_a^{\alpha+\beta} f$ ,  $Re(\alpha) > 0$ ,  $Re(\beta) > 0$ .
- ii)  $\frac{d}{dt}(I_a^\alpha f)(t) = (I_a^{\alpha+1} f)(t)$ ,  $Re(\alpha) > 0$ .
- iii)  $\lim_{\alpha \rightarrow 0^+} (I_a^\alpha f)(t) = f(t)$ ,  $Re(\alpha) > 0$ .

## 2.6 Definition

The caputo-type fractional derivative of order  $\alpha > 0$  of a function  $f \in C_{-1}^m$  ( $m = 1, 2, 3, \dots$ ) is given by [5;6;7]

$$\left\{ \begin{array}{l} {}^c D^\alpha = I^{n-\alpha} u^{(n)} = \frac{\partial^\alpha u(x, s)}{\partial s^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \frac{\partial^n u(x, s)}{\partial s^n} ds, \quad \text{if } n-1 < \alpha < n \\ \frac{d^n}{dt^n} f, \quad \text{if } \alpha = n \end{array} \right. \quad (7)$$

where  $n = [\alpha] + 1$  is the integer part of the real number.

## 3 The SBA numerical method

Consider the following functional equation [6;7;11]

$$A(u) = h \quad (8)$$

where  $A : H \rightarrow H$  is a linear or nonlinear operator,  $H$  a real Hilbert space,  $h$  a given function in  $H$  and  $u$  the unknown function in  $H$ .

By setting  $A = L - R - N$ , where  $L$  is a linear operator assumed to be invertible, invertible in the Adomian sense,  $R$  a linear operator and  $N$  a nonlinear operator.

We obtain

$$L(u) - R(u) - N(u) = h \quad (9)$$

By applying the inverse  $L^{-1}$  of  $L$  to (9), we obtain the following canonical form of Adomian

$$u = \theta + L^{-1}(h) + L^{-1}(R(u)) + L^{-1}(N(u)) \quad (10)$$

where  $\theta$  satisfies  $L\theta = 0$

Apply the method of successive approximations to (10), we obtain :

$$u^k = u^k(0) + L^{-1}(h^k) + L^{-1}(R(u^k)) + L^{-1}(N(u^{k-1})) \quad k \geq 1 \quad (11)$$

From (11), deduce the following SBA algorithm for a fixed  $k$  :

$$\left\{ \begin{array}{l} u_0^k = u^k(0) + L^{-1}(h^k) + L^{-1}(N(u^{k-1})) \quad k \geq 1 \\ u_n^k = L^{-1}(R(u_{n-1}^k)), \quad n \geq 1 \end{array} \right. \quad (12)$$

We then apply the Picard principle to (12) : in choosing  $u^0$ , such that  $Nu^0 = 0$ , this choice in fact allows the first iteration to solve only a linear problem and this principle will be checked at each iteration before continuing the calculations.

### First iteration

For  $k = 1$ , we calculate  $u^1$  using the following algorithm

$$\left\{ \begin{array}{l} u_0^1 = u^1(0) + L^{-1}(h^1) \\ u_n^1 = L^{-1}(R(u_{n-1}^1)), \quad n \geq 1 \end{array} \right. \quad (13)$$

If the series  $\sum_{n=0}^{+\infty} u_n^1$  is convergent, then we get :

$$u^1 = \sum_{n=0}^{+\infty} u_n^1 \quad (14)$$

Approximate solution of equation (8) in step  $k = 1$ .

**Second iteration**

For  $k = 2$ , we calculate  $u^2$  using the following algorithm

$$\begin{cases} u_0^2 = u^2(0) + L^{-1}(h^2) \\ u_n^2 = L^{-1}(R(u_{n-1}^2)), \quad n \geq 1 \end{cases} \quad (15)$$

If the series  $\sum_{n=0}^{+\infty} u_n^2$  is convergent, then we get :

$$u^2 = \sum_{n=0}^{+\infty} u_n^2 \quad (16)$$

Approximate solution of equation (8) in step  $k = 2$ .

**k-th iteration**

Recursively, if the series  $\sum_{n=0}^{+\infty} u_n^k$  is convergent for  $k \geq 1$ , then we get :

$$u^k = \sum_{n=0}^{+\infty} u_n^k \quad k \geq 1 \quad (17)$$

Approximate solution of equation (8) in step  $k$ .

So the solution of the problem (8) is then :

$$u = \lim_{k \rightarrow +\infty} u^k = \lim_{k \rightarrow +\infty} \left( \sum_{n=0}^{+\infty} u_n^k \right). \quad (18)$$

## 4 The homotopy perturbation method

To illustrate the fundamental ideas of homotopy perturbation, we consider the following nonlinear equation [5,8,9,10] :

$$A(u) - f(x) = 0, \quad x \in \Omega \quad (19)$$

with the boundary conditions

$$B \left( u, \frac{\partial u}{\partial n} \right) = 0, \quad x \in \partial\Omega \quad (20)$$

or

\* A is usually a differential operator

- \* B designates the boundary conditions
- \*  $f(x)$  is an analytically known function
- \*  $\partial\Omega$  is the domain boundary  $\Omega$

The method consists in decomposing the nonlinear operator in the form

$$A = L + R + N \quad (21)$$

or

$L + R$  is a linear operator,  $N$  is a nonlinear operator.

