

Common fixed point theorems for four weakly compatible self maps along with (CLR) property in fuzzy 2-metric spaces

Abstract: In this paper, we prove some common fixed point theorems for four weakly compatible self-maps along with (CLR) property in fuzzy 2- metric spaces. Our results are the improved version of the theorems proved by Shojaei et al. [7] in 2013, since our results does not require closedness of ranges of subsets of X .

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1.Introduction and Preliminaries: In 1965, L.A.Zadeh [10] introduced the notion of fuzzy sets. A lot of authors proved several fixed point theorems by using the concept of fuzzy set theory. The notion of 2-metric spaces was introduced by Gahler [1], [2], [3].

Definition 1.1: A triangular norm $*$ (shortly t-norm) is a binary operation on the unit interval $[0,1]$ such that for all $a, b, c, d \in [0,1]$. The following conditions are satisfied:

1. $a * 1 = a$;
2. $a * b = b * a$;
3. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$
4. $a * (b * c) = (a * b) * c$.

Definition 1.2 ([4]): The 3-tuple $(X, M, *)$ is called a fuzzy metric space, if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following condition:

for all $x, y, z \in X$ and $s, t > 0$

1. $M(x, y, 0) = 0$,
2. $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
5. $M(x, y, \cdot): [0, \infty) \rightarrow [0, 1]$ is left continuous,
6. $\lim_{n \rightarrow \infty} M(x, y, t) = 1$.

Example 1.3: Let (X, d) be a metric space. Define $a * b = ab$ (or $a * b = \min \{a, b\}$) and for all $x, y \in X$ and $t > 0$, $M(x, y, t) = \frac{t}{t+d(x,y)}$. Then $(X, M, *)$ is a fuzzy metric space and this metric d is the standard fuzzy metric.

Definition 1.4 ([6]): A binary operation $*$: $[0, 1] \times [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t - norm if $([0, 1], *)$ is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2$ whenever $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2$ for all a_1, a_2, b_1, b_2 and c_1, c_2 in $[0, 1]$.

Definition 1.5: The 3- tuple $(X, M, *)$ is called a fuzzy 2-metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^3 \times [0, \infty]$ satisfying the following conditions, for all $x, y, z, u \in X$ and $t_1, t_2, t_3 > 0$.

$$1 \ M(x, y, z, 0) = 0.$$

$$2 \ M(x, y, z, t) = 1, t > 0 \text{ and when at least two of the three points are equal,}$$

$$3 \ M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$$

$$4 \ M(x, y, z, t_1 + t_2 + t_3) \geq M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3)$$

(This correspond to tetrahedron inequality in 2-metric space)

The function value $M(x, y, z, t)$ may be interpreted as the probability that the area of triangle is less than t .

$$5 \ M(x, y, z, \cdot) : [0, \infty) \rightarrow [0, 1] \text{ is left continuous.}$$

Example 1.6: Let (X, d) be 2-metric space and denote $a * b = ab$ for all $a, b \in [0, 1]$.

$$\text{For each } h, m, n \in R^+ \text{ and } t > 0, \text{ define } M(x, y, z, t) = \frac{ht^n}{ht^n + md(x, y, z)}.$$

Then $(X, M, *)$ is an fuzzy 2-metric space.

Definition 1.7 ([4]): A sequence $\{x_n\}$ in a fuzzy 2-metric space $(X, M, *)$ is said to converge to x (in X) if and only if $\lim_{n \rightarrow \infty} M(x_n, x, a, t) = 1$ for all $a \in X$ and $t > 0$.

Definition 1.8: Let $(X, M, *)$ be a fuzzy 2-metric space. A sequence $\{x_n\}$ in X is called Cauchy sequence, if and only if $\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, a, t) = 1$ for all $a \in X, p > 0$ and $t > 0$.

Definition 1.9 ([4]): A fuzzy 2-metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent in X .

Definition 1.10: Let $(X, M, *)$ be a fuzzy 2-metric space. Suppose f and g be self maps on X . A point x in X is called a coincidence point of f and g iff $fx = gx$. In this case, $w = fx = gx$ is called a point of coincidence of f and g .

Definition 1.11 ([8]): A pair of self mapping $\{f, g\}$ of a fuzzy 2-metric space (X, d) is said to be weakly compatible if they commute at the coincidence point i.e., if $fu = gu$ for some $u \in X$, then $fgu = gfu$.

It is to see that two compatible maps are weakly compatible but converse is not true.

2. Main Results:

Definition 2.1 ([9]): Let f and g be two self-maps of a 2-metric space $(X, M, *)$, then

they are said to satisfy (CLR_g) property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = g x \text{ for some } x \in X.$$

Similarly, the property (CLR_T) and the property (CLR_S) hold if in the above definition the mapping $g: X \rightarrow X$ has been replaced by the mapping $T: X \rightarrow X$ and $S: X \rightarrow X$.

