The existence of periodic solutions for a two-enterprise interaction model with delays

Abstract: In this paper, a two-enterprise interaction model with four delays is investigated. By means of the mathematical analysis method, some sufficient conditions to guarantee the existence of periodic oscillatory solution for the model are obtained. An open problem is solved. Computer simulations are provided to demonstrate the proposed results.

Keywords: two-enterprise interaction model, delay, oscillation

1 Introduction

Recently, many researchers have investigated the dynamics among enterprises from biological point of view and described the development progress and the growth tendency of enterprises by analyzing the dynamics of mathematical models [1-15]. For example, in [1], assume that the interaction between two enterprises are continuous and the outputs of two enterprises satisfy a certain relation between resource and consumers, Xu proposed the following model:

$$\begin{cases} x_1'(t) = r_1 x_1(t)(1 - \frac{x_1(t)}{K_1} - \frac{\alpha(x_2(t)-c_2)^2}{K_2}), \\ x_2'(t) = r_2 x_2(t)(1 - \frac{x_2(t)}{K_2} + \frac{\beta(x_1(t)-c_1)^2}{K_1}), \\ x_1(0) \geq 0, x_2(0) \geq 0. \end{cases} \tag{1}$$

where variables $x_1(t)$ and $x_2(t)$ denote the output of two enterprises, respectively; r_1 and r_2 represent respectively the intrinsic growth rate of two enterprises; K_1 and K_2 are the natural market carrying capacity of two enterprises under the unlimited conditions; α is the consumption coefficient of the enterprise with the output $x_2(t)$ to the one with the output $x_1(t)$ and β denotes the transformation coefficient of the enterprise with the output $x_1(t)$ to the one with the output $x_2(t)$; c_1 and c_2 denote the initial output of two enterprises. By applying the coincidence degree theory, the existence of periodic solutions of the model (1) has been investigated.

By regarding the time delay effect among enterprises and on the basis of the model (1), Liao et al. extended model (1) into the following delayed differential system:

$$\begin{cases} x_1'(t) = r_1x_1(t)(1 - \frac{x_1(t-\tau)}{K_1} - \frac{\alpha(x_2(t-\tau)-c_2)^2}{K_2}), \\ x_2'(t) = r_2x_2(t)(1 - \frac{x_2(t-\tau)}{K_2} + \frac{\beta(x_1(t-\tau)-c_1)^2}{K_1}), \\ x_1(t) = \phi(t), x_2(t) = \varphi(t), t \in [-\tau, 0]. \end{cases} \quad (2)$$

By choosing the delay τ as the parameter and using the linearization method, the effect of the delay on the stability of the positive equilibrium of the model (2) and Hopf bifurcation also obtained [2]. Then Liao et al. [3] considered the following two delayed differential equations:

$$\begin{cases} x_1'(t) = r_1x_1(t)(1 - \frac{x_1(t-\tau_1)}{K_1} - \frac{\alpha(x_2(t-\tau_2)-c_2)^2}{K_2}), \\ x_2'(t) = r_2x_2(t)(1 - \frac{x_2(t-\tau_1)}{K_2} + \frac{\beta(x_1(t-\tau_2)-c_1)^2}{K_1}), \\ x_1(t) = \phi(t), x_2(t) = \varphi(t), t \in [-\max_{1\le i\le 2}\{\tau_i\}, 0]. \end{cases} \quad (3)$$

By choosing τ_1 or τ_2 as the bifurcation parameter, Liao et al. [3] investigated the dynamical behaviors of the system (3) and obtained some interesting results.

General speaking, the time delay effects in the interior of a certain enterprise and among different enterprises are completely different. Hence, a more reasonable model based on (2) and (3) should be described by the following delayed differential equations with multiple different delays [4]:

$$\begin{cases} x_1'(t) = r_1x_1(t)(1 - \frac{x_1(t-\tau_1)}{K_1} - \frac{\alpha(x_2(t-\tau_2)-c_2)^2}{K_2}), \\ x_2'(t) = r_2x_2(t)(1 - \frac{x_2(t-\tau_3)}{K_2} + \frac{\beta(x_1(t-\tau_4)-c_1)^2}{K_1}), \\ x_1(t) = \phi(t), x_2(t) = \varphi(t), t \in [-\max_{1\le i\le 4}\{\tau_i\}, 0]. \end{cases} \quad (4)$$

