Application of the SBA method to solving the time-fractional Schrödinger equation and comparison with the homotopic perturbation method.

Abstract

We have solved the Schrödinger equation with the HPM method and the SBA method. We have noticed that with these two methods we find the same result.

Key words : Fractional equation, Some Blaise Abbo (SBA) method, the homotopic perturbation method (HPM) ; Schrödinger equation ; fractional Partial Differential Equations (PDEs).

1 Introduction

For more than three decades, mathematicians and research authors have paid special attention to fractional calculus because a fractional derivative is not a derivative is not a derivative at the local point, it considers history and non-local distributed effects.

Therefore, fractional mathematical models are more realistic and practical than other classical models. Thus, while the latter do not include information on mechanisms related to memory and learning, fractional models consider past and distributed effects in the system under study any dynamic process modelled through fractional equations then has a memory effect. This is why applications of fractional calculus can be found in several scientific fields such as physics (especially quantum mechanics and electrical signal propagation), biomechanics (biological tissue mechanics and rheology), signal processing and bioengineering.

In the field of quantum mechanics, the solution of the Schrödinger equation, which allows the determination of the the different energies or eigenvalues of the energy of an atomic, nuclear or solid state system, has been the subject of several studies, among others, Wazwaz [1] for the variational iteration method (VIM), Ravi Kanth [2] for the differential transformation method (DTM), and Golmankhaneh et al [3] for the homotopy perturbatio method (HPM).

For the time-dependent Schrödinger equation many efforts are made to try to find robust numerical and analytical methods e.g. Rams Wroop et al using the homotopy analysis transformation method (HATM), Khan et al [4] the homotopy analysis method (HAM). Twenty years ago, the Some Blaise Abbo (SBA) method was proposed by an african team to solve linear and nonlinear ODEs and PDEs.

In this paper, we propose to extend the applications of the SBA method to the solution of the fractional time-dependent Schrödinger equation.

This paper is organized as follows.In section 2, time-fractional Schrödinger Equation is described and we give some basic definition and proprerties. In sections 3 and 4, we present some properties of SBA method and HPM method respectively. In section 5, we present and compare numerical solutions of time-fractional Schrödinger Equation using these two methods. Finally, the conclusions are given in section 6.

2 Fractional Schrödinger Equation and preliminaries

The Schrödinger Equation, ferst obtained in 1926 by Erwin Schrödinger, was an extention of de Broglie's hypotheses, proposed two years earlier, that each material particule has associated with it a wave-length λ related to the linear manentum p of the partide by the equation : $\lambda = \frac{h}{p}$ where h is planck's constant.

If the time-independent for the Schrödinger Equation is given by

$$H\left(-ih\frac{\partial}{\partial q_i}, q_i\right)u = -ih\frac{\partial u}{\partial t}$$

u is a wave function, q_i dynamical variable and H the Hamiltonian operateur.
In the peper, the time-fractional Schrödinger Equation (FSE) has the following form

$$\begin{cases} i\,{}^{c}D_t^{\alpha}u(x,t) = \gamma\nabla^2\left(u(x,t)\right) + V_d(x)u(x,t) + q|u(x,t)|^2u(x,t), \quad 0<\alpha\leq 1, \quad t>0 \\ \\ u(x,0) = f(x), \quad x\in\mathbb{R}^d \end{cases} \tag{1}$$

where $V_d(x)$ is the trapping potential and γ, q is a reals constants. The physical model (1) and its generalized forms occur in various areas of physics, including nonlinear optics, plasma physics, superconductivity, and quantum mechanics.

We give some basic definitions, notations, and properties of the fractional calculus theory, which will be used later in this work.

2.1 Definition

The Euler Gamma function is definied by [5 ;6 ;7 ;8 ;9]

$$\Gamma(z) = \int_0^{\infty} t^{z+1}e^{-t}dt \tag{2}$$

with z is any complex, number such that $Re(z) > 0$. The function is strictly decreasing on $]0;1]$.

2.2 Definition

The Beta function is defined by [5 ;6 ;7 ;8 ;9]

$$\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt \tag{3}$$

with $Re(p) > 0$, $Re(q) > 0$. The relation between the beta function and the gamma function is given by :

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{4}$$

2.3 Definition

The Mittag-Leffler function is defined by [5 ;6 ;7 ;8 ;9]

$$E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha+1)} \tag{5}$$

where z is a complex α is a strictly positive real.

2.4 Definition

Let $f \in C([a,b])$. The operator I_a^α defined on $[a,b]$ by :

$$(I_a^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad \alpha > 0 \tag{6}$$

is called Riemann-Liouville fractional integral of order α.

2.5 Propriety

Let α and β two complex numbers, $f \in C([a,b])$

i) $I_a^\alpha(I_a^\beta f) = I_a^{\alpha+\beta} f, \quad Re(\alpha) > 0, Re(\beta) > 0.$

ii) $\dfrac{d}{dt}\left(I_a^\alpha f\right)(t) = \left(I_a^{\alpha+1} f\right)(t), \quad Re(\alpha) > 0.$

iii) $\lim\limits_{a \to 0^+} (I_a^\alpha f)(t) = f(t), \quad Re(\alpha) > 0.$

2.6 Definition

The caputo-type fractional derivative of order $\alpha > 0$ of a function $f \in C_{-1}^{m}$ $(m = 1, 2, 3, ...)$ is given by [5 ;6 ;7 ;8 ;9]

$$\begin{cases} {}^{c}D^{\alpha} = I^{n-\alpha}u^{(n)} = \dfrac{\partial^{\alpha}u(x,s)}{\partial s^{\alpha}} = \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_{a}^{t}(t-s)^{n-\alpha-1}\frac{\partial^{n}u(x,s)}{\partial s^{n}}ds, \quad if \quad n-1<\alpha<n \\ \\ \dfrac{d^{n}}{dt^{n}}f, \quad if \quad \alpha = n \end{cases} \tag{7}$$

where $n = [\alpha] + 1$ is the integer part of the real number.

