

## Analysis of Generalization of a Problem of Vieta's Descend Method with Examples and Computing Support

**Abstract.** Let  $c$  and  $d$  be fixed integer numbers. Assume that  $(a^2 + b^2 + c)$  is divisible by  $(ab + d)$  for some natural numbers  $a$  and  $b$ . Then the value of the fraction  $k (= \frac{a^2+b^2+c}{ab+d})$  remains the same. Statement of this kind will be proved in pp. 1-3 and illustrated on some examples in pp. 3-10. The general method of proofs will be unified and simplified. Computing support will be provided: in pages

11-19 a simple program code is defined with the help of which one can hunt for natural numbers  $a, b$  with the same integer values of  $c, d$  and  $k$ . Here, a number of examples are given as well.

**Keywords:** Vieta's formulae, Vieta's descend method, divisibility, descending pairs of integers.

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The last problem of 1988 International Mathematical Olympiad and other interesting remarks in [1-8] have inspired me to check problems of this type:

**Problem.** Let  $c$  and  $d$  be fixed integers. Assume that

$$(a^2 + b^2 + c) \text{ is divisible by } (ab + d) \tag{1}$$

for some natural numbers  $a$  and  $b$ . Then the value of the fraction

$$k \doteq \frac{a^2+b^2+c}{ab+d} \tag{2}$$

does not change, no matter how  $a$  and  $b$  ( $\in \mathbb{N}$ ) vary.

When solving problems of this kind, some parts of all proofs can be done in a unified manner. Recognizing this fact forced me to formulate two lemmas and the so-called "proof-finishing procedure".

### **Rough Sketch of the Proof of our Problem.**

Without loss of generality we can assume that  $a \geq b$ .

First, we check two simple cases number 1 and 2.1.

Case 1:  $a = b$ . We have  $k = \frac{2a^2+c}{a^2+d} = 2 + \frac{c-2d}{a^2+d} \in \mathbb{Z}$ .

Case 2.1:  $a > b = 1$ . Then we have  $k = \frac{a^2+1+c}{a+d} = a + \frac{c+1-ad}{a+d} \in \mathbb{Z}$ .

Then in Case 2.2:  $a > b \geq 2$ , we will use Lemma 1, Lemma 2 and the so-called finishing procedure.

For  $ab + d \neq 0$ , (2) is equivalent to  $a^2 + b^2 + c = abk + dk$ .

By replacing  $a$  by  $x$ , we obtain the quadratic equation

$$x^2 - kbx + (b^2 - dk + c) = 0 \quad \text{with roots } x_1 (= a) \text{ and } x_2. \quad (3)$$

**Vieta's** formulae apply to give

$$a + x_2 = bk \quad \text{and} \quad ax_2 = b^2 - dk + c$$

from which we get  $x_2 = bk - a = \frac{b^2 - dk + c}{a} (\in \mathbb{Z})$ .

On the other hand,  $a > b \Leftrightarrow a = b + p$  for some  $p \in \mathbb{N}$ .

**Lemma 1.** If  $a > b$ ,  $ab + d > 0$  and  $S \doteq 2bd + p(b^2 + d) - bc > 0$  for  $a, b, p \in \mathbb{N}$  and  $c, d \in \mathbb{Z}$ , then  $x_2 < b$ .

**Proof.**  $x_2 = bk - a < b \Leftrightarrow b \frac{a^2 + b^2 + c}{ab + d} < a + b \Leftrightarrow$

$$\Leftrightarrow b(a^2 + b^2 + c) < (a + b)(ab + d) \Leftrightarrow S \doteq 2bd + p(b^2 + d) - bc > 0. \quad \blacksquare$$

**Lemma 2.** Assume that  $a > b$  and  $T \doteq b^2(b^2 + c) + d^2 \neq 0$  for  $a, b, p \in \mathbb{N}$ .

If we replace  $a$  by  $b$ , and  $b$  by  $x_2$  in (2), then the value of  $k$  remains the same:

$$k = \frac{a^2+b^2+c}{ab+d} = \frac{b^2+x_2^2+c}{bx_2+d}.$$

**Proof.** We form  $\frac{b^2+x_2^2+c}{bx_2+d} = \frac{b^2+(bk-a)^2+c}{b(bk-a)+d} = \frac{(b^2k^2-abk+dk)-abk-dk+a^2+b^2+c}{b^2k-ab+d} =$