Noting that there are four time delays in model (4), the existence of Hopf bifurcation of system (4) is more difficult and complex since the appearance of multiple delays leads to the analysis of the associated characteristic equation is more difficult. In particular, a complete analysis regarding the distribution of roots in the complex plane of the transcendental polynomial characteristic equation with multiple different exponential terms is still an open problem [16]. Therefore, Li et al. assume that $\tau_1 = \tau_3 = 0$, $\tau_2 + \tau_4 > 0$ and denote τ_2 and τ_4 by τ_1 and τ_2, namely, for the following model of the form [4]:

$$\begin{cases} y_1'(t) = (y_1(t) + c_1)(d_1 - a_1y_1(t) - b_1y_2^2(t-\tau_1)), \\ y_2'(t) = (y_2(t) + c_2)(d_2 - a_2y_2(t) + b_2y_1^2(t-\tau_2)), \\ y_1(t) = \phi(t), y_2(t) = \varphi(t), t \in [-\max_{1\le i\le 2}\{\tau_i\}, 0]. \end{cases} \quad (5)$$

where $a_1 = \frac{r_1}{K_1}, a_2 = \frac{r_2}{K_2}, b_1 = \frac{\alpha r_1}{K_1}, b_2 = \frac{\beta r_2}{K_2}, d_1 = r_1 - a_1c_1$, $d_2 = r_2 - a_2c_2$, $y_1(t) = x_1(t) - c_1$, and $y_2(t) = x_2(t) - c_2$. By choosing $\tau = \tau_1 + \tau_2$ as the bifurcation parameter and using the linearization method, the authors found that when τ is less than a certain critical value, the unique positive equilibrium of the system (5) is locally asymptotically stable while it becomes unstable when τ is greater than this critical value through Hopf bifurcation. Noting that the bifurcating method is so hard to deal with model (4). We must use another method to study the dynamic behavior of model (4). In this paper, by means of the mathematical analysis method, the existence of periodic solutions for four different time delays in model (4) is obtained. An open problem is solved. Computer simulation indicates that our result is correct.

2 Main result

We can rewrite model (4) as the following form:

$$\begin{cases} y_1'(t) = (y_1(t) + c_1)(d_1 - a_1 y_1(t-\tau_1) - b_1 y_2^2(t-\tau_2)), \\ y_2'(t) = (y_2(t) + c_2)(d_2 - a_2 y_2(t-\tau_3) + b_2 y_1^2(t-\tau_4)), \\ y_1(t) = \phi(t), y_2(t) = \varphi(t), t \in [-\max_{1\le i\le 4}\{\tau_i\}, 0]. \end{cases} \tag{6}$$

where the parameters a_i, b_i, c_i, and $d_i (i = 1, 2)$ are the same as in model (5).

Lemma 1[4] Assume that

$$a_1^2 d_1 > b_1 d_2^2 \tag{7}$$

holds, then system (6) has a unique positive equilibrium point (y_1^*, y_2^*).

Lemma 2 All solutions of system (6) are bounded.

Proof It is known that time delay affect the stability of the solutions, it can not affect the boundedness of the solutions. Therefore, we can only consider the boundedness of the following without time delay system:

$$\begin{cases} y_1'(t) = (y_1(t) + c_1)(d_1 - a_1 y_1(t) - b_1 y_2^2(t)), \\ y_2'(t) = (y_2(t) + c_2)(d_2 - a_2 y_2(t) + b_2 y_1^2(t)). \end{cases} \tag{8}$$

Noting that all parameters are positive real numbers in system (8). Fixed $y_2(t) = y_{20} > 0$ suitably large, then in the first equation of system (8) we have $y_1'(t) < 0 (t > t_0)$ for some t_0 since $b_1 > 0$. This means that $y_1(t)$ is bounded. Since $y_1(t)$ is bounded, in the second equation of system (8) we have $y_2'(t) < 0 (t > T_0)$ for some T_0 since $a_2 > 0$. This means that $y_2(t)$ also is bounded. The proof is completed.