Therefore equation (19) becomes :

$$L(u) + R(u) + N(u) - f(x) = 0, \quad x \in \Omega \quad (22)$$

with the homotopy technique, we construct a homoto

$$v(x, p) : \Omega \times [0, 1] \rightarrow \mathbb{R} \quad (23)$$

which satisfies :

$$H(v, p) = (1 - p)(L(v) - L(u_0)) + p(L(v) + R(v) + N(v) - f(x)) = 0 \quad (24)$$

$p \in [0, 1]$  and  $x \in \Omega$

by a simplification we find the following relation :

$$H(v, p) = L(v) - L(u_0) + p(L(u_0) + R(v) + N(v) - f(x)) = 0 \quad (25)$$

$p \in [0, 1]$  and  $x \in \Omega$

$\Leftrightarrow$

$$L(v) = L(u_0) + p[f(x) - L(u_0) - R(v) - N(v)] \quad (26)$$

$p \in [0, 1]$  and  $x \in \Omega$

or

\*  $p$  is a parameter that varies from 0 to 1.

\*  $u_0$  is an initial approximation of (24) which satisfies the boundary conditions.

Obviously, by considering equations (24) we have :

$$H(v, 0) = L(v) - L(u_0) = 0 \quad (27)$$

$$H(v, 1) = (L(v) + R(v) + N(v) - f(x)) = 0 \quad (28)$$

According to the homotopy perturbation method, we can first use the parameter  $p$  as a small parameter and assume that the solution of equation (19) can be written as a series of power  $p$ .

By varying  $p$  from 0 to 1, we change  $v(x, p)$  from  $u_0(x)$  to  $u(x)$ , in topology, this is called the deformation,  $L(v) - L(u_0)$  and  $f(x) - L(u_0) - R(v) - N(v)$  are homotopy. The basic principle and that the solution of equations (24) and (25) can be written as a series :

$$v(x, p) = \sum_{n=0}^{\infty} p^n v_n(x) \quad (29)$$

and

$$N(v(x, p)) = \sum_{n=0}^{\infty} p^n H_n \quad (30)$$

where  $H_n$  are polynomials of He which are defined by :

$$H_n(v_0, v_1, v_2, \dots, v_n) = \frac{1}{n!} \frac{d^n}{dp^n} \left[ N \left( \sum_{i=0}^{\infty} p^i v_i(x) \right) \right]_{p=0} ; \quad n \geq 0 \quad (31)$$

(29) and (30) in (26) gives

$$\sum_{i=0}^{\infty} p^i L(v_i) = L(u_0) + p(f(x) - L(u_0)) - \sum_{i=0}^{\infty} p^{i+1} (R(v_i) + H_i) \quad (32)$$

$\Leftrightarrow$

$$\begin{aligned} p^0 L(v_0) + p^1 L(v_1) + p^2 L(v_2) + p^3 L(v_3) + \dots &= p^0 L(u_0) + P^1(f(x) - L(u_0)) \\ &\quad - p^1 (R(v_0) + H_0) - p^2 (R(v_1) + H_1) \\ &\quad - p^3 (R(v_2) + H_2) - p^4 (R(v_3) + H_3) - \dots \end{aligned}$$

By identifying the terms with those of the same power of  $p$ , we find

$$\begin{cases} p^0 : L(v_0) = L(u_0) \\ p^1 : L(v_1) = f(x) - L(u_0) - R(v_0) - H_0 \\ p^2 : L(v_2) = -R(v_1) - H_1 \\ p^3 : L(v_3) = -R(v_2) - H_2 \\ \vdots \end{cases} \quad (33)$$

so we conclude that

$$\begin{cases} p^0 : v_0 = u_0 \\ p^1 : L(v_1) = f(x) - L(u_0) - R(v_0) - H_0 \\ p^2 : L(v_2) = -R(v_1) - H_1 \\ p^3 : L(v_3) = -R(v_2) - H_2 \\ \vdots \end{cases} \quad (34)$$

the approximate solution of equation (19) is written :

$$u = \lim_{p \rightarrow 1} v = \sum_{n=0}^{\infty} v_n(x).$$

## 5 Comparing the two methods

In this section, we apply the homotopy perturbation method (HPM) and Some Blaise Abbo (SBA) method to some nonlinear partial differential equations.

Consider the following nonlinear time-fractionar Schrödinger equation :

$$\begin{cases} i^c D_t^\alpha u(x, t) = -\frac{\partial^2 u(x, t)}{\partial x^2} - q|u(x, t)|^2 u(x, t) \\ u(x, 0) = e^{i\beta x} \end{cases} \quad (35)$$

or  $0 < \alpha \leq 1$ ,  $x \in \mathbb{R}$  and  $t > 0$ .