3 The SBA numerical method

Consider the following functional equation [6 ;7 ;8 ;9 ;10]

$$A(u) = h \tag{8}$$

where $A : H \to H$ is a linear or nonlinear operator, H a real Hilbert space, h a given function in H and u the unknown function in H.
By setting $A = L - R - N$, where L is a linear operator assumed to be invertible, invertible in the Adomian sense, R a linear operator and N a nonlinear operator.
We obtain

$$L(u) - R(u) - N(u) = h \tag{9}$$

By applying the inverse L^{-1} of L to (9), we obtain the following canonical form of Adomian

$$u = \theta + L^{-1}(h) + L^{-1}(R(u)) + L^{-1}(N(u)) \tag{10}$$

where θ satisfies $L\theta = 0$
Apply the method of successive approximations to (10), we obtain :

$$u^{k} = u^{k}(0) + L^{-1}(h^{k}) + L^{-1}(R(u^{k})) + L^{-1}(N(u^{k-1})) \quad k \geq 1 \tag{11}$$

From (11), deduce the following SBA algorithm for a fixed k :

$$\begin{cases} u_{0}^{k} = u^{k}(0) + L^{-1}(h^{k}) + L^{-1}(N(u^{k-1})) \quad k \geq 1 \\ u_{n}^{k} = L^{-1}(R(u_{n-1}^{k})), \quad n \geq 1 \end{cases} \tag{12}$$

We then apply the Picard principle to (12) : in choosing u^{0}, such that $Nu^{0} = 0$, this choice in fact allows the first iteration to solve only a linear problem and this principe will be checked at each iteration before continuing the calculations.
First iteration
For $k = 1$, we calculate u^{1} using the following algorithm

$$\begin{cases} u_{0}^{1} = u^{1}(0) + L^{-1}(h^{1}) \\ u_{n}^{1} = L^{-1}(R(u_{n-1}^{1})), \quad n \geq 1 \end{cases} \tag{13}$$

If the series $\sum_{n=0}^{+\infty} u_n^1$ is convergent, then we get :

$$u^1 = \sum_{n=0}^{+\infty} u_n^1 \tag{14}$$

Approximate solution of equation (8) in step $k = 1$.

Second iteration

For $k = 2$, we calculate u^2 using the following algorithm

$$\begin{cases} u_0^2 = u^2(0) + L^{-1}(h^2) \\ u_n^2 = L^{-1}(R(u_{n-1}^2)), \quad n \geq 1 \end{cases} \tag{15}$$

If the series $\sum_{n=0}^{+\infty} u_n^2$ is convergent, then we get :

$$u^2 = \sum_{n=0}^{+\infty} u_n^2 \tag{16}$$

Approximate solution of equation (8) in step $k = 2$.

k-th iteration

Recursively, if the series $\sum_{n=0}^{+\infty} u_n^k$ is convergent for $k \geq 1$, then we get :

$$u^k = \sum_{n=0}^{+\infty} u_n^k \quad k \geq 1 \tag{17}$$

Approximate solution of equation (8) in step k.

So the solution of the problem (8) is then :

$$u = \lim_{k \to +\infty} u^k = \lim_{k \to +\infty} \left(\sum_{n=0}^{+\infty} u_n^k \right). \tag{18}$$

4 The homotopy perturbation method

To illustrate the fundamental ideas of homotopic perturbation we consider the following non- linear equation [5] :

$$A(u) - f(x) = 0, \quad x \in \Omega \tag{19}$$

with the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad x \in \partial\Omega \tag{20}$$

or

* A is usually a differential operator

* B designates the boundary conditions
* $f(x)$ is an analytically known function
* $\partial\Omega$ is the domain boundary Ω
The method consists in decomposing the nonlinear operator in the form

$$A = L + R + N \tag{21}$$

or
$L + R$ is a linear operator, N is a nonlinear operator.
Therefore equation (19) becomes :

$$L(u) + R(u) + N(u) - f(x) = 0, \quad x \in \Omega \tag{22}$$

with the homotopy technique, we construct a homoto

$$v(x, p) : \Omega \times [0, 1] \to \mathbb{R} \tag{23}$$

which satisfies :

$$\begin{gathered} H(v, p) = (1 - p)(L(v) - L(u_0)) + p(L(v) + R(v) + N(v) - f(x)) = 0 \\ p \in [0, 1] \quad and \quad x \in \Omega \end{gathered} \tag{24}$$

by a simplification we find the following relation :

$$\begin{gathered} H(v, p) = L(v) - L(u_0) + p(L(u_0) + R(v) + N(v) - f(x)) = 0 \\ p \in [0, 1] \quad and \quad x \in \Omega \end{gathered} \tag{25}$$

