

# The existence of periodic solutions for a two-enterprise interaction model with delays

**Abstract:** Various enterprise interaction models with or without time delays were appeared in the literature. The stability and Hopf bifurcation for one or two delays in the models were studied by many researchers. However, the stability and oscillation for a four time delays enterprise interaction model is still an open problem due to the complexity of the bifurcating equation. In this paper, by means of the mathematical analysis method, some sufficient conditions to guarantee the existence of periodic oscillatory solution for the four time delays model are obtained. An open problem is solved. Computer simulation is given to demonstrate the present results.

**Keywords:** two-enterprise interaction model, delay, oscillation

**AMS Mathematical Subject Classification:** 34K13

## 1 Introduction

Recently, many researchers have investigated the dynamics of the enterprise mathematical models from biological point of view and described the development progress. One can also forecast the growth tendency of enterprises by analyzing the dynamics of the models [1-15]. For example, in [1], assume that the interaction between two enterprises are continuous and the outputs of two enterprises satisfy a certain relation between resource and consumers, Xu proposed the following model:

$x' =$

$$x_1(t) = r_1 x_1(t) \left(1 - \frac{x_1(t)}{K_1}\right) - \frac{c_2 x_2(t)}{K_2},$$

$x_2(t) = r_2 x_2(t) \left(1 - \frac{x_2(t)}{K_2}\right) + \frac{c_1 x_1(t)}{K_1},$

$$x_1(0) = x_1(0), \quad x_2(0) = x_2(0),$$

(1)

where

$x_1(t)$

$$x_2(t) = r_2 x_2(t) \left(1 - \frac{x_2(t)}{K_2}\right) + \frac{c_1 x_1(t)}{K_1},$$

$x_1(0)$

$$x_2(0) = x_2(0),$$

(1)

where

$$x_1(0) = x_1(0), \quad x_2(0) = x_2(0),$$

(1)

where variables  $x_1(t)$  and  $x_2(t)$  denote the output of two enterprises, respectively;  $r_1$  and  $r_2$  represent respectively the intrinsic growth rate of two enterprises;  $K_1$  and  $K_2$  are the natural market carrying capacity of two enterprises under the unlimited conditions;  $c_1$  denotes the consumption coefficient of the enterprise with the output  $x_2(t)$  to the one with the output  $x_1(t)$ ,  $c_2$  represents the transformation

coefficient of the enterprise with the output  $x_1(t)$  to the one with the output  $x_2(t)$ ;  $x_1(0)$  and  $x_2(0)$  denote the initial output of two enterprises. By applying the coincidence degree theory, the existence of periodic solutions of the model (1) has been investigated.

By regarding the time delay effect among enterprises and based on model (1), Liao et al. considered the following delayed differential system:

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$x' =$

$x_1(t)$

$$x_1(t) = r_1 x_1(t) \left(1 - \frac{x_1(t-\tau)}{K_1}\right) - \frac{c_2 x_2(t-\tau)}{K_2},$$

$x_2(t)$

$$x_2(t) = r_2 x_2(t) \left(1 - \frac{x_2(t-\tau)}{K_2}\right) + \frac{c_1 x_1(t-\tau)}{K_1},$$

$x_1(0)$

$x_2(0)$

(1)

$$\begin{aligned} \dot{x}_1(t) &= r_1 x_1(t) (1 - x_1(t - \tau_1)) \\ &\quad - \frac{c_1 x_1(t) x_2(t - \tau_2)}{K_1 + x_1(t - \tau_1) + c_1 x_2(t - \tau_2)}, \\ \dot{x}_2(t) &= r_2 x_2(t) (1 - x_2(t - \tau_2)) \\ &\quad + \frac{c_2 x_1(t - \tau_1) x_2(t)}{K_2 + x_2(t - \tau_2) + c_2 x_1(t - \tau_1)}, \end{aligned}$$

(2)