## 5.1 Solving by the HPM method

(35) can also be written

$$i \frac{\partial^2 u(x, t)}{\partial x^2} + iq|u(x, t)|^2 u(x, t) - {}^c D_t^\alpha u(x, t) = 0 \quad (36)$$

with

$$\begin{cases} L_t u = {}^c D_t^\alpha u \Leftrightarrow L_t^{-1} u = I_t^\alpha u \\ N(u(x, t)) = |u(x, t)|^2 u(x, t) \end{cases} \quad (37)$$

with the homotopy technique, we construct a homotopy according to

$${}^c D_t^\alpha v(x, t) - {}^c D_t^\alpha u_0(x, t) = p \left[ i \frac{\partial^2 v(x, t)}{\partial x^2} + iqN(v(x, t)) - {}^c D_t^\alpha u_0(x, t) \right] \quad (38)$$

with  $u_0(x, t) = u(x, 0) = e^{i\beta x}$  the initial approximation of (35) that is

$$\begin{cases} {}^c D_t^\alpha v(x, t) - {}^c D_t^\alpha u_0(x, t) = p \left[ i \frac{\partial^2 v(x, t)}{\partial x^2} + iqN(v(x, t)) - {}^c D_t^\alpha u_0(x, t) \right] \\ v(x, 0) = e^{i\beta x} \end{cases} \quad (39)$$

with

$$N(v) = |v(x, t)|^2 v(x, t) \quad (40)$$

we pose

$$\begin{cases} v(x, t) = \sum_{n=0}^{\infty} p^n v_n(x, t) \\ N(v(x, t)) = \sum_{n=0}^{\infty} p^n H_n \end{cases} \quad (41)$$

and

$$H_n = \frac{1}{n} \frac{d^n}{dp^n} \left[ N \left( \sum_{i=0}^{\infty} p^i v_i \right) \right]_{p=0} \quad (42)$$

then (41) in (39) gives

$$\begin{cases} \sum_{n=0}^{\infty} p^n ({}^c D_t^\alpha v_n(x, t)) - {}^c D_t^\alpha u_0(x, t) = \sum_{n=0}^{\infty} p^{n+1} \left[ i \frac{\partial^2 v_n(x, t)}{\partial x^2} + iqH_n \right] - p [{}^c D_t^\alpha u_0(x, t)] \\ \sum_{n=0}^{\infty} p^n v_n(x, 0) = e^{i\beta x} \end{cases} \quad (43)$$

where the terms of the sequence  $(H_n)$  of the polynomials of he are

$$\begin{cases} H_0 = v_0^2(x, t) \bar{v}_0(x, t) \\ H_1 = 2v_0(x, t)v_1(x, t)\bar{v}_0(x, t) + v_0^2(x, t)\bar{v}_1(x, t) \\ H_2 = 2v_0(x, t)v_2(x, t)\bar{v}_0(x, t) + v_1^2(x, t)\bar{v}_0(x, t) + 2v_0(x, t)v_1(x, t)\bar{v}_1(x, t) + v_0^2(x, t)\bar{v}_2(x, t) \\ \vdots \end{cases}$$

by identifying terms with the same powers of  $p$ , we get

$$p^0 : \begin{cases} {}^c D_t^\alpha v_0(x, t) - {}^c D_t^\alpha u_0(x, t) = 0 \\ v_0(x, 0) = e^{i\beta x} \end{cases} \quad (44)$$

$$p^1 : \begin{cases} {}^c D_t^\alpha v_1(x, t) = i \frac{\partial^2 v_0(x, t)}{\partial x^2} + iqH_0 - {}^c D_t^\alpha u_0(x, t) \\ v_1(x, 0) = 0 \end{cases} \quad (45)$$

$$p^2 : \begin{cases} {}^c D_t^\alpha v_2(x, t) = i \frac{\partial^2 v_1(x, t)}{\partial x^2} + iqH_1 \\ v_2(x, 0) = 0 \end{cases} \quad (46)$$

$$p^3 : \begin{cases} {}^c D_t^\alpha v_3(x, t) = i \frac{\partial^2 v_2(x, t)}{\partial x^2} + iqH_2 \\ v_3(x, 0) = 0 \end{cases} \quad (47)$$

etc. Thus the resolution of the systems (44)-(47) gives

$$v_0(x, t) = u_0(x, t) = e^{i\beta x}$$

$$p^1 : \begin{cases} H_0 = e^{i2\beta x} e^{-i\beta x} = e^{i\beta x} \\ {}^c D_t^\alpha v_1(x, t) = i \frac{\partial^2}{\partial x^2} (e^{i\beta x}) - {}^c D_t^\alpha (e^{i\beta x}) \\ {}^c D_t^\alpha v_1(x, t) = (iq - i\beta^2) e^{i\beta x} \\ I_t^\alpha ({}^c D_t^\alpha v_1(x, t)) = I_t^\alpha ((iq - i\beta^2) e^{i\beta x}) \\ v_1(x, t) = (iq - i\beta^2) e^{i\beta x} I_t^\alpha(1) \\ v_1(x, t) = \frac{i(q - \beta^2)}{\Gamma(\alpha + 1)} e^{i\beta x} t^\alpha \end{cases}$$