Example 2.2: let $X = [3, \infty)$. Define $f, g: X \rightarrow X$ by $g x = x + 2$ and $f x = 4x + 2$, for all $x \in X$. Suppose that the (CLR_g) property holds. Then, there exists a sequence $\{x_n\}$ in X satisfying

$$\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = g x \text{ for some } x \in X.$$

Therefore, $\lim_{n \rightarrow \infty} x_n = g x - 2$ and $\lim_{n \rightarrow \infty} x_n = \frac{g x - 2}{4}$.

Thus, $g x = 2$, which is a contradiction, since 2 is not in X .

Hence, f and g do not satisfy (CLR_g) property.

Lemma 2.3 ([5]). Let $(X, M, *)$ be a fuzzy 2-metric space. If there exists $k \in (0, 1)$ such that $M(x, y, z, kt) \geq M(x, y, z, t)$, for all $x, y, z \in X$ with $z \neq x, z \neq y$ and $t > 0$, then $x = y$.

Theorem 2.4. Let A, B, S and T be self-maps of a fuzzy 2-metric spaces $(X, M, *)$ satisfying the following condition :

$$(2.1) \quad AX \subset TX \text{ and } BX \subset SX,$$

$$(2.2) \quad M(Ax, By, z, kt) \geq \phi(M(Sx, Ty, z, t), M(Ax, Sx, z, t), M(By, Ty, z, t), \\ M(Sx, By, z, t), M(Ax, Ty, z, t)),$$

for all x, y, z in X and $t > 0$, where $k \in (0, 1)$.

$$(2.3) \quad \text{the pairs } (A, S) \text{ and } (B, T) \text{ are weakly compatible.}$$

$$(2.4) \quad \text{the pair } (A, S) \text{ satisfies } (CLR_S) \text{ property or the pair } (B, T) \text{ satisfies the } (CLR_T) \text{ property.}$$

Then A, B, S and T have a unique common fixed point in X .

Proof. Suppose that $BX \subset SX$ and (B, T) satisfies property (CLR_T) , then there exists a sequence

$\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} B x_n = \lim_{n \rightarrow \infty} T x_n = T x, \text{ for some } x \in X.$$

Since $BX \subset SX$, therefore there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n \rightarrow \infty} B x_n = \lim_{n \rightarrow \infty} S y_n = T x$$

Hence, $\lim_{n \rightarrow \infty} Sy_n = Tx$

Now, We shall show that $\lim_{n \rightarrow \infty} Ay_n = Tx$

Suppose that

$$\lim_{n \rightarrow \infty} Ay_n = l$$

Putting $x = y_n$ and $y = x_n$ in (2.2), we have

$$M(Ay_n, Bx_n, z, kt) \geq \phi(M(Sy_n, Tx_n, z, t), M(Ay_n, Sy_n, z, t), M(Bx_n, Tx_n, z, t), \\ M(Sy_n, Bx_n, z, t), M(Ay_n, Tx_n, z, t)).$$

Proceeding limit when $n \rightarrow \infty$, we have

$$M(l, Tx, z, kt) \geq \phi(1, M(l, Tx, z, t), 1, 1, M(l, Tx, z, kt)) \geq M(l, Tx, z, t).$$

By Lemma 2.3, we have

$$l = Tx.$$

Therefore, we have $\lim_{n \rightarrow \infty} Ay_n = Tx$.

$$\lim_{n \rightarrow \infty} Ay_n = \lim_{n \rightarrow \infty} Bx_n = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sy_n = Tx = Sv.$$

Now, we shall show that $Av = Tx$.

From (2.2), we have

$$M(Av, Bx_n, z, kt) \geq \phi(M(Sv, Tx_n, z, t), M(Av, Sv, z, t), M(Bx_n, Tx_n, z, t), \\ M(Sv, Bx_n, z, t), M(Av, Tx_n, z, t)).$$

Letting limit as $n \rightarrow \infty$,

$$M(Av, Tx, z, kt) \geq \phi(1, M(Av, Tx, z, t), 1, 1, M(Av, Tx, z, t)) \geq M(Av, Tx, z, t).$$

By Lemma 2.3, we have $Av = Sv = Tx$.

Since $AX \subset TX$, so, there exists $w \in X$ such that $Tx = Av = Tw$.