By choosing the delay  $\tau_1$  as the parameter and using the linearized method, the effect of the delay on the stability of the positive equilibrium of the model (2) and Hopf bifurcation also obtained [2]. Then Liao et al. [3] considered the following two delayed differential equations:

$$\begin{aligned} \dot{x}_1(t) &= r_1 x_1(t) (1 - x_1(t - \tau_1)) \\ &\quad - \frac{c_2 x_1(t) x_2(t - \tau_2)}{K_1 + x_1(t - \tau_1) + c_2 x_2(t - \tau_2)}, \\ \dot{x}_2(t) &= r_2 x_2(t) (1 - x_2(t - \tau_2)) \\ &\quad + \frac{c_1 x_1(t - \tau_1) x_2(t)}{K_2 + x_2(t - \tau_2) + c_1 x_1(t - \tau_1)}, \end{aligned}$$

(3)

By choosing  $\tau_1$  or  $\tau_2$  as the bifurcation parameter, the authors investigated the dynamical behaviors of the system (3) and obtained some new results.

General speaking, the time delay effects in the interior of a certain enterprise and among different enterprises are completely different. Hence, a more reasonable model based on (2) and (3) should be described by the following delayed differential equations with multiple different delays [4]:

$$\begin{aligned} \dot{x}_1(t) &= r_1 x_1(t) (1 - x_1(t - \tau_1)) \\ &\quad - \frac{c_2 x_1(t) x_2(t - \tau_2)}{K_1 + x_1(t - \tau_1) + c_2 x_2(t - \tau_2)}, \\ \dot{x}_2(t) &= r_2 x_2(t) (1 - x_2(t - \tau_3)) \\ &\quad + \frac{c_1 x_1(t - \tau_4) x_2(t)}{K_2 + x_2(t - \tau_3) + c_1 x_1(t - \tau_4)}, \end{aligned}$$

(4)

Noting that there are four time delays in model (4), the existence of Hopf bifurcation of system (4) is more difficult due to the complexity of the associated characteristic equation. In particular, an analysis regarding the distribution of roots in the complex plane of the transcendental polynomial characteristic equation with multiple different exponential terms is still an open problem [4]. Therefore, Li et al. assume that  $\tau_1 = \tau_3 = 0$ ,  $\tau_2 + \tau_4 > 0$  and denote  $\tau_2$  and  $\tau_4$  by  $\tau_1$  and  $\tau_2$ , namely, for the following model of the form:

$$\begin{aligned} \dot{x}_1(t) &= r_1 x_1(t) (1 - x_1(t - \tau_1)) \\ &\quad - \frac{c_2 x_1(t) x_2(t - \tau_2)}{K_1 + x_1(t - \tau_1) + c_2 x_2(t - \tau_2)}, \\ \dot{x}_2(t) &= r_2 x_2(t) (1 - x_2(t - \tau_2)) \\ &\quad + \frac{c_1 x_1(t - \tau_1) x_2(t)}{K_2 + x_2(t - \tau_2) + c_1 x_1(t - \tau_1)}, \end{aligned}$$

$$\begin{aligned}
y_1'(t) &= (y_1(t) + c_1)(d_1 - a_1 y_1(t) - b_1 y_2(t - \tau_1)), \\
y_2'(t) &= (y_2(t) + c_2)(d_2 - a_2 y_2(t) + b_2 y_1(t - \tau_2)), \\
y_1(t) &= \varphi_1(t), \quad y_2(t) = \varphi_2(t), \quad t \in [-\max\{\tau_1, \tau_2\}, 0],
\end{aligned}
\tag{5}$$

where  $a_1 = r_1$

$K_1$

,  $a_2 = r_2$

$K_2$

,  $b_1 = \frac{r_1}{K_1}$

$K_1$

,  $b_2 = \frac{r_2}{K_2}$

$K_2$

,  $d_1 = r_1 - a_1 c_1$ ,  $d_2 = r_2 - a_2 c_2$ ,  $y_1(t) = x_1(t) - c_1$ , and