$$= k + \frac{a^2+b^2+c-k(ab+d)}{b^2k-ab+d} = k + \frac{a^2+b^2+c-\left(\frac{a^2+b^2+c}{ab+d}\right)(ab+d)}{b^2k-ab+d} = k, \text{ provided}$$

$$b^2k - ab + d = b^2 \left( \frac{a^2+b^2+c}{ab+d} \right) - ab + d = \frac{b^2(b^2+c)+d^2}{ab+d} \neq 0, \text{ i.e.}$$

$$T \doteq b^2(b^2 + c) + d^2 \neq 0 \text{ [for } ab + d \neq 0]. \quad \blacksquare$$

### **Procedure Finishing (C, K)**

So, we can replace the ordered pair  $(a, b)$  by the pair  $(b, x_2)$  with  $a > b > x_2$ , and the value of  $k$  remains the same. After a finite number of "descending" and "k-preserving" steps  $(a, b) \succ (b, x_2)$ , we will arrive at case **C**, where  $k = \mathbf{K}$ , whenever  $(ab + d) | (a^2 + b^2 + c)$ . ■

Let me present some examples of applications of our Lemmas and "proof-finishing" procedure. These examples will cover many combinations of  $sign(c)$  and  $sign(d)$  but, in each example,  $k \in \mathbb{N}$ .

**Example 1.** If  $(a^2 + b^2 + 2)$  is divisible by  $(ab + 4)$  for some  $a, b \in \mathbb{N}$ , then  $\frac{a^2+b^2+2}{ab+4} = 12$ .

**Proof.** Without loss of generality we can assume that  $a \geq b$ .

**Case 1:**  $a = b$ . We have

$$k = \frac{2a^2+2}{a^2+4} = 2 - \frac{6}{a^2+4} \in \mathbb{Z} \implies (a^2 + 4) | 6. \text{ Then}$$

$a^2 + 4 = \{1 \text{ OR } 2 \text{ OR } 3 \text{ OR } 6\} \Rightarrow a^2 = \{-3 \text{ OR } -2 \text{ OR } -1 \text{ OR } 2\}$   
contradicting  $a \in \mathbb{N}$ .

Case 2:  $a > b$ .

Case 2.1:  $a > b = 1$ . Then we have

$$k = \frac{a^2+3}{a+4} = a - 4 + \frac{19}{a+4} \in \mathbb{Z} \Rightarrow (a + 4) | 19. \text{ Thus,}$$

$$a + 4 = \{1 \text{ OR } 19\}.$$

$$a + 4 = 1 \Rightarrow a = -3, \text{ contradicting } a \in \mathbb{N}.$$

$$a + 4 = 19 \Rightarrow a = 15. \text{ Then } k = \frac{15^2+3}{15+4} = 12.$$

Case 2.2:  $a > b \geq 2$ . Lemma 1 applies to give  $x_2 < b$ , because  $ab + 4 > 0$  and  $S = 2b(4) + p(b^2 + 4) - b(2) = 6b + p(b^2 + 4) > 0$  for  $b, p \in \mathbb{N}$ .

Then, Lemma 2 applies to give  $k = \frac{a^2+b^2+2}{ab+4} = \frac{b^2+x_2^2+2}{bx_2+4}$ , because

$$T = b^2(b^2 + 2) + 4^2 > 0, \text{ for } b \in \mathbb{N}.$$

Finishing (2.1, 12) applies, whenever  $(ab + 4) | (a^2 + b^2 + 2)$ . ■

The first few pairs  $(a, b)$  in Ex. 1 are as follows:

15, 1; 37, 3; 179, 15; 441, 37; 2133, 179; 5255, 441.

**Example 2.** If  $(a^2 + b^2 + 1)$  is divisible by  $(ab + 3)$  for some  $a, b \in \mathbb{N}$ ,

$$\text{then } \frac{a^2+b^2+1}{ab+3} = 6.$$

**Proof.** Without loss of generality we can assume that  $a \geq b$ .

Case 1:  $a = b$ . Then

$$k = \frac{2a^2+1}{a^2+3} = 2 - \frac{5}{a^2+3} \in \mathbb{Z} \Rightarrow (a^2 + 3) | 5. \text{ Then}$$

$$a^2 + 3 = \{1 \text{ OR } 5\} \Rightarrow a^2 = \{-2 \text{ OR } 2\}, \text{ contradicting } a \in \mathbb{N}.$$

Case 2:  $a > b$ .