Now, we claim that

$$Tx = Bw.$$

From (2.2), we have

$$M(Av, Bw, z, kt) \geq \phi(M(Sv, Tw, z, t), M(Av, Sv, z, t), M(Bw, Tw, z, t), \\ M(Sv, Bw, z, t), M(Av, Tw, z, t)).$$

Letting limit as $n \rightarrow \infty$,

$$M(Tx, Bw, z, kt) \geq \phi(1, 1, M(Bw, Tx, z, t), M(Tx, Bw, z, t), 1) \geq M(Tx, Bw, z, t).$$

By Lemma 2.3, we have

$$Tx = Bw.$$

Thus, we have $Av = Sv = Tw = Bw = Tx$.

Since the pair (A, S) is weakly compatible, therefore $ASv = SAV$, i.e., $ATx = STx$.

Now, we show that $ATx = Tx$

Since,

$$\begin{aligned} M(ATx, Bw, z, kt) &\geq \phi(M(STx, Tw, z, t), M(ATx, STx, z, t), M(Bw, Tw, z, t), \\ &\quad M(STx, Bw, z, t), M(ATx, Tw, z, t)), \text{ that is,} \\ M(ATx, Tx, z, kt) &\geq \phi(M(ATx, Tx, z, t), 1, 1, M(ATx, Tx, z, t), M(ATx, Tx, z, t)) \\ &\geq M(ATx, Tx, z, t). \end{aligned}$$

By Lemma 2.3, we have

$$ATx = STx = Tx.$$

The weak compatibility of B and T implies that

$$\begin{aligned} BTw &= TBw \\ \text{i.e. } BTx &= TTx. \end{aligned}$$

Now we shall further show that Tx is the common fixed point of B .

From (2.2), we have

$$\begin{aligned} M(ATx, BTx, z, kt) &\geq \phi(M(STx, TTx, z, t), M(ATx, STx, z, t), M(BTx, TTx, z, t), \\ &\quad M(STx, BTx, z, t), M(ATx, TTx, z, t)). \end{aligned}$$

or

$$M(ATx, BTx, z, kt) \geq \phi(M(ATx, BTx, z, t), 1, 1, M(ATx, BTx, z, t), 1).$$

By Lemma 2.3, we have

$$BTx = Tx.$$

Hence, $ATx = BTx = STx = TTx = Tx$.

Therefore, Tx is the common fixed point of A, B, S and T .

Corollary 2.5. Let A, B, S and T be self-maps of a fuzzy 2-metric space $(X, M, *)$ with continuous t-norm satisfying (2.1), (2.3), (2.4) and the following :

$$(2.5) \quad M(Ax, By, z, t) \geq \min \{M(Sx, Ty, z, t), M(Ax, Sx, z, t), M(Sx, By, z, t), M(Ax, Ty, z, t)\}$$

Holds, for all x, y, z in X and $t > 0$.

Then A, B, S and T have a unique common fixed point in X .

Proof : Taking in the Theorem 2.2

$$\phi(x_1, x_2, x_3, x_4, x_5) = \min\{x_1, x_2, x_3, x_4, x_5\}$$

Now, we consider a function $\psi: [0,1] \rightarrow [0,1]$ satisfying the conditions

(*) ψ is continuous and non-decreasing on $[0,1]$ and $\psi(t) > t$ for all $t \in (0,1)$

Note that $\psi(1) = 1$ and $\psi(t) \geq t$ for all $t \in [0,1]$,

i.e. $\psi(M(x, y, z, t)) \geq M(x, y, z, t)$ holds for every $t > 0$ and for all $x, y \in X$.

Theorem 2.6. Let A, B, S and T be self maps of a fuzzy 2 – metric space $(X, M, *)$

with continuous t-norm $*$ satisfying (2.1), (2.3), (2.4) and the following :

$$(2.6) M(Ax, By, z, t) \geq \psi(\min\{M(Sx, Ty, z, t), M(Ax, Sx, z, t), M(By, Ty, z, t), \\ M(Sx, By, z, t), M(Ax, Ty, z, t)\}).$$

with $M(x, y, z, t) > 0$ for all $x, y, z \in X$ and $t > 0$.

Then A, B, S and T have a unique common fixed point in X .

Proof. Let (A, B) satisfies the (CLR) property.

Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Bx_n = \lim_{n \rightarrow \infty} Tx_n = Tx, \text{ for some } x \in X$$

Since $BX \subset SX$, there exists a sequence $\{y_n\} \in X$ such that

$$Bx_n = Sy_n = Tx.$$

Hence, $\lim_{n \rightarrow \infty} Sy_n = Tx$.

Now, we show that $\lim_{n \rightarrow \infty} Ax_n = Tx$.

