Original Research Article

Common fixed point theorems for four weakly compatible self maps along with (CLR) property in fuzzy 2-metric spaces

Abstract: In this paper, we prove some common fixed point theorems for four weakly compatible self-maps along with (CLR) property in fuzzy 2- metric spaces. Our results are the improved version of the theorems proved by Shojaei et al. [7] in 2013, since our results does not require closedness of ranges of subsets of X.

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1.Introduction and Preliminaries: In 1965, L.A.Zadeh [10] introduced the notion of fuzzy sets. A lot of authors proved several fixed point theorems by using the concept of fuzzy set theory. The notion of 2-metric spaces was introduced by Gahler [1], [2], [3].

Definition 1.1: A triangular norm * (shortly t-norm) is a binary operation on the unit interval [0,1] such that for all $a, b, c, d \in [0,1]$. The following conditions are satisfied:

1. $a * 1 = a$;
2. $a * b = b * a$;
3. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$
4. $a * (b * c) = (a * b) * c$.

Definition 1.2 ([4]**):** The 3-tuple $(X, M, *)$ is called a fuzzy metric space, if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty]$ satisfying the following condition:

for all $x, y, z \in X$ and $s, t > 0$

1 $M(x, y, 0) = 0$

2 $M(x, y, t) = 1, for\ all\ t > 0, if\ and\ only\ if\ x = y$

3 $M(x, y, t) = M(y, x, t)$

4 $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

5 $M(x, y, .) : [0, \infty) \to [0,1]$ $is\ left\ continuous,$

6 $\lim_{n\to\infty} M(x, y, t) = 1.$

Example 1.3: Let (X, d) be a metric space. Define $a * b = ab(\; or \; a * b = min\{a, b\})$ and for all $x, y \in X$ and $t > 0$, $M(x, y, t) = \frac{t}{t+d(x,y).}$ Then $(X, M,*)$ is a fuzzy metric space and this metric d is the standard fuzzy metric.

Definition 1.4 ([6])**:** A binary operation $*: [0,1] \times [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t - norm if ([0, 1], *) is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2$ whenever $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$ for all a_1, a_2, b_1, b_2 and c_1, c_2 in [0, 1].

Definition 1.5: The 3- tuple $(X, M,*)$ is called a fuzzy 2-metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^3 \times [0, \infty]$ satisfying the following conditions, $for \; all \; x, y, z, u \in X \; and \; t_1, t_2, t_3 > 0$.

1 $M(x, y, z, 0) = 0.$

2 $M(x, y, z, t) = 1, t > 0 \; and \; when \; at \; least \; two \; of \; the \; three \; point \; are \; equal,$

3 $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$

4 $M(x, y, z, t_1 + t_2 + t_3) \geq M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3)$

(This correspond to tetrahedron inequality in 2-metric space)

The function value $M(x, y, z, t)$ may be interpreted as the probability that the area of triangle is less than t.

5 $M(x, y, z, .) : [0, \infty) \rightarrow [0,1]$ is left continuous.

Example 1.6: Let (X, d) be 2-metric space and denote $a * b = ab$ for all $a, b \in [0,1]$.

For each $h, m, n \in R^+$ and $t > 0$, define $M(x, y, z, t) = \frac{ht^n}{ht^n + md(x,y,z)}$.

Then $(X, M,*)$ is an fuzzy 2-metric space.

Definition 1.7 ([4])**:** A sequence $\{x_n\}$ in a fuzzy 2-metric space $(X, M,*)$ is said to converge to x (in X) if and only if $\lim_{n\to\infty} M(x_n, x, a, t) = 1$ for all $a \in X$ and $t > 0$.

Definition 1.8: Let $(X, M,*)$ be a fuzzy 2-metric space. A sequence $\{x_n\}$ in X is called Cauchy sequence, if and only if $\lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1$ for all $a \in X, p > 0$ and $t > 0$.

Definition 1.9 ([4])**:** A fuzzy 2-metric space $(X, M,*)$ is said to be complete if and only if every Cauchy sequence in X is convergent in X.

Definition 1.10: Let $(X, M,*)$ be a fuzzy 2-metric space. Suppose f and g be self maps on X. A point x in X is called a coincidence point of $f \; and \; g$ iff $fx = gx$. In this case, $w = fx = gx$ is called a point of coincidence of $f \; and \; g$.