$y_2(t) = x_2(t) - c_2$ . By choosing  $\tau = \tau_1 + \tau_2$  as the bifurcation parameter and using the linearized method,

the authors have discussed the asymptotical stability of the unique positive equilibrium and the Hopf bifurcation periodic solution of the system (5). Apart from the bifurcation method, Hamoud et al.

used some iterative methods to deal with different integro-differential equations [17-20]. Noting that the bifurcating method or iterative method are so hard to deal with model (4). We must use another

method to study the dynamic behavior of model (4). In this paper, by means of the mathematical

analysis method, the existence of periodic oscillatory solutions for four different time delays in model

(4) is obtained. An open problem is solved. Computer simulation indicates that our result is correct.

## 2 Main result

We can rewrite model (4) as the following form:

$\mathbb{R}^2 \times \mathbb{C}^2 \rightarrow \mathbb{R}^2 \times \mathbb{C}^2$ :

$$y_1'(t) = (y_1(t) + c_1)(d_1 - a_1 y_1(t - \tau_1) - b_1 y_2(t - \tau_2)),$$

$$y_2'(t) = (y_2(t) + c_2)(d_2 - a_2 y_2(t - \tau_3) + b_2 y_1(t - \tau_4)),$$

$$y_1(t) = \varphi_1(t), \quad y_2(t) = \varphi_2(t), \quad t \in [-\max\{\tau_1, \tau_2, \tau_3, \tau_4\}, 0],$$

$$\tag{6}$$

$$\tag{6}$$

where the parameters  $a_i$ ,  $b_i$ ,  $c_i$ , and  $d_i$  ( $i = 1, 2$ ) are the same as in model (5).

**Lemma 1** [4] Assume that

$a_{21}$

$d_1 > b_1 d_2$

, (7)

holds, then system (6) has a unique positive equilibrium point  $(y_1^*, y_2^*)$ .

1,  $y_1^*$

2).

**Lemma 2** All solutions of system (6) are bounded.

**Proof** It is known that time delay affect the stability of the solutions, it can not affect the boundedness

of the solutions. Therefore, we can only consider the boundedness of the following without time

delay system:

$\mathbb{R}^2 \times \mathbb{C}^2$ :

$$y_1'(t) = (y_1(t) + c_1)(d_1 - a_1 y_1(t) - b_1 y_2(t)),$$

$$y_2'(t) = (y_2(t) + c_2)(d_2 - a_2 y_2(t) + b_2 y_1(t)).$$

$$y_2'(t) = (y_2(t) + c_2)(d_2 - a_2 y_2(t) + b_2 y_1(t)).$$

$$y_1(t).$$

(8)

Noting that all parameters are positive real numbers in system (8), and we only consider the boundedness of positive solutions. We first fixed  $y_2(t) = y_{20} > 0$  suitably large, then in the first equation of system (8) we have  $y_1'(t) < 0 (t > t_0)$  for some  $t_0$  since  $b_1 > 0$ . This means that  $y_1(t)$  is bounded. Since  $y_1(t)$  is bounded, in the second equation of system (8) we have  $y_2'(t) < 0 (t > t_1)$  for some  $t_1$  since  $a_2 > 0$ . This means that  $y_2(t)$  also is bounded. The proof is completed.

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If system (6) has a unique equilibrium point  $(y^*$

$1, y^*$

2), setting that  $u(t) = y_1(t) - y^*$

$1, v(t) = y_2(t) - y^*$

2,

then system (6) has the following equivalent form:

8>>>><<>>>>:

$u'(t) = (u(t) + y^*$

$1 + c_1)[d_1 - a_1(u(t - \tau_1) + y^*$

$1) - b_1(v_2(t - \tau_2) + y^*$

2)],

$v'(t) = (v(t) + y^*$

$2 + c_2)[d_2 - a_2(v(t - \tau_3) + y^*$

2) +  $b_2(u_2(t - \tau_4) + y^*$

1)],

$u(t) = u_1(t), v(t) = v_1(t), t \in [-\max_{1 \leq i \leq 4} \{\tau_i\}, 0]$ .