Case 2.1:  $a > b = 1$ . Then we have

$$k = \frac{a^2+2}{a+3} = a - 3 + \frac{11}{a+3} \in \mathbb{Z} \Rightarrow (a+3)|11. \text{ So,}$$

$$a+3 = \{1 \text{ OR } 11\}.$$

$$a+3 = 1 \Rightarrow a = -2, \text{ contradicting } a \in \mathbb{N}.$$

On the other hand,  $a+3 = 11 \Rightarrow a = 8$ . Then  $k = \frac{8^2+2}{8+3} = 6$ .

Case 2.2:  $a > b \geq 2$ . Then Lemma 1 applies to give  $x_2 < b$ , because

$ab+3 > 0$  and  $S = 2b(3) + p(b^2+3) - b(1) = 5b + p(b^2+3) > 0$  for  $b, p \in \mathbb{N}$ . Then, Lemma 2 applies to give  $k = \frac{a^2+b^2+1}{ab+3} = \frac{b^2+x_2^2+1}{bx_2+3}$ , because

$$T = b^2(b^2+1) + 3^2 > 0, \text{ for } b \in \mathbb{N}.$$

Finishing (2.1, 6) applies, whenever  $(ab+3)|(a^2+b^2+1)$ . ■

The first few pairs  $(a, b)$  in Ex. 2 are as follows:

8, 1; 13, 2; 47, 8; 76, 13; 274, 47; 443, 76; 1597, 274; 2582, 443; 9308, 1597.

**Example 3.** If  $(a^2 + b^2 - 6)$  is divisible by  $(ab + 4)$  for some  $a, b \in \mathbb{N}$ ,

then  $\frac{a^2+b^2-6}{ab+4} = 4$ .

**Proof.** Without loss of generality we can assume that  $a \geq b$ .

Case 1:  $a = b$ . Then we have

$$k = \frac{2a^2-6}{a^2+4} = 2 - \frac{14}{a^2+4} \in \mathbb{Z} \Rightarrow (a^2+4)|14. \text{ Then}$$

$a^2+4 = \{1 \text{ OR } 2 \text{ OR } 7 \text{ OR } 14\} \Rightarrow a^2 = \{-3 \text{ OR } -2 \text{ OR } 3 \text{ OR } 10\}$ ,  
contradicting  $a \in \mathbb{N}$ .

Case 2:  $a > b$ .

Case 2.1:  $a > b = 1$ . Then we have

$$k = \frac{a^2-5}{a+4} = a - \frac{4a+5}{a+4} = a - 4 + \frac{11}{a+4} \in \mathbb{Z} \Rightarrow (a+4)|11.$$

Then  $a+4 = \{1 \text{ OR } 11\}$ .

$a+4 = 1 \Rightarrow a = -3$ , contradicting  $a \in \mathbb{N}$ .

$a+4 = 11 \Rightarrow a = 7$ . Then  $k = \frac{7^2+1-6}{7+4} = 4$ .

Case 2.2:  $a > b \geq 2$ . Then Lemma 1 applies to give  $x_2 < b$ , because

$$ab + 4 > 0 \text{ and } S = 2b(4) + p(b^2 + 4) - b(-6) = 14b + p(b^2 + 4) > 0$$

for  $b, p \in \mathbb{N}$ . Then, Lemma 2 applies to give  $k = \frac{a^2+b^2-6}{ab+4} = \frac{b^2+x_2^2-6}{bx_2+4}$  because,

for  $b \geq 3$ :  $T = b^2(b^2 - 6) + 4^2 > 0$ ;

for  $b = 2$ :  $T = 2^2(2^2 - 6) + 4^2 = 8 > 0$ ; and

for  $b = 1$ :  $T = 1^2(1^2 - 6) + 4^2 = 11 > 0$ .

Thus, for  $b \in \mathbb{N}$ ,  $T > 0$ .

Finishing (2.1, 4) applies, whenever  $(ab + 4)|(a^2 + b^2 - 6)$ . ■

The first few pairs  $(a, b)$  in Ex. 3 are as follows:

7, 1; 13, 3; 27, 7; 49, 13; 101, 27; 183, 49; 377, 101; 683, 183; 1407, 377;

2549, 683; 5251, 1407; 9513, 2549.