If system (6) has a unique equilibrium point (y_1^*, y_2^*), setting that $u(t) = y_1(t) - y_1^*, v(t) = y_2(t) - y_2^*$, then system (6) has the following equivalent form:

$$\begin{cases} u'(t) = (u(t) + y_1^* + c_1)[d_1 - a_1(u(t-\tau_1) + y_1^*) - b_1(v^2(t-\tau_2) + y_2^*)], \\ v'(t) = (v(t) + y_2^* + c_2)[d_2 - a_2(v(t-\tau_3) + y_2^*) + b_2(u^2(t-\tau_4) + y_1^*)], \\ u(t) = \phi_1(t), v(t) = \varphi_1(t), t \in [-\max_{1\le i\le 4}\{\tau_i\}, 0]. \end{cases} \tag{9}$$

In order to discuss the instability of the equilibrium point (y_1^*, y_2^*) in system (9), consider an auxiliary system of (9) as the following:

$$\begin{cases} u'(t) = (u(t) + y_1^* + c_1)[d_1 - a_1(u(t-\tau) + y_1^*) - b_1(v^2(t-\tau) + y_2^*)], \\ v'(t) = (v(t) + y_2^* + c_2)[d_2 - a_2(v(t-\tau) + y_2^*) + b_2(u^2(t-\tau) + y_1^*)], \\ u(t) = \phi_1(t), v(t) = \varphi_1(t), t \in [-\tau, 0]. \end{cases} \tag{10}$$

where $\tau = \min_{1\le i\le 4}\{\tau_i\}$. Obviously, (y_1^*, y_2^*) is also a unique positive equilibrium point in system (10). According to the basic theory of functional differential equation, if the unique positive equilibrium point (y_1^*, y_2^*) in system (10) is unstable, when the time delays increase in the system and the instability of the solutions still maintain. In other words, the instability of the positive equilibrium point (y_1^*, y_2^*) in system (10) implying that the instability of the positive equilibrium point (y_1^*, y_2^*) in system (9) [16]. The linearized system of (10) is the form:

$$\begin{cases} u'(t) = p_1 u(t) - q_{11} u(t-\tau) - q_{12} v(t-\tau), \\ v'(t) = p_2 v(t) - q_{21} v(t-\tau) + q_{22} u(t-\tau). \end{cases} \tag{11}$$

where $p_1 = d_1 - a_1 y_1^* - b_1 y_2^{*2}$, $p_2 = d_2 - a_2 y_2^* + b_2 y_1^{*2}$; $q_{11} = a_1(y_1^* + c_1)$, $q_{12} = 2b_1 y_2^*(y_1^* + c_1)$, $q_{21} = a_2(y_2^* + c_2)$, $q_{22} = 2b_2 y_1^*(y_2^* + c_2)$. System (11) also can be written as a matrix form:

$$U'(t) = PU(t) + QU(t-\tau) \tag{12}$$

where $U(t) = [u(t), v(t)]^T, U(t-\tau) = [u(t-\tau), v(t-\tau)]^T$, P=diag(p_1, p_2), and

$$Q = \begin{pmatrix} -q_{11} & -q_{12} \\ -q_{21} & q_{22} \end{pmatrix}.$$

Theorem 1 Assume that the lemma 1 holds for selecting parameter values. At least one of two eigenvalues γ_1 and γ_2 of matrix Q satisfies

$$7\tau^2|\gamma_i||p_i| > 4e^{|p_i|\tau}(i = 1, 2) \tag{13}$$

Then there exists a limit cycle in system (10), implying that system (9) has a periodic solution.

Proof We will show that the trivial solution of linearized system (11) is unstable. Let γ_1 and γ_2 be two eigenvalues of matrix Q, then the characteristic equation of system (11) are

$$\lambda - p_i - \gamma_i e^{-\lambda\tau} = 0(i = 1, 2) \tag{14}$$

Thus, we are led to an investigation of the nature of the roots of

$$\lambda = p_i + \gamma_i e^{-\lambda\tau} (i = 1, 2) \tag{15}$$

Suppose that the trivial solution of system (11) is stable, then there exists a negative root say λ^* such that

$$\lambda^* = p_i + \gamma_i e^{-\lambda^*\tau} \tag{16}$$

for some $i(= 1, \text{ or } 2)$. Then

$$|\lambda^*| + |p_i| \geq |\gamma_i| e^{|\lambda^*|\tau} \tag{17}$$

Using the formula $e^x \geq \frac{7}{4}x^2 (x > 0)$ we have

$$1 \geq \frac{|\gamma_i| e^{|\lambda^*|\tau}}{|\lambda^*| + |p_i|} = \frac{\tau|\gamma_i| e^{(|\lambda^*|+|p_i|)\tau}}{\tau(|\lambda^*| + |p_i|) \cdot e^{|p_i|\tau}} = \frac{7\tau^2|\gamma_i|(|\lambda^*| + |p_i|)}{4e^{|p_i|\tau}} > \frac{7\tau^2|\gamma_i||p_i|}{4e^{|p_i|\tau}} \tag{18}$$