$\Leftrightarrow$

$$\begin{gathered} L(v) = L(u_0) + p[f(x) - L(u_0) - R(v) - N(v)] \\ p \in [0, 1] \quad and \quad x \in \Omega \end{gathered} \tag{26}$$

or
* p is a parameter that varies from 0 to 1.
* u_0 is an initial approximation of (24) which satisfies the boundary conditions.
Obviously, by considering equations (24) we have :

$$H(v, 0) = L(v) - L(u_0) = 0 \tag{27}$$

$$H(v, 1) = (L(v) + R(v) + N(v) - f(x)) = 0 \tag{28}$$

According to the homotopic perturbation method, we can first use the parameter p as a small parameter and assume that the solution of equation (19) can be written as a series of power p.
By varying p from 0 to 1, we change $v(x, p)$ from $u_0(x)$ to $u(x)$, in topology, this is called the deformation, $L(v) - L(u_0)$ and $f(x) - L(u_0) - R(v) - N(v)$ are homotopy. The basic principle and that the solution of equations (24) and (25) can be written as a series :

$$v(x, p) = \sum_{n=0}^{\infty} p^n v_n(x) \tag{29}$$

and

$$N(v(x,p)) = \sum_{n=0}^{\infty} p^n H_n \tag{30}$$

where H_n are polynomials of He which are defined by :

$$H_n(v_0, v_1, v_2, ..., v_n) = \frac{1}{n!}\frac{d^n}{dp^n}\left[N\left(\sum_{i=0}^{\infty} p^i v_i(x)\right)\right]_{p=0} \; ; \quad n \geq 0 \tag{31}$$

(29) and (30) in (26) gives

$$\sum_{i=0}^{\infty} p^i L(v_i) = L(u_0) + p(f(x) - L(u_0)) - \sum_{i=0}^{\infty} p^{i+1}(R(v_i) + H_i) \tag{32}$$

$\Leftrightarrow$

$$\begin{aligned} p^0 L(v_0) + p^1 L(v_1) + p^2 L(v_2) + p^3 L(v_3) + ... \quad = \quad & p^0 L(u_0) + P^1(f(x) - L(u_0)) \\ & -p^1(R(v_0) + H_0) - p^2(R(v_1) + H_1) \\ & -p^3(R(v_2) + H_2) - p^4(R(v_3) + H_3) - ... \end{aligned}$$

By identifying the terms with those of the same power of p, we find

$$\left\{ \begin{array}{lcl} p^0 & : & L(v_0) = L(u_0) \\ p^1 & : & L(v_1) = f(x) - L(u_0) - R(v_0) - H_0 \\ p^2 & : & L(v_2) = -R(v_1) - H_1 \\ p^3 & : & L(v_3) = -R(v_2) - H_2 \\ \vdots & & \end{array} \right. \tag{33}$$

so we conclude that

$$\left\{ \begin{array}{lcl} p^0 & : & v_0 = u_0 \\ p^1 & : & L(v_1) = f(x) - L(u_0) - R(v_0) - H_0 \\ p^2 & : & L(v_2) = -R(v_1) - H_1 \\ p^3 & : & L(v_3) = -R(v_2) - H_2 \\ \vdots & & \end{array} \right. \tag{34}$$

the approximate solution of equation (19) is written :

$$u = \lim_{p \to 1} v = \sum_{n=0}^{\infty} v_n(x).$$

5 Comparing the two methods

In this section, we apply the homotopy perturbation method (HPM) and Some Blaise Abbo (SBA) method to some nonlinear partial differential equations.
Consider the following nonlinear time-fractionar Schrödinger equation :

$$\left\{ \begin{array}{l} i\,{}^c D_t^{\alpha} u(x,t) = -\dfrac{\partial^2 u(x,t)}{\partial x^2} - q|u(x,t)|^2 u(x,t) \\ u(x,0) = e^{i\beta x} \end{array} \right. \tag{35}$$

or $0 < \alpha \leq 1$, $x \in \mathbb{R}$ and $t > 0$.

5.1 Solving by the HPM method

(35) can be written

$$i\frac{\partial^2 u(x,t)}{\partial x^2} + iq|u(x,t)|^2u(x,t) - {}^cD_t^\alpha u(x,t) = 0 \tag{36}$$

with

$$\begin{cases} L_t u = {}^cD_t^\alpha u \Leftrightarrow L_t^{-1}u = I_t^\alpha u \\ N(u(x,t)) = |u(x,t)|^2u(x,t) \end{cases} \tag{37}$$

with the homotopy technique, we construct a homotopy according to

$${}^cD_t^\alpha v(x,t) - {}^cD_t^\alpha u_0(x,t) = p\left[i\frac{\partial^2 v(x,t)}{\partial x^2} + iqN(v(x,t)) - {}^cD_t^\alpha u_0(x,t)\right] \tag{38}$$

with $u_0(x,t) = u(x,0) = e^{i\beta x}$ the initial approximation of (35) that is

$$\begin{cases} {}^cD_t^\alpha v(x,t) - {}^cD_t^\alpha u_0(x,t) = p\left[i\dfrac{\partial^2 v(x,t)}{\partial x^2} + iqN(v(x,t)) - {}^cD_t^\alpha u_0(x,t)\right] \\ v(x,0) = e^{i\beta x} \end{cases} \tag{39}$$

with

$$N(v) = |v(x,t)|^2v(x,t) \tag{40}$$

we pose

$$\begin{cases} v(x,t) = \displaystyle\sum_{n=0}^{\infty} p^n v_n(x,t) \\ N(v(x,t)) = \displaystyle\sum_{n=0}^{\infty} p^n H_n \end{cases} \tag{41}$$