(9)

Thus, the zero equilibrium point in system (9) corresponds to the positive equilibrium point  $(y^*$

$1, y^*$

2)

in system (6). In order to discuss the instability of the equilibrium point  $(y^*$

$1, y^*$

2) in system (6) we

only need to consider the instability of the trivial solution in system (9), consider an auxiliary system of (9) as the following:

8>>>><<>>>>:

$u'(t) = (u(t) + y^*$

$1 + c_1)[d_1 - a_1(u(t - \tau) + y^*$

$1) - b_1(v_2(t - \tau) + y^*$

2)],

$v'(t) = (v(t) + y^*$

$2 + c_2)[d_2 - a_2(v(t - \tau) + y^*$

2) +  $b_2(u_2(t - \tau) + y^*$

1)],

$u(t) = u_1(t), v(t) = v_1(t), t \in [-\tau, 0]$ ,

(10)

where  $\tau = \max_{1 \leq i \leq 4} \{\tau_i\}$ . According to the basic theory of functional differential equation, if the trivial solution in system (10) is unstable, when the time delays increase in the system and the instability of the solutions still maintain [16]. In other words, the instability of the trivial solution in system (10) implies that the trivial solution in system (9) is unstable, thus the positive equilibrium point  $(y^*$

$1, y^*$

2)

in system (6) is unstable. The linearized system of (10) is the form:

$\delta \dot{x} =$

$$u'(t) = p_1 u(t) - q_{11} u(t - \tau) - q_{12} v(t - \tau),$$

$$v'(t) = p_2 v(t) - q_{21} v(t - \tau) + q_{22} u(t - \tau),$$

(11)

where  $p_1 = d_1 - a_1 y^*$

$$1 - b_1 y^{*2}$$

$$2, p_2 = d_2 - a_2 y^*$$

$$2 + b_2 y^{*2}$$

$$1; q_{11} = a_1 (y^*$$

$$1 + c_1), q_{12} = 2b_1 y^*$$

$$2(y^*$$

$$1 + c_1),$$

$$q_{21} = a_2 (y^*$$

$$2 + c_2), q_{22} = 2b_2 y^*$$

$$1(y^*$$

$2 + c_2)$ . System (11) also can be written as a matrix form:

$$U'(t) = PU(t) + QU(t - \tau), \quad (12)$$

where  $U(t) = [u(t), v(t)]^T, U(t - \tau) = [u(t - \tau), v(t - \tau)]^T, P = \text{diag}(p_1, p_2)$ , and

$$Q = \begin{pmatrix} -q_{11} & -q_{12} \\ -q_{21} & q_{22} \end{pmatrix}$$

$$1A$$

.

**Theorem 1** Assume that the lemma 1 holds for selecting parameter values. At least one of two eigenvalues  $\lambda_1$  and  $\lambda_2$  of matrix  $Q$  satisfies

$$\tau |\lambda_i| > 4e^{|\lambda_i| \tau} \quad (i = 1, 2). \quad (13)$$

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Then there exists a limit cycle in system (10), implying that system (6) has a periodic solution.

**Proof** We will show that the trivial solution of linearized system (11) is unstable. Let  $\lambda_1$  and  $\lambda_2$  be two eigenvalues of matrix  $Q$ , then the characteristic equations of system (11) are

$$\lambda - p_i - i e^{-\lambda \tau} = 0 \quad (i = 1, 2). \quad (14)$$

Thus, we are led to an investigation of the nature of the roots of

$$\lambda = p_i + i e^{-\lambda \tau} \quad (i = 1, 2). \quad (15)$$

Suppose that the trivial solution of system (11) is stable, then there exists a negative root say  $\lambda^*$  such that