$$p^2 : \begin{cases} H_1 = 2i \frac{(q - \beta^2)}{\Gamma(\alpha + 1)} e^{i\beta x} t^\alpha - i \frac{(q - \beta^2)}{\Gamma(\alpha + 1)} e^{i\beta x} t^\alpha = i \frac{(q - \beta^2)}{\Gamma(\alpha + 1)} e^{i\beta x} t^\alpha \\ {}^c D_t^\alpha v_2(x, t) = \left( -\frac{(q - \beta^2)^2}{\Gamma(\alpha + 1)} \right) e^{i\beta x} t^\alpha \\ v_2(x, t) = \left( -\frac{(q - \beta^2)^2}{\Gamma(\alpha + 1)} \right) e^{i\beta x} I_t^\alpha(t^\alpha) \\ v_2(x, t) = \left( -\frac{(q - \beta^2)^2}{\Gamma(\alpha + 1)} \right) e^{i\beta x} \frac{\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} t^{2\alpha} \\ v_2(x, t) = \frac{[i(q - \beta^2)]^2}{\Gamma(2\alpha + 1)} e^{i\beta x} t^{2\alpha} \end{cases}$$

$$v_n(x, t) = \frac{[i(q - \beta^2)]^n}{\Gamma(n\alpha + 1)} e^{i\beta x} t^{n\alpha}$$

$$v(x, t) = \left( \sum_{n=0}^{\infty} p^n \frac{[i(q - \beta^2) t^\alpha]^n}{\Gamma(n\alpha + 1)} \right) e^{i\beta x}$$

Finally the exact solution is :

$$u(x, t) = \lim_{p \rightarrow 1} v(x, t) = \left( \sum_{n=0}^{\infty} \frac{[i(q - \beta^2)t^\alpha]^n}{\Gamma(n\alpha + 1)} \right) e^{i\beta x} = e^{i\beta x} E_\alpha(i(q - \beta^2)t^\alpha).$$

the exact solution of the Schrödinger equation for  $\alpha = 1$  is :

$$u(x, t) = e^{i[\beta x + (q - \beta^2)t]}$$

## 5.2 Solving by the SBA method

Consider the following problem :

$$(P) : \begin{cases} {}^c D_t^\alpha u(x, t) = i \frac{\partial^2 u(x, t)}{\partial x^2} + iq|u(x, t)|^2 u(x, t) \\ u(x, 0) = g(x) \end{cases} \quad (48)$$

let's put  $L(u) = {}^c D_t^\alpha u(x, t)$ ;  $R(u) = i \frac{\partial^2 u(x, t)}{\partial x^2}$  and  $N(u) = iq|u(x, t)|^2 u(x, t)$   
we have :

$$Lu(x, t) = Ru(x, t) + N(u(x, t)) \quad (49)$$

### Theorem

If  $\forall k \geq 1$ ,  $N(u^{k-1}(x, t)) = 0$ ,  $\left| \frac{MT^\alpha}{\Gamma(\alpha + 1)} \right| < 1$ ,  $g \in C(\mathbb{R})$ ,  $u(x, t) \in C^2(\Omega)$  such as  
 $\exists m = \sup_{x \in \Omega} g(x)$  and  $\exists M = \sup_{(x, t) \in \Omega} u(x, t) > 0$  ou  $\Omega = \mathbb{R} \times [0; T]$  then the SBA algorithm is  
convergent and the problem (P) has a unique solution.

Proof. we have the following SBA algorithm :

$$\begin{cases} u_0^k(x, t) = g(x) + N(u^{k-1}(x, t)); & k \geq 1 \\ u_{n+1}^k(x, t) = I_0^\alpha(R(u_n^k(x, t))); & n \geq 0 \end{cases} \quad (50)$$

or even

$$\begin{cases} u_0^k(x, t) = g(x); & k \geq 1 \\ u_{n+1}^k(x, t) = I_0^\alpha(R(u_n^k(x, t))); & n \geq 0 \end{cases} \quad (51)$$

$$\left\{ \begin{array}{l} |u_0^k(x, t)| = |g(x)| \leq m; \quad k \geq 1 \\ |u_1^k(x, t)| = |I_0^\alpha(R(u_0^k(x, t)))| \leq \frac{MT^\alpha}{\Gamma(\alpha + 1)} \\ |u_2^k(x, t)| = |I_0^\alpha(R(u_1^k(x, t)))| \leq \left( \frac{MT^\alpha}{\Gamma(\alpha + 1)} \right)^2 \\ |u_3^k(x, t)| = |I_0^\alpha(R(u_2^k(x, t)))| \leq \left( \frac{MT^\alpha}{\Gamma(\alpha + 1)} \right)^3 \\ \vdots = \vdots \\ |u_n^k(x, t)| = |I_0^\alpha(R(u_n^k(x, t)))| \leq \left( \frac{MT^\alpha}{\Gamma(\alpha + 1)} \right)^n; \quad n > 0 \end{array} \right.$$

Summing member by member, we get :

$$\sum_{n=0}^{+\infty} |u_n^k(x, t)| = m + \frac{MT^\alpha}{\Gamma(\alpha + 1) - MT^\alpha}$$

from where  $\sum_{n=0}^{+\infty} |u_n^k(x, t)|$  is absolutely convergent by sequence  $\sum_{n=0}^{+\infty} u_n^k(x, t)$  simply convergent.