and

$$H_n = \frac{1}{n}\frac{d^n}{dp^n}\left[N\left(\sum_{i=0}^{\infty} p^i v_i\right)\right]_{p=0} \tag{42}$$

then (41) in (39) gives

$$\begin{cases} \displaystyle\sum_{n=0}^{\infty} p^n({}^cD_t^\alpha v_n(x,t)) - {}^cD_t^\alpha u_0(x,t) = \sum_{n=0}^{\infty} p^{n+1}\left[i\frac{\partial^2 v_n(x,t)}{\partial x^2} + iqH_n\right] - p\left[{}^cD_t^\alpha u_0(x,t)\right] \\ \displaystyle\sum_{n=0}^{\infty} p^n v_n(x,0) = e^{i\beta x} \end{cases} \tag{43}$$

where the terms of the sequence (H_n) of the polynomials of he are

$$\begin{cases} H_0 &= v_0^2(x,t)\overline{v_0}(x,t) \\ H_1 &= 2v_0(x,t)v_1(x,t)\overline{v_0}(x,t) + v_0^2(x,t)\overline{v_1}(x,t) \\ H_2 &= 2v_0(x,t)v_2(x,t)\overline{v_0}(x,t) + v_1^2(x,t)\overline{v_0}(x,t) + 2v_0(x,t)v_1(x,t)\overline{v_1}(x,t) + v_0^2(x,t)\overline{v_2}(x,t) \\ \vdots & \end{cases}$$

by identifying terms with the same powers of p, we get

$$p^0 : \begin{cases} {}^cD_t^\alpha v_0(x,t) - {}^c D_t^\alpha u_0(x,t) = 0 \\ v_0(x,0) = e^{i\beta x} \end{cases} \tag{44}$$

$$p^1 : \begin{cases} {}^cD_t^\alpha v_1(x,t) = i\dfrac{\partial^2 v_0(x,t)}{\partial x^2} + iqH_0 - {}^c D_t^\alpha u_0(x,t) \\ v_1(x,0) = 0 \end{cases} \tag{45}$$

$$p^2 : \begin{cases} {}^cD_t^\alpha v_2(x,t) = i\dfrac{\partial^2 v_1(x,t)}{\partial x^2} + iqH_1 \\ v_2(x,0) = 0 \end{cases} \tag{46}$$

$$p^3 : \begin{cases} {}^cD_t^\alpha v_3(x,t) = i\dfrac{\partial^2 v_2(x,t)}{\partial x^2} + iqH_2 \\ v_3(x,0) = 0 \end{cases} \tag{47}$$

etc. Thus the resolution of the systems (44)-(47) gives

$$v_0(x,t) = u_0(x,t) = e^{i\beta x}$$

$$p^1 : \begin{cases} H_0 = e^{i2\beta x} e^{-i\beta x} = e^{i\beta x} \\ {}^cD_t^\alpha v_1(x,t) = i\dfrac{\partial^2}{\partial x^2}(e^{i\beta x}) - {}^c D_t^\alpha (e^{i\beta x}) \\ {}^cD_t^\alpha v_1(x,t) = (iq - i\beta^2)e^{i\beta x} \\ I_t^\alpha({}^cD_t^\alpha v_1(x,t)) = I_t^\alpha((iq - i\beta^2)e^{i\beta x}) \\ v_1(x,t) = (iq - i\beta^2)e^{i\beta x} I_t^\alpha(1) \\ v_1(x,t) = \dfrac{i(q-\beta^2)}{\Gamma(\alpha+1)} e^{i\beta x} t^\alpha \end{cases}$$

$$p^2 : \begin{cases} H_1 = 2i\dfrac{(q-\beta^2)}{\Gamma(\alpha+1)} e^{i\beta x} t^\alpha - i\dfrac{(q-\beta^2)}{\Gamma(\alpha+1)} e^{i\beta x} t^\alpha = i\dfrac{(q-\beta^2)}{\Gamma(\alpha+1)} e^{i\beta x} t^\alpha \\ {}^cD_t^\alpha v_2(x,t) = \left(-\dfrac{(q-\beta^2)^2}{\Gamma(\alpha+1)}\right) e^{i\beta x} t^\alpha \\ v_2(x,t) = \left(-\dfrac{(q-\beta^2)^2}{\Gamma(\alpha+1)}\right) e^{i\beta x} I_t^\alpha(t^\alpha) \\ v_2(x,t) = \left(-\dfrac{(q-\beta^2)^2}{\Gamma(\alpha+1)}\right) e^{i\beta x} \dfrac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} t^{2\alpha} \\ v_2(x,t) = \dfrac{\left[i(q-\beta^2)\right]^2}{\Gamma(2\alpha+1)} e^{i\beta x} t^{2\alpha} \end{cases}$$

$$v_n(x,t) = \frac{\left[i(q-\beta^2)\right]^n}{\Gamma(n\alpha+1)} e^{i\beta x} t^{n\alpha}$$

$$v(x,t) = \left(\sum_{n=0}^{\infty} p^n \frac{\left[i(q-\beta^2)t^\alpha\right]^n}{\Gamma(n\alpha+1)}\right) e^{i\beta x}$$