A contradiction with (13) and hence the trivial solution of system (11) is unstable. This implies that the unique positive equilibrium point (y_1^*, y_2^*) in system (10) is unstable. Also this suggests that the unique positive equilibrium point (y_1^*, y_2^*) in system (9) is unstable. Since all solutions in system (9) are bounded, based on the extended Chafee's limit cycle criterion [17, 18], system (9) generates a limit cycle, namely, a periodic solution. the proof is completed.

Theorem 2 Assume that the lemma 1 holds for selecting parameter values. At least one of two eigenvalues γ_1 and γ_2 of matrix Q satisfies

$$\gamma_i + p_i > 0, i \in \{1, 2\} \tag{19}$$

then the trivial solution of system (11) is unstable, implying that system (9) generates a limit cycle, namely, a periodic solution.

Proof We still consider the characteristic equation (14). Noting that the characteristic equation (14) is a transcendental equation, one cannot calculate its roots explicitly. However, we claim that equation (14) has a real positive root. Let $f(\lambda) = \lambda - p_i - \gamma_i e^{-\lambda\tau}$. Then $f(\lambda)$ is a continuous function of λ. Noting that $f(0) = -p_i - \gamma_i = -(p_i + \gamma_i) < 0$ since $\gamma_i + p_i > 0$. On the other hand, $f(\lambda) \to +\infty$ as $\lambda \to +\infty$. Therefore, there exists a suitably large positive number L such that $L - p_i - \gamma_i e^{-L\tau} > 0$. According to the Intermediate Value Theorem of continuous function, there exists a $\lambda_0 \in (0, L)$ such that $f(\lambda_0) = 0$. In other words, there exists a positive characteristic root of the characteristic equation (14). Thus, the trivial solution of system (11) is unstable, implying that the unique positive equilibrium point (y_1^*, y_2^*) in system (10), also in system (9) is unstable. This instability of the unique equilibrium combine with the boundedness of the solutions will force system (9) to generate a limit cycle, namely, a periodic solution.

3 Simulation results

This simulation is based on system (6). We first select $a_1 = 0.2, a_2 = 0.5, b_1 = 0.1, b_2 = 0.08, c_1 = 1.2, c_2 = 1.2, d_1 = 0.35, d_2 = 0.35$. Then $a_1^2 d_1 = 0.014, b_1 d_2^2 = 0.0121$. Therefore, $a_1^2 d_1 > b_1 d_2^2$, the condition of lemma 1 holds. The unique positive equilibrium point is $(y_1^*, y_2^*) = (1.1372, 1.1069)$. We have $p_1 = d_1 - a_1 y_1^* - b_1 y_2^{*2} = 0.0016, p_2 = d_2 - a_2 y_2^* + b_2 y_1^{*2} = 0.00046$; Two eigenvalues of matrix Q are $\gamma_1 = 1.2172$, and $\gamma_2 = -0.3288$, respectively. So we get $\gamma_1 + p_1 = 1.2188 > 0$. The condition of Theorem 2 is satisfied. When we select time delays as $\tau_1 = 1.80, \tau_2 = 1.45, \tau_3 = 1.55, \tau_4 = 1.25$, and $\tau_1 = 1.35, \tau_2 = 1.55, \tau_3 = 1.65, \tau_4 = 1.68$, respectively, there exists a periodic oscillatory solution (see figure 1). However, when we select $\tau = 1.25$, The condition (13) of Theorem 1 is not satisfied, implying that condition (13) is a stronger restrictive condition. Then we select $a_1 = 0.25, a_2 = 0.65, b_1 = 0.20, b_2 = 0.12, c_1 = 0.95, c_2 = 0.85, d_1 = 0.38, d_2 = 0.42$. We have $a_1^2 d_1 = 0.0548, b_1 d_2^2 = 0.0352$. So $a_1^2 d_1 > b_1 d_2^2$ still holds. The unique positive equilibrium point is $(y_1^*, y_2^*) = (0.9781, 0.8232)$. We have $p_1 = d_1 - a_1 y_1^* - b_1 y_2^{*2} = 0.0003, p_2 = d_2 - a_2 y_2^* + b_2 y_1^{*2} = -0.00017$. Two eigenvalues of matrix Q are $\gamma_1 = 1.2746$, and $\gamma_2 = -0.3999$. So we have $\gamma_1 + p_1 = 1.2749 > 0$. The condition of Theorem 2 is still satisfied. When we select time delays as $\tau_1 = 1.45, \tau_2 = 1.35, \tau_3 = 1.55, \tau_4 = 1.38$, and $\tau_1 = 1.95, \tau_2 = 1.24, \tau_3 = 1.28, \tau_4 = 1.20$, respectively, there exists a periodic oscillatory solution (see figure 2).