### The solution uniformity

Let  $u(x, t)$  and  $v(x, t)$  be two solutions of (48) with  $u(x, t) \neq v(x, t)$  and for  $u$  and  $v$  we have the following algorithms

$$\begin{cases} u_0^k(x, t) = e^{i\beta x}; & k \geq 1 \\ u_{n+1}^k(x, t) = I_0^\alpha(R(u_n^k(x, t))); & n \geq 0 \end{cases} \quad (52)$$

and

$$\begin{cases} v_0^k(x, t) = e^{i\beta x}; & k \geq 1 \\ v_{n+1}^k(x, t) = I_0^\alpha(R(v_n^k(x, t))); & n \geq 0 \end{cases} \quad (53)$$

by making the difference of the two algorithms we obtain :

$$\left\{ \begin{array}{l} u_0^k(x, t) - v_0^k(x, t) = e^{i\beta x} - e^{i\beta x} = 0 \\ u_1^k(x, t) - v_1^k(x, t) = \frac{-i\beta^2 t^\alpha}{\Gamma(\alpha + 1)} e^{i\beta x} - \frac{-i\beta^2 t^\alpha}{\Gamma(\alpha + 1)} e^{i\beta x} = 0 \\ u_2^k(x, t) - v_2^k(x, t) = \frac{(-i\beta^2 t^\alpha)^2}{\Gamma(2\alpha + 1)} e^{i\beta x} - \frac{(-i\beta^2 t^\alpha)^2}{\Gamma(2\alpha + 1)} e^{i\beta x} = 0 \\ u_3^k(x, t) - v_3^k(x, t) = \frac{(-i\beta^2 t^\alpha)^3}{\Gamma(3\alpha + 1)} e^{i\beta x} - \frac{(-i\beta^2 t^\alpha)^3}{\Gamma(3\alpha + 1)} e^{i\beta x} = 0 \\ \vdots = \vdots \\ u_n^k(x, t) - v_n^k(x, t) = \frac{(-i\beta^2 t^\alpha)^n}{\Gamma(n\alpha + 1)} e^{i\beta x} - \frac{(-i\beta^2 t^\alpha)^n}{\Gamma(n\alpha + 1)} e^{i\beta x} = 0 \end{array} \right.$$

so  $u_n^k(x, t) - v_n^k(x, t) = 0 \Rightarrow u(x, t) = v(x, t)$  or according to the hypothesis  $u(x, t) \neq v(x, t)$  which is contradictory, so the solution is unique.

### Application

of (35) on a so :

$$L(u) = R(u) + N(u) \quad (54)$$

apply  $L^{-1} = I_0^\alpha(\cdot)$  the fractional integral to (54), we have :

$$u(x, t) = e^{i\beta x} + I_0^\alpha(R(u(x, t))) + I_0^\alpha(N(u(x, t))) \quad (55)$$

Applying the method of successive approximations to (55), we have

$$u^k(x, t) = e^{i\beta x} + I_0^\alpha(R(u^k(x, t))) + I_0^\alpha(N(u^{k-1}(x, t))); \quad k \geq 1 \quad (56)$$

of (56), we obtain the following SBA algorithm :

$$\begin{cases} u_0^k(x, t) = e^{i\beta x} + I_0^\alpha(N(u^{k-1}(x, t))); & k \geq 1 \\ u_{n+1}^k(x, t) = I_0^\alpha(R(u_n^k(x, t))); & n \geq 0 \end{cases} \quad (57)$$

At step  $k = 1$ , we have :

$$\begin{cases} u_0^1(x, t) = e^{i\beta x} + I_0^\alpha(N(u^0(x, t))) \\ u_{n+1}^1(x, t) = I_0^\alpha(R(u_n^1(x, t))); \quad n \geq 0 \end{cases} \quad (58)$$

Applying Picard's principle, (58), we will choose  $u^0$  such that  $N(u^0) = 0$  so we will take  $u^0 = 0$

The algorithm above becomes :

$$\begin{cases} u_0^1(x, t) = e^{i\beta x} \\ u_{n+1}^1(x, t) = I_0^\alpha(R(u_n^1(x, t))); \quad n \geq 0 \end{cases} \quad (59)$$

let's calculate  $u^1(x, t)$

for  $n = 0$ , we have

$$\begin{aligned} u_1^1(x, t) &= I_0^\alpha(R(u_0^1(x, t))) = iI_0^\alpha\left(\frac{\partial^2 u_0^1(x, t)}{\partial x^2}\right) \\ &= \frac{-i\beta^2 t^\alpha}{\Gamma(\alpha + 1)} e^{i\beta x} \end{aligned}$$

for  $n = 1$ , we have

$$\begin{aligned} u_2^1(x, t) &= I_0^\alpha(R(u_1^1(x, t))) = iI_0^\alpha\left(\frac{\partial^2 u_1^1(x, t)}{\partial x^2}\right) \\ &= \frac{(-i\beta^2 t^\alpha)^2}{\Gamma(2\alpha + 1)} e^{i\beta x} \end{aligned}$$