Finally the exact solution is :

$$u(x,t) = \lim_{p\to 1} v(x,t) = \left(\sum_{n=0}^{\infty} \frac{\left[i(q-\beta^2)t^\alpha\right]^n}{\Gamma(n\alpha+1)}\right) e^{i\beta x} = e^{i\beta x} E_\alpha(i(q-\beta^2)t^\alpha).$$

the exact solution of the Schrödinger equation for $\alpha = 1$ is :

$$u(x,t) = e^{i\left[\beta x + (q-\beta^2)t\right]}$$

5.2 Solving by the SBA method

Consider the following problem :

$$(P) : \begin{cases} {}^cD_t^\alpha u(x,t) = i\dfrac{\partial^2 u(x,t)}{\partial x^2} + iq|u(x,t)|^2 u(x,t) \\ u(x,0) = g(x) \end{cases} \tag{48}$$

let's put $L(u) ={}^c D_t^\alpha u(x,t)$; $R(u) = i\dfrac{\partial^2 u(x,t)}{\partial x^2}$ and $N(u) = iq\left|u(x,t)\right|^2 u(x,t)$
we have :

$$Lu(x,t) = Ru(x,t) + N(u(x,t)) \tag{49}$$

Theorem
If $\forall k \geq 1$, $N(u^{k-1}(x,t)) = 0$, $\left|\dfrac{MT^\alpha}{\Gamma(\alpha+1)}\right| < 1$, $g \in C(\mathbb{R})$, $u(x,t) \in C^2(\Omega)$ such as $\exists m = \underset{x\in\Omega}{sup} g(x)$ and $\exists M = \underset{(x,t)\in\Omega}{sup}\ u(x,t) > 0$ ou $\Omega = \mathbb{R}\times[0;T]$ then the SBA algorithm is convergent and the problem (P) has a unique solution.
Proof. we have the following SBA algorithm :

$$\begin{cases} u_0^k(x,t) = g(x) + N(u^{k-1}(x,t)); \quad k \geq 1 \\ u_{n+1}^k(x,t) = I_0^\alpha(R(u_n^k(x,t))); \quad n \geq 0 \end{cases} \tag{50}$$

or even

$$\begin{cases} u_0^k(x,t) = g(x); \quad k \geq 1 \\ u_{n+1}^k(x,t) = I_0^\alpha(R(u_n^k(x,t))); \quad n \geq 0 \end{cases} \tag{51}$$

$$\begin{cases} \left|u_0^k(x,t)\right| &=& |g(x)| \leq m;\ k \geq 1 \\ \left|u_1^k(x,t)\right| &=& \left|I_0^\alpha(R(u_0^k(x,t)))\right| \leq \dfrac{MT^\alpha}{\Gamma(\alpha+1)} \\ \left|u_2^k(x,t)\right| &=& \left|I_0^\alpha(R(u_1^k(x,t)))\right| \leq \left(\dfrac{MT^\alpha}{\Gamma(\alpha+1)}\right)^2 \\ \left|u_3^k(x,t)\right| &=& \left|I_0^\alpha(R(u_2^k(x,t)))\right| \leq \left(\dfrac{MT^\alpha}{\Gamma(\alpha+1)}\right)^3 \\ \vdots &=& \vdots \\ \left|u_n^k(x,t)\right| &=& \left|I_0^\alpha(R(u_n^k(x,t)))\right| \leq \left(\dfrac{MT^\alpha}{\Gamma(\alpha+1)}\right)^n; \quad n > 0 \end{cases}$$

Summing member by member, we get :

$$\sum_{n=0}^{+\infty} |u_n^k(x,t)| = m + \frac{MT^\alpha}{\Gamma(\alpha+1) - MT^\alpha}$$

from where $\sum_{n=0}^{+\infty} |u_n^k(x,t)|$ is absolutely convergent by sequence $\sum_{n=0}^{+\infty} u_n^k(x,t)$ simply convergent.

Unicité de la solution

Let $u(x,t)$ and $v(x,t)$ be two solutions of (48) with $u(x,t) \neq v(x,t)$ and for u and v we have the following algorithms

$$\begin{cases} u_0^k(x,t) = e^{i\beta x}; \quad k \geq 1 \\ u_{n+1}^k(x,t) = I_0^\alpha(R(u_n^k(x,t))); \quad n \geq 0 \end{cases} \tag{52}$$

and

$$\begin{cases} v_0^k(x,t) = e^{i\beta x}; \quad k \geq 1 \\ v_{n+1}^k(x,t) = I_0^\alpha(R(v_n^k(x,t))); \quad n \geq 0 \end{cases} \tag{53}$$

by making the difference of the two algorithms we obtain :

$$\begin{cases} u_0^k(x,t) - v_0^k(x,t) & = & e^{i\beta x} - e^{i\beta x} = 0 \\ u_1^k(x,t) - v_1^k(x,t) & = & \dfrac{-i\beta^2 t^\alpha}{\Gamma(\alpha+1)} e^{i\beta x} - \dfrac{-i\beta^2 t^\alpha}{\Gamma(\alpha+1)} e^{i\beta x} = 0 \\ u_2^k(x,t) - v_2^k(x,t) & = & \dfrac{(-i\beta^2 t^\alpha)^2}{\Gamma(2\alpha+1)} e^{i\beta x} - \dfrac{(-i\beta^2 t^\alpha)^2}{\Gamma(2\alpha+1)} e^{i\beta x} = 0 \\ u_3^k(x,t) - v_3^k(x,t) & = & \dfrac{(-i\beta^2 t^\alpha)^3}{\Gamma(3\alpha+1)} e^{i\beta x} - \dfrac{(-i\beta^2 t^\alpha)^3}{\Gamma(3\alpha+1)} e^{i\beta x} = 0 \\ \vdots & = & \vdots \\ u_n^k(x,t) - v_n^k(x,t) & = & \dfrac{(-i\beta^2 t^\alpha)^n}{\Gamma(n\alpha+1)} e^{i\beta x} - \dfrac{(-i\beta^2 t^\alpha)^n}{\Gamma(n\alpha+1)} e^{i\beta x} = 0 \end{cases}$$