Definition 1.11 ([8])**:** A pair of self mapping $\{f, g\}$ of a fuzzy 2-metric space (X, d) is said to be weakly compatible if they commute at the coincidence point i.e., if $fu = gu$ for some $u \in X$, then $fgu = gfu$.

It is to see that two compatible maps are weakly compatible but converse is not true.

2. Main Results:

Definition 2.1 ([9])**:** Let f *and* g be two self-maps of a 2-metric space $(X, M, *)$, then they are said to satisfy (CLR_g) property if there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some $x \in X$.

Similarly, the property (CLR_T) and the property (CLR_S) hold if in the above definition the mapping $g: X \to X$ has been replaced by the mapping $T: X \to X$ and $S: X \to X$.

Example 2.2: let $X = [3, \infty)$. Define $f, g : X \to X$ by $gx = x + 2$ and $fx = 4x + 2$, for all $x \in X$. Suppose that the (CLR_g) property holds. Then, there exists a sequence $\{x_n\}$ in X satisfying

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx \text{ for some } gx \in X.$$

Therefore, $\lim_{n\to\infty} x_n = gx - 2$ *and* $\lim_{n\to\infty} x_n = \frac{gx-2}{4}$.

Thus, $gx = 2$, which is a contradiction, since 2 is not in X.

Hence, f *and* g do not satisfy (CLR_g) property.

Lemma 2.3 ([5])**.** Let $(X, M, *)$ be a fuzzy 2-metric space. If there exists $k \in (0,1)$ such that $M(x, y, z, kt) \geq M(x, y, z, t)$, for all $x, y, z \in X$ with $z \neq x, z \neq y$ and $t > 0$, then $x = y$.

Theorem 2.4. Let A, B, S *and* T be self-maps of a fuzzy 2-metric spaces $(X, M, *)$ satisfying the following condition :

(2.1) $AX \subset TX$ *and* $BX \subset SX$,

(2.2) $$M(Ax, By, z, kt) \geq \phi\,(M(Sx, Ty, z, t), M(Ax, Sx, z, t), M(By, Ty, z, t), M(Sx, By, z, t), M(Ax, Ty, z, t)),$$

for all x, y, z *in* X and $t > 0$, where $k \in (0,1)$.

(2.3) the pairs (A, S) and (B, T) are weakly compatible.

(2.4) the pair (A, S) satisfies (CLR_S) property or the pair (B, T) satisfies the (CLR_T) property.

Then A, B, S *and* T have a unique common fixed point in X.

Proof. Suppose that $BX \subset SX$ and (B,T) satisfies property (CLR_T), then there exists a sequence

$\{x_n\}$ in X such that

$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = Tx$, for some $x \in X$.

Since $BX \subset SX$, therefore there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Sy_n = Tx$$

Hence, $\lim_{n\to\infty} Sy_n = Tx$

Now, We shall show that $\lim_{n\to\infty} Ay_n = Tx$

Suppose that

$$\lim_{n\to\infty} Ay_n = l$$

Putting $x = y_n$ and $y = x_n$ in (2.2), we have

$$M(Ay_n, Bx_n, z, kt) \geq \phi(M(Sy_n, Tx_n, z, t), M(Ay_n, Sy_n, z, t), M(Bx_n, Tx_n, z, t),$$
$$M(Sy_n, Bx_n, z, t), M(Ay_n, Tx_n, z, t)).$$

Proceeding limit $n \to \infty$, we have

$$M(l, Tx, z, kt) \geq \phi(1, M(l, Tx, z, t), 1, 1, M(l, Tx, z, kt)) \geq M(l, Tx, z, t).$$

By Lemma 2.3, we have

$l = Tx.$

Therefore, we have $\lim_{n\to\infty} Ay_n = Tx.$

$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = Tx = Sv.$

Now, we shall show that $Av = Tx.$

From (2.2), we have

$$M(Av, Bx_n, z, kt) \geq \phi(M(Sv, Tx_n, z, t), M(Av, Sv, z, t), M(Bx_n, Tx_n, z, t),$$
$$M(Sv, Bx_n, z, t), M(Av, Tx_n, z, t)).$$

Letting limit as $n \to \infty$,

$$M(Av, Tx, z, kt) \geq \phi(1, M(Av, Tx, z, t), 1, 1, M(Av, Tx, z, t) \geq M(Av, Tx, z, t)).$$

By Lemma 2.3, we have $Av = Sv = Tx.$

Since $AX \subset TX$, so, there exists $w \in X$ such that $Tx = Av = Tw.$

Now, we claim that

$$Tx = Bw.$$

From (2.2), we have

$$M(Av, Bw, z, kt) \geq \phi(M(Sv, Tw, z, t), M(Av, Sv, z, t), M(Bw, Tw, z, t), M(Sv, Bw, z, t), M(Av, Tw, z, t)).$$

Letting limit $n \to \infty$,

$M(Tx, Bw, z, kt) \geq \phi(1, 1, M(Bw, Tx, z, t), M(Tx, Bw, z, t), 1) \geq M(Tx, Bw, z, t))$.