for  $n = 2$ , we have

$$\begin{aligned} u_3^1(x, t) &= I_0^\alpha(R(u_2^1(x, t))) = iI_0^\alpha\left(\frac{\partial^2 u_2^1(x, t)}{\partial x^2}\right) \\ &= \frac{(-i\beta^2 t^\alpha)^3}{\Gamma(3\alpha + 1)} e^{i\beta x} \end{aligned}$$

in a recursive way we have :

$$\begin{cases} u_0^1(x, t) = e^{i\beta x} \\ u_1^1(x, t) = \frac{-i\beta^2 t^\alpha}{\Gamma(\alpha + 1)} e^{i\beta x} \\ u_2^1(x, t) = \frac{(-i\beta^2 t^\alpha)^2}{\Gamma(2\alpha + 1)} e^{i\beta x} \\ u_3^1(x, t) = \frac{(-i\beta^2 t^\alpha)^3}{\Gamma(3\alpha + 1)} e^{i\beta x} \\ \vdots = \vdots \\ u_n^1(x, t) = \frac{(-i\beta^2 t^\alpha)^n}{\Gamma(n\alpha + 1)} e^{i\beta x} \end{cases}$$

we have :

$$\begin{aligned} u^1(x, t) &= \sum_{n=0}^{+\infty} \frac{(-i\beta^2 t^\alpha)^n}{\Gamma(n\alpha + 1)} e^{i\beta x} = e^{i\beta x} \sum_{n=0}^{+\infty} \frac{(-i\beta^2 t^\alpha)^n}{\Gamma(n\alpha + 1)} \\ &= e^{i\beta x} . E_\alpha(-i\beta^2 t^\alpha) \end{aligned}$$

with  $E_\alpha(-i\beta^2 t^\alpha)$  the Mittag-Leffler function  
so the solution at step  $k = 1$  is :

$$u^1(x, t) = e^{i\beta x} . E_\alpha(-i\beta^2 t^\alpha)$$

At step  $k = 2$ , we have :

$$\begin{cases} u_0^2(x, t) = e^{i\beta x} + I_0^\alpha(N(u^1(x, t))) \\ u_{n+1}^2(x, t) = I_0^\alpha(R(u_n^2(x, t))); \quad n \geq 0 \end{cases} \quad (60)$$

let's calculate  $N(u^1(x, t))$

$$\begin{aligned} N(u^1(x, t)) &= iq |u^1(x, t)|^2 u^1(x, t) \\ &= iq |e^{i\beta x} . E_\alpha(-i\beta^2 t^\alpha)|^2 e^{i\beta x} . E_\alpha(-i\beta^2 t^\alpha) \\ &= iq [e^{i\beta x} . E_\alpha(-i\beta^2 t^\alpha) . \overline{e^{i\beta x} . E_\alpha(-i\beta^2 t^\alpha)}] . e^{i\beta x} . E_\alpha(-i\beta^2 t^\alpha) \\ &= iq \times u^1(x, t) \\ &\neq 0 \end{aligned}$$

as  $N(u^1(x, t)) \neq 0$ , then we will modify the initial problem, we have

$$\begin{cases} {}^c D_t^\alpha u(x, t) = i \frac{\partial^2 u(x, t)}{\partial x^2} + iq |u(x, t)|^2 u(x, t) + iqu(x, t) - iqu(x, t) \\ u(x, 0) = e^{i\beta x} \end{cases} \quad (61)$$

let's put  $L(u) = {}^c D_t^\alpha u(x, t)$ ;  $R(u) = i \frac{\partial^2 u(x, t)}{\partial x^2} + iqu(x, t)$  and

$$N(u) = iq |u(x, t)|^2 u(x, t) - iqu(x, t)$$

so we have :

$$L(u) = R(u) + N(u) \quad (62)$$

apply  $L^{-1} = I_0^\alpha$  the fractional integral to (62), we have :

$$u(x, t) = e^{i\beta x} + I_0^\alpha(R(u(x, t))) + I_0^\alpha(N(u(x, t))) \quad (63)$$

Applying the method of successive approximations to (63), we have

$$u^k(x, t) = e^{i\beta x} + I_0^\alpha(R(u^k(x, t))) + I_0^\alpha(N(u^{k-1}(x, t))); \quad k \geq 1 \quad (64)$$

of (64), we obtain the following SBA algorithm :

$$\begin{cases} u_0^k(x, t) = e^{i\beta x} + I_0^\alpha(N(u^{k-1}(x, t))); \quad k \geq 1 \\ u_{n+1}^k(x, t) = I_0^\alpha(R(u_n^k(x, t))); \quad n \geq 0 \end{cases} \quad (65)$$