so $u_n^k(x,t) - v_n^k(x,t) = 0 \Rightarrow u(x,t) = v(x,t)$ or according to the hypothesis $u(x,t) \neq v(x,t)$ which is contradictory, so the solution is unique.

Application

of (35) on a so :

$$L(u) = R(u) + N(u) \tag{54}$$

apply $L^{-1} = I_0^\alpha(.)$ the fractional integral to (54), we have :

$$u(x,t) = e^{i\beta x} + I_0^\alpha(R(u(x,t))) + I_0^\alpha(N(u(x,t))) \tag{55}$$

Applying the method of successive approximations to (55), we have

$$u^k(x,t) = e^{i\beta x} + I_0^\alpha(R(u^k(x,t))) + I_0^\alpha(N(u^{k-1}(x,t))); \quad k \geq 1 \tag{56}$$

of (56), we obtain the following SBA algorithm :

$$\begin{cases} u_0^k(x,t) = e^{i\beta x} + I_0^\alpha(N(u^{k-1}(x,t))); \quad k \geq 1 \\ u_{n+1}^k(x,t) = I_0^\alpha(R(u_n^k(x,t))); \quad n \geq 0 \end{cases} \tag{57}$$

At step $k=1$, we have :

$$\left\{\begin{array}{l} u_0^1(x,t) = e^{i\beta x} + I_0^\alpha(N(u^0(x,t))) \\ u_{n+1}^1(x,t) = I_0^\alpha(R(u_n^1(x,t))); \quad n \geq 0 \end{array}\right. \tag{58}$$

Applying Picard's principle, (58), we will choose u^0 such that $N(u^0)=0$ so we will take $u^0=0$
The algorithm above becomes :

$$\left\{\begin{array}{l} u_0^1(x,t) = e^{i\beta x} \\ u_{n+1}^1(x,t) = I_0^\alpha(R(u_n^1(x,t))); \quad n \geq 0 \end{array}\right. \tag{59}$$

let's calculate $u^1(x,t)$
for $n=0$, we have

$$\begin{array}{rcl} u_1^1(x,t) & = & I_0^\alpha(R(u_0^1(x,t))) = iI_0^\alpha(\dfrac{\partial^2 u_0^1(x,t)}{\partial x^2}) \\ & = & \dfrac{-i\beta^2 t^\alpha}{\Gamma(\alpha+1)} e^{i\beta x} \end{array}$$

for $n=1$, we have

$$\begin{array}{rcl} u_2^1(x,t) & = & I_0^\alpha(R(u_1^1(x,t))) = iI_0^\alpha(\dfrac{\partial^2 u_1^1(x,t)}{\partial x^2}) \\ & = & \dfrac{(-i\beta^2 t^\alpha)^2}{\Gamma(2\alpha+1)} e^{i\beta x} \end{array}$$

for $n=2$, we have

$$\begin{array}{rcl} u_3^1(x,t) & = & I_0^\alpha(R(u_2^1(x,t))) = iI_0^\alpha(\dfrac{\partial^2 u_2^1(x,t)}{\partial x^2}) \\ & = & \dfrac{(-i\beta^2 t^\alpha)^3}{\Gamma(3\alpha+1)} e^{i\beta x} \end{array}$$

in a recursive way we have :

$$\left\{\begin{array}{rcl} u_0^1(x,t) & = & e^{i\beta x} \\ u_1^1(x,t) & = & \dfrac{-i\beta^2 t^\alpha}{\Gamma(\alpha+1)} e^{i\beta x} \\ u_2^1(x,t) & = & \dfrac{(-i\beta^2 t^\alpha)^2}{\Gamma(2\alpha+1)} e^{i\beta x} \\ u_3^1(x,t) & = & \dfrac{(-i\beta^2 t^\alpha)^3}{\Gamma(3\alpha+1)} e^{i\beta x} \\ \vdots & = & \vdots \\ u_n^1(x,t) & = & \dfrac{(-i\beta^2 t^\alpha)^n}{\Gamma(n\alpha+1)} e^{i\beta x} \end{array}\right.$$

we have :

$$\begin{aligned} u^1(x,t) &= \sum_{n=0}^{+\infty} \frac{(-i\beta^2 t^\alpha)^n}{\Gamma(n\alpha+1)} e^{i\beta x} = e^{i\beta x} \sum_{n=0}^{+\infty} \frac{(-i\beta^2 t^\alpha)^n}{\Gamma(n\alpha+1)} \\ &= e^{i\beta x}.E_\alpha(-i\beta^2 t^\alpha) \end{aligned}$$

with $E_\alpha(-i\beta^2 t^\alpha)$ the Mittag-Leffler function
so the solution at step $k = 1$ is :

$$u^1(x,t) = e^{i\beta x}.E_\alpha(-i\beta^2 t^\alpha)$$

At step $k = 2$, we have :