Putting $x = y_n, y = x_n$ in (2.6), we have

$$M(Ay_n, Bx_n, z, t) \geq \psi(\min\{M(Sy_n, Tx_n, z, t), M(Ay_n, Sy_n, z, t), M(Bx_n, Tx_n, z, t), \\ M(Sy_n, Bx_n, z, t), M(Ay_n, Tx_n, z, t)\}).$$

Proceeding limit when $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} Ay_n = Tx.$$

Now,

$$\lim_{n \rightarrow \infty} Ay_n = \lim_{n \rightarrow \infty} Bx_n = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sy_n = Tx = Sv.$$

Now, we shall show that $Av = Sv$

From (2.6),

$$M(Av, Bx_n, z, t) \geq \psi(\min\{M(Sv, Tx_n, z, t), M(Av, Sv, z, t), M(Bx_n, Tx_n, z, t),$$

$$M(Sv, Bx_n, z, t), M(Av, Tx_n, z, t)).$$

Letting limit as $n \rightarrow \infty$,

$$\begin{aligned} & M(Av, Tx, z, t) \\ & \geq \psi(\min\{M(Tx, Tx, z, t), M(Av, Tx, z, t), M(Tx, Tx, z, t), M(Tx, Tx, z, t), M(Av, Tx, z, t)\}). \end{aligned}$$

Using (*), we have $Av = Sv = Tx$.

Since $AX \subset TX$ (from (2.1)),

So, there exists $w \in X$ such that $Tx = Av = Tw$.

Now, we prove that $Tx = Tw = Bw$.

From(2.6),

$$\begin{aligned} & M(Av, Bw, z, t) \geq \\ & \psi(\min\{M(Sv, Tw, z, t), M(Av, Sv, z, t), M(Bw, Tw, z, t), M(Sv, Bw, z, t), M(Av, Tw, z, t)\}), \end{aligned}$$

i.e.,

$$\begin{aligned} & M(Tx, Bw, z, t) = \\ & \psi(\min\{M(Tx, Tx, z, t), M(Tx, Tx, z, t), M(Bw, Tx, z, t), M(Tx, Bw, z, t), M(Tx, Tx, z, t)\}). \end{aligned}$$

Using (*) we have $Tx = Bw$.

Thus we have

$$Av = Sv = Tw = Bw = Tx.$$

Since the pair (A, S) is weak compatible

Therefore, $ASv = SAV$

$$\text{i.e. } ATx = STx.$$

From (2.6)

$$\begin{aligned} & M(ATx, Bw, z, t) \\ & \geq \psi(\min\{M(STx, Tw, z, t), M(ATx, STx, z, t), M(Bw, Tw, z, t), M(STx, Bw, z, t), M(ATx, Tw, z, t)\}). \end{aligned}$$

From (*), we get

$$ATx = STx = Tx.$$

As (B, T) is weakly compatible, which gives $BTw = TBw$ i.e., $BTx = TTx$.

Now, we show that Tx is the common fixed point of A, B, T and S .

Consider $BTx \neq Tx$, then using (2.6), we get

$$M(ATx, BTx, z, t)$$

Using (*), we have $BTx = Tx$.

Hence, $ATx = BTx = STx = TTx = Tx$.

Therefore, Tx is a common fixed point of A, B, S and T .

Theorem 2.7: Let A, B, S and T be self maps of a fuzzy 2-metric space $(X, M, *)$ satisfying (2.1), (2.2), (2.3) and the following conditions :

(2.7) The pair (A, S) satisfies property (CLR_S) and the pair (B, T) also satisfies property (CLR_T) .

Then A, B, S and T have a unique common fixed point in X .

Proof . Consider that (A, S) and (B, T) satisfy a common (CLR) property.

Then there exists sequences $\{X_n\}$ and $\{y_n\}$ in X such that

$\lim_{n \rightarrow \infty} A x_n = \lim_{n \rightarrow \infty} S x_n = \lim_{n \rightarrow \infty} B y_n = \lim_{n \rightarrow \infty} T y_n = Tx$ for some $Tx \in X$.

We get, $Tx = Sv = Tw$ for some v, w in X ,

From (2.6)

$$M(Av, By_n, z, t) \geq \psi(\min\{M(Sv, Ty_n, z, t), M(Av, Sv, z, t), M(By_n, Ty_n, z, t), \\ M(Sv, By_n, z, t), M(Av, Ty_n, z, t)\}).$$

Letting limit as $n \rightarrow \infty$ and by (*), we get

$$Tx = Av = Sv = Tw.$$

Thus, from Theorem (2.4),

A, B, S and T have a unique common fixed point Tx in X .

3. Conclusion:

In this paper, we have extended the results of Shojaei *et al.* [7] by using (CLR) property for four weakly compatible self maps in fuzzy 2-metric spaces.

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