$$\lambda^* = p_i + i e^{-\lambda^* \tau}, \quad (16)$$

for some  $i (= 1, \text{ or } 2)$ . Then

$$|\lambda^*| + |p_i| \leq |i| e^{|\lambda^*| \tau}. \quad (17)$$

Using the formula  $e^x = 1 + x + \frac{x^2}{2} + \dots$

$x > 0$ ) we have

$$1 +$$

$$|i| e^{|\lambda^*| \tau}$$

$$|\lambda^*| + |p_i|$$

$$=$$

$$1 + |\lambda^*| \tau + |p_i| \tau$$

$$= (|\lambda^*| + |p_i|) \cdot e^{|\lambda^*| \tau} =$$

$$\begin{aligned} & 7_{-2}|i|(|_{-}^*| + |p_i|) \\ & 4e^{|p_i|_{-}} \\ & > \\ & 7_{-2}|i||p_i| \\ & 4e^{|p_i|_{-}} \\ & . \quad (18) \end{aligned}$$

A contradiction with (13) and hence the trivial solution of system (11) is unstable. This implies that the trivial solution in system (9) is unstable. Also this suggests that the unique positive equilibrium point  $(y^*$

1,  $y^*$

2) in system (6) is unstable. Since all solutions in system (6) are bounded, based on the extended Chafee's limit cycle criterion [21, 22], system (6) generates a limit cycle, namely, a periodic solution. the proof is completed.

**Theorem 2** Assume that the lemma 1 holds for selecting parameter values. At least one of two eigenvalues  $\lambda_1$  and  $\lambda_2$  of matrix  $Q$  satisfies

$$p_i + i > 0, \quad i \in \{1, 2\}, \quad (19)$$

then the trivial solution of system (11) is unstable, implying that system (6) generates a limit cycle, namely, a periodic solution.

**Proof** We still consider the characteristic equation (14). Noting that the characteristic equation (14) is a transcendental equation, one cannot calculate its roots explicitly. However, we claim that equation (14) has a real positive root. Let  $f(\lambda) = \lambda - p_i - i e^{-\lambda}$ . Then  $f(\lambda)$  is a continuous function of  $\lambda$ . Noting that  $f(0) = -p_i - i = -(p_i + i) < 0$  since  $p_i + i > 0$ . On the other hand,  $f(\lambda) \rightarrow +\infty$  as

$\lambda \rightarrow +\infty$ . Therefore, there exists a suitably large positive number  $L$  such that  $L - p_i - i e^{-L} > 0$ . According to the Intermediate Value Theorem of continuous function, there exists a  $\lambda_0 \in (0, L)$  such that  $f(\lambda_0) = 0$ . In other words, there exists a positive characteristic root of the characteristic equation (14). Thus, the trivial solution of system (11) is unstable, implying that the unique positive equilibrium point  $(y^*$

1,  $y^*$

2) in system (6) is unstable. This instability of the unique equilibrium together with the boundedness of the solutions will force system (6) to generate a limit cycle, namely, a periodic solution.

### 3 Simulation results

This simulation is based on system (6). We first select  $a_1 = 0.2$ ,  $a_2 = 0.5$ ,  $b_1 = 0.1$ ,  $b_2 = 0.08$ ,  $c_1 = 1.2$ ,  $c_2 = 1.2$ ,  $d_1 = 0.35$ ,  $d_2 = 0.35$ . Then  $a_2 d_1$

$$d_1 = 0.0140, \quad b_1 d_2$$

$$= 0.0121. \quad \text{Therefore, } a_2 d_1$$

$$d_1 > b_1 d_2$$

, the

condition of lemma 1 holds. The unique positive equilibrium point is  $(y^*$

1,  $y^*$

$$2) = (1.1372, 1.1069). \quad \text{We}$$

$$\text{have } p_1 = d_1 - a_1 y^*$$

$$1 - b_1 y^*$$

$$2 = 0.0016, \quad p_2 = d_2 - a_2 y^*$$

$$2 + b_2 y^*$$

$$1 = 0.0005; \quad \text{Two eigenvalues of matrix } Q \text{ are}$$

$$\lambda_1 = 1.2172, \quad \text{and } \lambda_2 = -0.3288, \quad \text{respectively. So we get } p_1 + 1 = 1.2188 > 0. \quad \text{The condition of Theorem}$$