**Example 4.** If  $(a^2 + b^2 - 11)$  is divisible by  $(ab + 3)$  for some  $a, b \in \mathbb{N}$ ,

then  $\frac{a^2+b^2-11}{ab+3} = 3$ .

**Proof.** Without loss of generality we can assume that  $a \geq b$ .

Case 1:  $a = b$ . Then

$$k = \frac{2a^2-11}{a^2+3} = 2 - \frac{17}{a^2+3} \in \mathbb{Z} \Rightarrow (a^2 + 3)|17. \text{ Then}$$

$a^2 + 3 = \{1 \text{ OR } 17\} \Rightarrow a^2 = \{-2 \text{ OR } 14\}$ , contradicting  $a \in \mathbb{N}$ .

Case 2:  $a > b$ .

Case 2.1:  $a > b = 1$ . Then we have

$$k = \frac{a^2+1-11}{a+3} = a - 3 - \frac{1}{a+3} \in \mathbb{N} \implies (a+3)|1. \text{ Then}$$

$$a+3 = 1 \implies a = -2, \text{ contradicting } a \in \mathbb{N}.$$

Case 2.2:  $a > b = 2$ . Then  $k = \frac{a^2+2^2-11}{2a+3} = \frac{a^2-7}{2a+3} \in \mathbb{Z}$ . Checking the values

$a = 3, 4, 5, 6, 7$ , none of them will do, i.e.  $k \notin \mathbb{Z}$ . The first value that will do is  $a = 8$ , for which  $k = \frac{8^2-7}{2(8)+3} = 3$ .

Case 2.3:  $a > b \geq 2$ . Then Lemma 1 applies to give  $x_2 < b$ , because

$$ab + 3 > 0 \text{ and } S = 2b(3) + p(b^2 + 3) - b(-11) = 17b + p(b^2 + 3) > 0$$

for  $b, p \in \mathbb{N}$ . Then, Lemma 2 applies to give  $k = \frac{a^2+b^2-11}{ab+3} = \frac{b^2+x_2^2-11}{bx_2+3}$

because,

$$\text{for } b \geq 4: T = b^2(b^2 - 11) + 3^2 > 0;$$

$$\text{for } b = 3: T = 3^2(3^2 - 11) + 3^2 = -9 \neq 0;$$

$$\text{for } b = 2: T = 2^2(2^2 - 11) + 3^2 = -19 \neq 0; \text{ and}$$

$$\text{for } b = 1: T = 1^2(1^2 - 11) + 3^2 = -1 \neq 0.$$

Thus, for  $b \in \mathbb{N}$ ,  $T \neq 0$ .

Finishing (2.2, 3) applies, whenever  $(ab + 3)|(a^2 + b^2 - 11)$ . ■

The first few pairs  $(a, b)$  in Ex. 4 are as follows:

8, 2; 22, 8; 58, 22; 152, 58; 398, 152; 1042, 398; 2728, 1042; 7142, 2728.

**Example 5.** If  $(a^2 + b^2 - 6)$  is divisible by  $(ab + 8)$  for some  $a, b \in \mathbb{N}$ ,

$$\text{then } \frac{a^2+b^2-6}{ab+8} = 44.$$

**Proof.** Without loss of generality we can assume that  $a \geq b$ .

Case 1:  $a = b$ .

Then  $k = \frac{2a^2-6}{a^2+8} = 2 - \frac{22}{a^2+8} \in \mathbb{Z} \Rightarrow (a^2 + 8) | 22$ , and

$a^2 + 8 = \{1 \text{ OR } 2 \text{ OR } 11 \text{ OR } 22\} \Rightarrow a^2 = \{-7 \text{ OR } -6 \text{ OR } 3 \text{ OR } 14\}$ ,  
contradicting  $a \in \mathbb{N}$ .

Case 2:  $a > b$ .

Case 2.1:  $a > b = 1$ . Then we have

$k = \frac{a^2-5}{a+8} = a - 8 + \frac{59}{a+8} \in \mathbb{Z} \Rightarrow (a + 8) | 59$ . Then

$a + 8 = \{1 \text{ OR } 59\}$ .

$a + 8 = 1 \Rightarrow a = -7$ , contradicting  $a \in \mathbb{N}$ .

$a + 8 = 59 \Rightarrow a = 51$ . Then  $k = \frac{51^2-5}{51+8} = 44$ .