By Lemma 2.3, we have

$Tx = Bw$.

Thus, we have $Av = Sv = Tw = Bw = Tx$.

Since the pair (A, S) is weakly compatible, therefore $ASv = SAv$, $i.e.$, $ATx = STx$.

Now, we show that $ATx = Tx$

Since,

$$M(ATx, Bw, z, kt) \geq \phi(M(STx, Tw, z, t), M(ATx, STx, z, t), M(Bw, Tw, z, t), M(STx, Bw, z, t), M(ATx, Tw, z, t)), that\ is,$$

$$M(ATx, Tx, z, kt) \geq \phi(M(ATx, Tx, z, t), 1, 1, M(ATx, Tx, z, t), M(ATx, Tx, z, t))$$

$$\geq M(ATx, Tx, z, t).$$

By Lemma 2.3, we have

$$ATx = STx = Tx.$$

The weak compatibility of B and T implies that

$$BTw = TBw$$

i.e. $BTx = TTx$.

Now, we shall further show that Tx is the common fixed point of B.

From (2.2), we have

$$M(ATx, BTx, z, kt) \geq \phi(M(STx, TTx, z, t), M(ATx, STx, z, t), M(BTx, TTx, z, t), M(STx, BTx, z, t), M(ATx, TTx, z, t)).$$

or

$$M(ATx, BTx, z, kt) \geq \phi(M(ATx, BTx, z, t), 1, 1, M(ATx, BTx, z, t), 1).$$

By Lemma 2.3, we have

$$BTx = Tx.$$

Hence, $ATx = BTx = STx = TTx = Tx$.

Therefore, Tx is the common fixed point of $A, B, S\ and\ T$.

Corollary 2.5. Let $A, B, S\ and\ T$ be self-maps of a fuzzy 2-metric space $(X, M,*)$ with continuous t-norm satisfying (2.1), (2.3), (2.4) and the followings:

(2.5) $M(Ax, By, z, t) \geq min\{M(Sx, Ty, z, t), M\ (Ax, Sx, z, t), M\ (Sx, By, z, t), M\ (Ax, Ty, z, t)\}$

holds, for all x, y, z in X and $t > 0$.

Then $A, B, S\ and\ T$ have a unique common fixed point in X.

Proof : Take in the Theorem 2.2

$$\phi\ (x_1, x_2, x_3, x_4, x_5) = min\{x_1, x_2, x_3, x_4, x_5\}$$

Now, we consider a function ψ: [0,1] $\to$ [0,1] satisfying the conditions

(*) ψ if continuous and non-decreasing on [0,1] and ψ(t) > t for all t $\in$ (0,1)

Note that $\psi(1) = 1$ and $\psi(t) \geq t$ for all $t \in [0,1]$,

i.e. $\psi(M\ (x, y, z, t) \geq M(x, y, z, t)$ holds for every $t > 0$ and for all $x, y \in X$.

Theorem 2.6. Let $A, B, S\ and\ T$ be self maps of a fuzzy 2 – metric space $(X, M,*)$ with continuous t-norm * satisfying (2.1), (2.3), (2.4) and the following :

(2.6) $M\ (Ax, By, z, t) \geq \psi(min\{M(Sx, Ty, z, t), M(Ax, Sx, z, t), M(By, Ty, z, t),$

$$M(Sx, By, z, t), M(Ax, Ty, z, t)\}).$$

with $M\ (x, y, z, t) > 0$ for all $x, y, z \in X$ and $t > 0$.

Then $A, B, S\ and\ T$ have a unique common fixed point in X.

Proof. Let (A, B) satisfies the (CLR) property.

Then there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = Tx, for\ some\ x \in X$

Since $BX \subset SX$, there exists a sequence $\{y_n\} \in X$ such that

$Bx_n = Sy_n = Tx$.

Hence, $\lim_{n\to\infty} Sy_n = Tx$.

Now, we show that $\lim_{n\to\infty} Ax_n = Tx$.