At step  $k = 1$ , we have :

$$\begin{cases} u_0^1(x, t) = e^{i\beta x} + I_0^\alpha(N(u^0(x, t))) \\ u_{n+1}^1(x, t) = I_0^\alpha(R(u_n^1(x, t))); \quad n \geq 0 \end{cases} \quad (66)$$

let's assume that  $u^0(x, t) = e^{i\beta x} \cdot E_\alpha(-i\beta^2 t^\alpha)$

let's calculate  $N(u^0(x, t))$

$$\begin{aligned} N(u^0(x, t)) &= iq \left| u^0(x, t) \right|^2 u^0(x, t) - iqu^0(x, t) \\ &= iq \left| e^{i\beta x} \cdot E_\alpha(-i\beta^2 t^\alpha) \right|^2 e^{i\beta x} \cdot E_\alpha(-i\beta^2 t^\alpha) - iqe^{i\beta x} \cdot E_\alpha(-i\beta^2 t^\alpha) \\ &= iq \left[ e^{i\beta x} \cdot E_\alpha(-i\beta^2 t^\alpha) \cdot \overline{e^{i\beta x} \cdot E_\alpha(-i\beta^2 t^\alpha)} \right] \cdot e^{i\beta x} \cdot E_\alpha(-i\beta^2 t^\alpha) - iqe^{i\beta x} \cdot E_\alpha(-i\beta^2 t^\alpha) \\ &= iqe^{i\beta x} \cdot E_\alpha(-i\beta^2 t^\alpha) - iqe^{i\beta x} \cdot E_\alpha(-i\beta^2 t^\alpha) \\ &= 0 \end{aligned}$$

The algorithm above becomes :

$$\begin{cases} u_0^1(x, t) = e^{i\beta x} \\ u_{n+1}^1(x, t) = I_0^\alpha(R(u_n^1(x, t))); \quad n \geq 0 \end{cases} \quad (67)$$

let's calculate  $u^1(x, t)$

for  $n = 0$ ; we have :

$$\begin{aligned} u_1^1(x, t) &= I_0^\alpha(R(u_0^1(x, t))) = iI_0^\alpha \left( \frac{\partial^2 u_0^1(x, t)}{\partial x^2} + qu_0^1(x, t) \right) \\ &= \frac{i(q - \beta^2)t^\alpha}{\Gamma(\alpha + 1)} e^{i\beta x} \end{aligned}$$

for  $n = 1$ ; we have :

$$\begin{aligned} u_2^1(x, t) &= I_0^\alpha(R(u_1^1(x, t))) = iI_0^\alpha \left( \frac{\partial^2 u_1^1(x, t)}{\partial x^2} + qu_1^1(x, t) \right) \\ &= \frac{[i(q - \beta^2)t^\alpha]^2}{\Gamma(2\alpha + 1)} e^{i\beta x} \end{aligned}$$

for  $n = 2$ ; we have :

$$\begin{aligned} u_3^1(x, t) &= I_0^\alpha(R(u_2^1(x, t))) = iI_0^\alpha \left( \frac{\partial^2 u_2^1(x, t)}{\partial x^2} + qu_2^1(x, t) \right) \\ &= \frac{[i(q - \beta^2)t^\alpha]^3}{\Gamma(3\alpha + 1)} e^{i\beta x} \end{aligned}$$

in a recursive way we have :

$$\left\{ \begin{array}{l} u_0^1(x, t) = e^{i\beta x} \\ u_1^1(x, t) = \frac{i(q - \beta^2)t^\alpha}{\Gamma(\alpha + 1)} e^{i\beta x} \\ u_2^1(x, t) = \frac{[i(q - \beta^2)t^\alpha]^2}{\Gamma(2\alpha + 1)} e^{i\beta x} \\ u_3^1(x, t) = \frac{[i(q - \beta^2)t^\alpha]^3}{\Gamma(3\alpha + 1)} e^{i\beta x} \\ \vdots = \vdots \\ u_n^1(x, t) = \frac{[i(q - \beta^2)t^\alpha]^n}{\Gamma(n\alpha + 1)} e^{i\beta x} \end{array} \right.$$

we have :

$$\begin{aligned} u^1(x, t) &= \sum_{n=0}^{+\infty} \frac{[i(q - \beta^2)t^\alpha]^n}{\Gamma(n\alpha + 1)} e^{i\beta x} = e^{i\beta x} \sum_{n=0}^{+\infty} \frac{[i(q - \beta^2)t^\alpha]^n}{\Gamma(n\alpha + 1)} \\ &= e^{i\beta x} . E_\alpha(i(q - \beta^2)t^\alpha) \end{aligned}$$

so the solution at step  $k = 1$  is :

$$u^1(x, t) = e^{i\beta x} . E_\alpha(i(q - \beta^2)t^\alpha)$$

with  $E_\alpha(i(q - \beta^2)t^\alpha)$  the Mittag-Leffler function

At step  $k = 2$ , we have :

$$\left\{ \begin{array}{l} u_0^2(x, t) = e^{i\beta x} + I_0^\alpha(N(u^1(x, t))) \\ u_{n+1}^2(x, t) = I_0^\alpha(R(u_n^2(x, t))); \quad n \geq 0 \end{array} \right. \quad (68)$$