4 Conclusion

In this paper, we have discussed a two-enterprise interaction model with four different delays. The existence of periodic oscillatory solution which is easy to check, as compared to the bifurcating method has been proposed. An open problem has been solved. Some simulations are provided to indicate the effectness of the criterion. In this paper, theorem 1 is a stronger sufficient condition.

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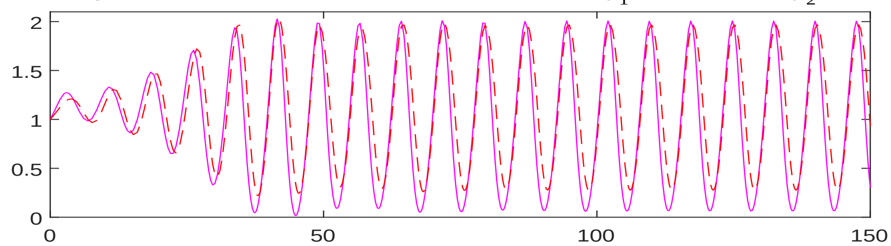
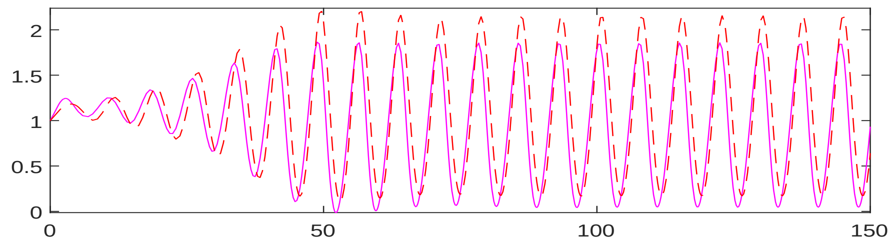
Fig. 1 Periodic oscillation of the solutions, solid line: $y_1(t)$, dashed line: $y_2(t)$.

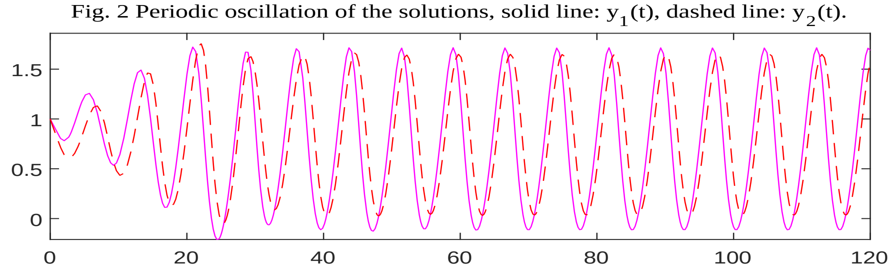


(a) Delays: 1.80, 1.45, 1.55, 1.25.

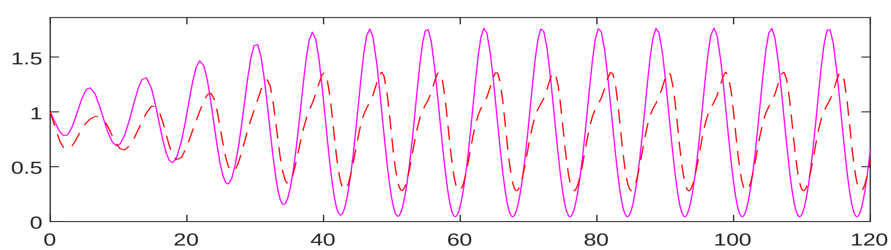


(b) Delays: 1.35, 1.55, 1.65, 1.68.

Fig. 2 Periodic oscillation of the solutions, solid line: $y_1(t)$, dashed line: $y_2(t)$.



(a) Delays: 1.45, 1.35, 1.55, 1.38.



(b) Delays: 1.95, 1.24, 1.28, 1.20.