$$2 \text{ is satisfied. When we select different time delays as } \tau_1 = 1.80, \quad \tau_2 = 1.45, \quad \tau_3 = 1.55, \quad \tau_4 = 1.25, \quad \text{and}$$

$$\tau_1 = 1.35, \quad \tau_2 = 1.55, \quad \tau_3 = 1.65, \quad \tau_4 = 1.68, \quad \text{respectively, there exists a periodic oscillatory solution (see$$

figure 1). However, when we select  $\tau = 1.25$ , The condition (13) of Theorem 1 is not satisfied, implying that condition (13) is a stronger restrictive condition. Then we select  $a_1 = 0.25$ ,  $a_2 = 0.65$ ,  $b_1 = 0.20$ ,  $b_2 = 0.12$ ,  $c_1 = 0.95$ ,  $c_2 = 0.85$ ,  $d_1 = 0.38$ ,  $d_2 = 0.42$ . We have  $a_2 d_1$

$$d_1 = 0.0548, b_1 d_2$$

$$= 0.0352. \text{ So}$$

$$a_2 d_1$$

$$d_1 > b_1 d_2$$

still holds. The unique positive equilibrium point is  $(y^*$

$$1, y^*$$

$$2) = (0.9781, 0.8232). \text{ We have}$$

$$p_1 = d_1 - a_1 y^*$$

$$1 - b_1 y^2$$

$$2 = 0.0003, p_2 = d_2 - a_2 y^*$$

$$2 + b_2 y^2$$

$\lambda_1 = -0.0002$ . Two eigenvalues of matrix  $Q$  are

$$\lambda_1 = 1.2746, \text{ and } \lambda_2 = -0.3999. \text{ So we have } p_1 + \lambda_1 = 1.2749 > 0. \text{ The condition of Theorem 2 is}$$

still satisfied. When we select different time delays as  $\tau_1 = 1.45$ ,  $\tau_2 = 1.35$ ,  $\tau_3 = 1.55$ ,  $\tau_4 = 1.38$ , and  $\tau_1 = 1.95$ ,  $\tau_2 = 1.24$ ,  $\tau_3 = 1.28$ ,  $\tau_4 = 1.20$ , respectively, there exists a periodic oscillatory solution (see figure 2).

## 4 Conclusion

In this paper, we have discussed a two-enterprise interaction model with four different delays by means of the mathematical analysis method. The existence of periodic oscillatory solution is proposed. An open problem in the literature is solved. We believe that the mathematical analysis method can be used to deal with higher dimensional enterprise interaction models with multiple time delays. This is our future research direction. The two theorems are only sufficient conditions. Some simulation is

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provided to indicate that the criterion is an effective method. From the simulation we know that the theorem 1 is a stronger sufficient condition.

### Competing Interests

Authors have declared that no competing interests exist.

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0 50 100 150

(a) Delays: 1.80, 1.45, 1.55, 1.25.

0

0.5

1

1.5

2

Fig. 1 Periodic oscillation of the solutions, solid line:  $y_1(t)$ , dashed line:  $y_2(t)$ .

0 50 100 150

(b) Delays: 1.35, 1.55, 1.65, 1.68.

0

0.5

1

1.5

2

0 20 40 60 80 100 120

(b) Delays: 1.95, 1.24, 1.28, 1.20.

0

0.5

1

1.5

0 20 40 60 80 100 120

(a) Delays: 1.45, 1.35, 1.55, 1.38.

0

0.5

1

1.5

Fig. 2 Periodic oscillation of the solutions, solid line:  $y_1(t)$ , dashed line:  $y_2(t)$ .

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