let's calculate  $N(u^1(x, t))$

$$\begin{aligned} N(u^1(x, t)) &= iq \left| u^1(x, t) \right|^2 u^1(x, t) - iqu^1(x, t) \\ &= iq \left| e^{i\beta x} . E_\alpha(i(\beta^2 - q)t^\alpha) \right|^2 e^{i\beta x} . E_\alpha(i(\beta^2 - q)t^\alpha) - iqe^{i\beta x} . E_\alpha(i(\beta^2 - q)t^\alpha) \\ &= iq \left[ e^{i\beta x} . E_\alpha(i(\beta^2 - q)t^\alpha) . \overline{e^{i\beta x} . E_\alpha(i(\beta^2 - q)t^\alpha)} \right] . e^{i\beta x} . E_\alpha(i(\beta^2 - q)t^\alpha) - \\ &\quad iqe^{i\beta x} . E_\alpha(i(\beta^2 - q)t^\alpha) \\ &= iqe^{i\beta x} . E_\alpha(i(\beta^2 - q)t^\alpha) - iqe^{i\beta x} . E_\alpha(i(\beta^2 - q)t^\alpha) \\ &= 0 \end{aligned}$$

The algorithm above becomes :

$$\left\{ \begin{array}{l} u_0^2(x, t) = e^{i\beta x} \\ u_{n+1}^2(x, t) = I_0^\alpha(R(u_n^2(x, t))); \quad n \geq 0 \end{array} \right. \quad (69)$$

The algorithm at step  $k = 2$  is the same as the algorithm at step  $k = 1$ .  
so we have

$$u^2(x, t) = u^1(x, t) = e^{i\beta x} . E_\alpha(i(q - \beta^2)t^\alpha)$$

In a recursive way we have

$$u^k(x, t) = e^{i\beta x} \cdot E_\alpha(i(q - \beta^2)t^\alpha); \quad t \geq 1$$

$$u(x, t) = \lim_{k \rightarrow +\infty} u^k(x, t) = u^1(x, t) = e^{i\beta x} \cdot E_\alpha(i(q - \beta^2)t^\alpha)$$

the exact solution of the modified problem for  $\alpha = 1$  is therefore

$$u(x, t) = e^{i[\beta x + (q - \beta^2)t]}$$

and since the modified problem is equivalent to the initial problem, then the exact solution of the Schrödinger equation for  $\alpha = 1$  is :

$$u(x, t) = e^{i[\beta x + (q - \beta^2)t]}$$

## 6 Conclusion

This work shows that SBA method is a mathematical tool able to calculate the time-dependent wave function of the time-fractional Schrödinger Equation in one dimension. We have successfully solved the Schrödinger Equation by the homotopy perturbation method (HPM) and the Some Blaise Abbo method (SBA). We find the same result with these two methods.

## Références

- [1] A. M. Wazwaz, *A study on linear and nonlinear Schrödinger Equation by the variational iteration method*, *Chaos, Solitons and Fractals*, 2008, 37, 1136-1142.
- [2] A.S.V. Ravi kanth and K. Aruna, *Two dimensional differential transform method for solving linear and non-linear Schrödinger Equation*, *Chaos, Solitons and Fractals*, 2009, 41, 2277-2281.
- [3] A.K. Golmankhaneh, Ali K. Golmankhaneh, *Homotopy perturbation method for solving a system of Schrödinger-korteweg de vries, equation*, *Romanian Reports in Physics*, 2011, 63, 609-623.
- [4] N.A. Khan, M. Jamil and A. Ara., *Approximate solution to Time Fractional Schrödinger Equation via Homotopy analysis method*, *ISRN Mathematical Physics*, 2012, Article ID 197068, 11 pages.
- [5] Abdoul Wassiha Nebie, *Résolution de quelques équations fonctionnelles d'ordre fractionnaire au sens de Caputo par la méthode SBA et par la méthode HPM*, Thèse de Doctorat unique, Université Joseph Ki-Zerbo, Soutenue le 11 mars 2022, UFR SEA.
- [6] Ouedraogo Seny, Abbo Bakari, Rasmene Yaro and Youssouf Pare, *Convergence of a new approach of Adomian's method for solving volterra fractional integral of second kind*, volume 127, number 2, 2020, pp 107-120.

- [7] Brahim Tellab, *Résolution des équations différentielles fractionnaires*, Thèse de doctorat, Université des Frères Mentouri Constantine-1 soutenu le 11-01-2018.
- [8] AMS Mahdy, AS Mohamed, AAH Mtawal *Variational homotopy perturbation method for solving the generalized time-space fractional Schrödinger equation* International Journal of Physical Science 10(11), 342-350.
- [9] AAH Mtawal, SR Alkaleeli, AAH Mtawal, *A new modified homotopy perturbation method for fractional partial differential equations with proportional delay*, Journal of Advances In Mathematics 19,58-73.
- [10] AMS Mahdy, AS Mohamed, AAH Mtawal, *Implementation of the homotopy perturbation sumudu transform method for solving Klein-Gordon equation*, Applied Mathematics 6(03), 617-628.
- [11] Youssef Pare, *Equations intégrales et intégréo-différentielles*, Generis publishing (2021).