$$\begin{cases} u_0^2(x,t) = e^{i\beta x} + I_0^\alpha(N(u^1(x,t))) \\ u_{n+1}^2(x,t) = I_0^\alpha(R(u_n^2(x,t))); \quad n \geq 0 \end{cases} \tag{60}$$

let's calculate $N(u^1(x,t))$

$$\begin{aligned} N(u^1(x,t)) &= iq \left| u^1(x,t) \right|^2 u^1(x,t) \\ &= iq \left| e^{i\beta x}.E_\alpha(-i\beta^2 t^\alpha) \right|^2 e^{i\beta x}.E_\alpha(-i\beta^2 t^\alpha) \\ &= iq \left[e^{i\beta x}.E_\alpha(-i\beta^2 t^\alpha).\overline{e^{i\beta x}.E_\alpha(-i\beta^2 t^\alpha)} \right] .e^{i\beta x}.E_\alpha(-i\beta^2 t^\alpha) \\ &= iq \times u^1(x,t) \\ &\neq 0 \end{aligned}$$

as $N(u^1(x,t)) \neq 0$, then we will modify the initial problem, we have

$$\begin{cases} {}^cD_t^\alpha u(x,t) = i\dfrac{\partial^2 u(x,t)}{\partial x^2} + iq \left| u(x,t) \right|^2 u(x,t) + iqu(x,t) - iqu(x,t) \\ u(x,0) = e^{i\beta x} \end{cases} \tag{61}$$

let's put $L(u) =^c D_t^\alpha u(x,t)$; $R(u) = i\dfrac{\partial^2 u(x,t)}{\partial x^2} + iqu(x,t)$ and
$N(u) = iq \left| u(x,t) \right|^2 u(x,t) - iqu(x,t)$
so we have :

$$L(u) = R(u) + N(u) \tag{62}$$

apply $L^{-1} = I_0^{alpha}$ the fractional integral to (62), we have :

$$u(x,t) = e^{i\beta x} + I_0^\alpha(R(u(x,t))) + I_0^\alpha(N(u(x,t))) \tag{63}$$

Applying the method of successive approximations to (63), we have

$$u^k(x,t) = e^{i\beta x} + I_0^\alpha(R(u^k(x,t))) + I_0^\alpha(N(u^{k-1}(x,t))); \quad k \geq 1 \tag{64}$$

of (64), we obtain the following SBA algorithm :

$$\begin{cases} u_0^k(x,t) = e^{i\beta x} + I_0^\alpha(N(u^{k-1}(x,t))); \quad k \geq 1 \\ u_{n+1}^k(x,t) = I_0^\alpha(R(u_n^k(x,t))); \quad n \geq 0 \end{cases} \tag{65}$$

At step $k = 1$, we have :

$$\left\{ \begin{array}{c} u_0^1(x,t) = e^{i\beta x} + I_0^\alpha(N(u^0(x,t))) \\ u_{n+1}^1(x,t) = I_0^\alpha(R(u_n^1(x,t))); \quad n \geq 0 \end{array} \right. \tag{66}$$

let's assume that $u^0(x,t) = e^{i\beta x}.E_\alpha(-i\beta^2 t^\alpha)$
let's calculate $N(u^0(x,t))$

$$\begin{array}{rcl} N(u^0(x,t)) & = & iq\left|u^0(x,t)\right|^2 u^0(x,t) - iqu^0(x,t) \\ & = & iq\left|e^{i\beta x}.E_\alpha(-i\beta^2 t^\alpha)\right|^2 e^{i\beta x}.E_\alpha(-i\beta^2 t^\alpha) - iqe^{i\beta x}.E_\alpha(-i\beta^2 t^\alpha) \\ & = & iq\left[e^{i\beta x}.E_\alpha(-i\beta^2 t^\alpha).\overline{e^{i\beta x}.E_\alpha(-i\beta^2 t^\alpha)}\right].e^{i\beta x}.E_\alpha(-i\beta^2 t^\alpha) - iqe^{i\beta x}.E_\alpha(-i\beta^2 t^\alpha) \\ & = & iqe^{i\beta x}.E_\alpha(-i\beta^2 t^\alpha) - iqe^{i\beta x}.E_\alpha(-i\beta^2 t^\alpha) \\ & = & 0 \end{array}$$

The algorithm above becomes :

$$\left\{ \begin{array}{c} u_0^1(x,t) = e^{i\beta x} \\ u_{n+1}^1(x,t) = I_0^\alpha(R(u_n^1(x,t))); \quad n \geq 0 \end{array} \right. \tag{67}$$

let's calculate $u^1(x,t)$
for $n = 0$; we have :

$$\begin{array}{rcl} u_1^1(x,t) & = & I_0^\alpha(R(u_0^1(x,t))) = iI_0^\alpha\left(\dfrac{\partial^2 u_0^1(x,t)}{\partial x^2} + qu_0^1(x,t)\right) \\ & = & \dfrac{i(q-\beta^2)t^\alpha}{\Gamma(\alpha+1)}e^{i\beta x} \end{array}$$

for $n = 1$; we have :

$$\begin{array}{rcl} u_2^1(x,t) & = & I_0^\alpha(R(u_1^1(x,t))) = iI_0^\alpha\left(\dfrac{\partial^2 u_1^1(x,t)}{\partial x^2} + qu_1^1(x,t)\right) \\ & = & \dfrac{\left[i(q-\beta^2)t^\alpha\right]^2}{\Gamma(2\alpha+1)}e^{i\beta x} \end{array}$$