Putting $x = y_n, y = x_n$ in (2.6), we have

$$M\ (Ay_n, Bx_n, z, t) \geq \psi(min\{M(Sy_n, Tx_n, z, t), M(Ay_n, Sy_n, z, t), M(Bx_n, Tx_n, z, t)$$
$$M\ (Sy_n, Bx_n, z, t), M(Ay_n, Tx_n, z, t)\}).$$

Proceeding limit $n \to \infty$, we have

$$\lim_{n\to\infty} Ay_n = Tx.$$

Now,

$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = Tx = Sv.$

Now, we shall show that $Av = Sv$

From (2.6),

$$M\ (Av, Bx_n, z, t) \geq \psi(min\{M(Sv, Tx\ n, z, t), M(Av, Sv, z, t), M(Bx_n, Tx_n, z, t),$$
$$M\ (Sv, Bx_n, z, t), M\ (Av, Tx_n, z, t)\}).$$

Letting limit $n \to \infty$,

$M(Av, Tx, z, t)$

$\geq\ \psi(min\{M(Tx, Tx, z, t), M(Av, Tx, z, t), M(Tx, Tx, z, t), M\ (Tx, Tx, z, t), M\ (Av, Tx, z, t)\}).$

Using (*), we have $Av = Sv = Tx.$

Since $AX \subset TX$ (from (2.1)),

So, there exists $w \in X$ such that $Tx = Av = Tw.$

Now, we prove that $Tx = Tw = Bw.$

From(2.6), $M(Av, Bw, z, t) \geq$

$\psi(min\{M(Sv, Tw, z, t), M(Av, Sv, z, t),\ \ M(Bw, Tw, z, t), M\ (Sv, Bw, z, t), M\ (Av, Tw, z, t)\}),$

i.e., $M(Tx, Bw, z, t) =$

$\psi(min\{M(Tx, Tx, z, t), M(Tx, Tx, z, t), M(Bw, Tx, z, t), M(Tx, Bw, z, t), M\ (Tx, Tx, z, t)\}).$

Using (*) we have $Tx = Bw.$

Thus we have

$$Av = Sv = Tw = Bw = Tx.$$

Since the pair (A, S) is weak compatible

Therefore, $ASv = SAv$

i.e. $ATx = STx.$

From (2.6)

$M(ATx, Bw, z, t)$

$\geq\ \psi(min\{M(STx, Tw, z, t), M(ATx, STx, z, t), M(Bw, Tw, z, t), M\ (STx, Bw, z, t), M\ (ATx, Tw, z, t)\}).$

From (*), we get

$ATx = STx = Tx.$

As (B, T) is weakly compatible, which gives $BTw = TBw\ \ i.e., BTx = TTx.$

Now, we show that Tx is the common fixed point of $A, B, T\ and\ S$.

Consider $BTx \neq Tx$, then using (2.6), we get

$M(ATx, BTx, z, t)$

$\geq \psi(min\{M(STx, TTx, z, t), M(ATx, STx, z, t), M(BTx, TTx, z, t), M\ (STx, BTx, z, t), M\ (ATx, TTx, z, t)$.

Using (*), we have $BTx = Tx$.

Hence, $ATx = BTx = STx = TTx = Tx$.

Therefore, Tx is a common fixed point of $A, B, S\ and\ T$.

Theorem 2.7: Let $A, B, S\ and\ T$ be self maps of a fuzzy 2-metric space $(X, M,*)$ satisfying (2.1), (2.2), (2.3) and the following conditions :

(2.7) The pair (A, S) satisfies property (CLR_S) and the pair (B, T) also satisfies property (CLR_T).

Then $A, B, S\ and\ T$ have a unique common fixed point in X.

Proof . Consider that (A, S) and (B, T) satisfy a common (CLR) property.

Then there exists sequences $\{X_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} A\, x_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = Tx$ for some $Tx \in X$.

We get, $Tx = Sv = Tw$ for some $v, w\ in\ X$,

From (2.6)

$$M\ (Av, By_n, z, t) \geq \psi(min\{M(Sv, Ty_n, z, t), M(Av, Sv, z, t), M(By_n, Ty_n, z, t),$$
$$M\ (Sv, By_n, z, t), M(Av, Ty_n, z, t)\}).$$

Letting limit $n \to \infty$ and by (*), we get

$$Tx = Av = Sv = Tw.$$

Thus, from Theorem (2.4),

$A, B, S\ and\ T$ have a unique common fixed point Tx in X.

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