for $n = 2$; we have :

$$\begin{array}{rcl} u_3^1(x,t) & = & I_0^\alpha(R(u_2^1(x,t))) = iI_0^\alpha\left(\dfrac{\partial^2 u_2^1(x,t)}{\partial x^2} + qu_2^1(x,t)\right) \\ & = & \dfrac{\left[i(q-\beta^2)t^\alpha\right]^3}{\Gamma(3\alpha+1)}e^{i\beta x} \end{array}$$

in a recursive way we have :

$$\left\{\begin{array}{lcl} u_0^1(x,t) & = & e^{i\beta x} \\ u_1^1(x,t) & = & \dfrac{i(q-\beta^2)t^\alpha}{\Gamma(\alpha+1)}e^{i\beta x} \\ u_2^1(x,t) & = & \dfrac{\left[i(q-\beta^2)t^\alpha\right]^2}{\Gamma(2\alpha+1)}e^{i\beta x} \\ u_3^1(x,t) & = & \dfrac{\left[i(q-\beta^2)t^\alpha\right]^3}{\Gamma(3\alpha+1)}e^{i\beta x} \\ \vdots & = & \vdots \\ u_n^1(x,t) & = & \dfrac{\left[i(q-\beta^2)t^\alpha\right]^n}{\Gamma(n\alpha+1)}e^{i\beta x} \end{array}\right.$$

we have :

$$\begin{array}{rcl} u^1(x,t) & = & \displaystyle\sum_{n=0}^{+\infty}\frac{\left[i(q-\beta^2)t^\alpha\right]^n}{\Gamma(n\alpha+1)}e^{i\beta x} = e^{i\beta x}\sum_{n=0}^{+\infty}\frac{\left[i(q-\beta^2)t^\alpha\right]^n}{\Gamma(n\alpha+1)} \\ & = & e^{i\beta x}.E_\alpha(i(q-\beta^2)t^\alpha) \end{array}$$

so the solution at step $k=1$ is :

$$u^1(x,t) = e^{i\beta x}.E_\alpha(i(q-\beta^2)t^\alpha)$$

with $E_\alpha(i(q-\beta^2)t^\alpha)$ the Mittag-Leffler function
At step $k=2$, we have :

$$\left\{\begin{array}{l} u_0^2(x,t) = e^{i\beta x} + I_0^\alpha(N(u^1(x,t))) \\ u_{n+1}^2(x,t) = I_0^\alpha(R(u_n^2(x,t))); \quad n \geq 0 \end{array}\right. \tag{68}$$

let's calculate $N(u^1(x,t))$

$$\begin{array}{rcl} N(u^1(x,t)) & = & iq\left|u^1(x,t)\right|^2 u^1(x,t) - iqu^1(x,t) \\ & = & iq\left|e^{i\beta x}.E_\alpha(i(\beta^2-q)t^\alpha)\right|^2 e^{i\beta x}.E_\alpha(i(\beta^2-q)t^\alpha) - iqe^{i\beta x}.E_\alpha(i(\beta^2-q)t^\alpha) \\ & = & iq\left[e^{i\beta x}.E_\alpha(i(\beta^2-q)t^\alpha).\overline{e^{i\beta x}.E_\alpha(i(\beta^2-q)t^\alpha)}\right].e^{i\beta x}.E_\alpha(i(\beta^2-q)t^\alpha) - \\ & & iqe^{i\beta x}.E_\alpha(i(\beta^2-q)t^\alpha) \\ & = & iqe^{i\beta x}.E_\alpha(i(\beta^2-q)t^\alpha) - iqe^{i\beta x}.E_\alpha(i(\beta^2-q)t^\alpha) \\ & = & 0 \end{array}$$

The algorithm above becomes :

$$\left\{\begin{array}{l} u_0^2(x,t) = e^{i\beta x} \\ u_{n+1}^2(x,t) = I_0^\alpha(R(u_n^2(x,t))); \quad n \geq 0 \end{array}\right. \tag{69}$$

The algorithm at step $k=2$ is the same as the algorithm at step $k=1$.
so we have

$$u^2(x,t) = u^1(x,t) = e^{i\beta x}.E_\alpha(i(q-\beta^2)t^\alpha)$$

In a recursive way we have

$$u^k(x,t) = e^{i\beta x}.E_\alpha(i(q-\beta^2)t^\alpha); \qquad t \geq 1$$

$$u(x,t) = \lim_{k \to +\infty} u^k(x,t) = u^1(x,t) = e^{i\beta x}.E_\alpha(i(q-\beta^2)t^\alpha)$$

the exact solution of the modified problem for $\alpha = 1$ is therefore

$$u(x,t) = e^{i\left[\beta x + (q-\beta^2)t\right]}$$

and since the modified problem is equivalent to the initial problem, then the exact solution of the Schrödinger equation for $\alpha = 1$ is :

$$u(x,t) = e^{i\left[\beta x + (q-\beta^2)t\right]}$$

6 Conclusion

This work show that SBA method is a mathematical tool able to calculate the time-dependent wave function of time fractional Schrödinger Equation in one dimension. We have successfully solved the Schrödinger Equation by the homotopy perturbation method (HPM) and the Some Blaise Abbo method (SBA). We find the same result with these two methods.

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