Case 2.2:  $a > b \geq 2$ . Then Lemma 1 applies to give  $x_2 < b$ , because

$ab + 8 > 0$  and  $S = 2b(8) + p(b^2 + 8) - b(-6) = 22b + p(b^2 + 8) > 0$

for  $b, p \in \mathbb{N}$ .

Then, Lemma 2 applies to give  $k = \frac{a^2+b^2-6}{ab+8} = \frac{b^2+x_2^2-6}{bx_2+8}$  because,

for  $b \geq 3$ :  $T = b^2(b^2 - 6) + 8^2 > 0$ ;

for  $b = 2$ :  $T = 2^2(2^2 - 6) + 8^2 = 56 > 0$ ; and

for  $b = 1$ :  $T = 1^2(1^2 - 6) + 8^2 = 59 > 0$ .

Thus, for  $b \in \mathbb{N}$ ,  $T \neq 0$ .

Finishing (2.1, 44) applies, whenever  $(ab + 8) | (a^2 + b^2 - 6)$ . ■

The first few pairs  $(a, b)$  in Ex. 5 are as follows:

51, 1; 309, 7; 2243, 51.

**Example 6.** If  $(a^2 + b^2)$  is divisible by  $(ab - 1)$  for some  $a, b \in \mathbb{N}$ , then  $\frac{a^2+b^2}{ab-1} = 5$ .

**Proof.** Without loss of generality we can assume that  $a \geq b$ .

Case 1:  $a = b$ . Then

$$k = \frac{2a^2}{a^2-1} = 2 + \frac{2}{a^2-1} \in \mathbb{Z} \implies (a^2 - 1) | 2. \text{ Then}$$

$$a^2 - 1 = \{1 \text{ OR } 2\} \implies a^2 = \{2 \text{ OR } 3\}, \text{ contradicting } a \in \mathbb{N}.$$

Case 2:  $a > b$ .

Case 2.1:  $a > b = 1$ . Then we have

$$k = \frac{a^2+1}{a-1} = a + 1 + \frac{2}{a-1} \in \mathbb{Z} \implies (a-1) | 2. \text{ Then}$$

$$a - 1 = \{1 \text{ OR } 2\} \implies a = \{2 \text{ OR } 3\}.$$

$$\text{Case 2.1.1: } a = 2, b = 1. \text{ Then } k = \frac{2^2+1}{2-1} = 5.$$

$$\text{Case 2.1.2: } a = 3, b = 1. \text{ Then } k = \frac{3^2+1}{3-1} = 5.$$

Case 2.2:  $a > b \geq 2$ . Then  $a = b + p$  for some  $p \in \mathbb{N}$ .

If  $p = 1$ , then  $a = b + 1$ , and we have

$$ab - 1 = (b^2 + b - 1) \nmid a^2 + b^2 = 2(b^2 + b - 1) + 3, \text{ since}$$

$$b^2 + b - 1 \geq 2^2 + 2 - 1 = 5.$$

For  $p \geq 2$ , Lemma 1 applies to give  $x_2 < b$ , because

$$\begin{aligned} ab - 1 &\geq 2a - 1 > 0 \text{ and } S = 2b(-1) + p(b^2 - 1) - b(0) = \\ &= b(bp - 2) - p \geq 2(2p - 2) - p = 3p - 4 \geq 3(2) - 4 = 2 > 0. \end{aligned}$$

Then, Lemma 2 applies to give  $k = \frac{a^2+b^2}{ab-1} = \frac{b^2+x_2^2}{bx_2-1} \in \mathbb{N}$ , because

$$T = b^4 + 1 > 0, \text{ for } b \in \mathbb{N}.$$

Finishing (2.1, 5) applies, whenever  $(ab - 1) | (a^2 + b^2)$ . ■

The first few pairs  $(a, b)$  in Ex. 6 are as follows:

2, 1; 3, 1; 9, 2; 14, 3; 43, 9; 67, 14; 206, 43; 321, 67; 987, 206; 1538, 321;  
4729, 987; 7369, 1538.

**Example 7.** If  $(a^2 + b^2 - 1)$  is divisible by  $(ab - 1)$  for some  $a, b \in \mathbb{N}$ ,  
then  $\frac{a^2+b^2-1}{ab-1} = 4$ .

**Proof.** Without loss of generality we can assume that  $a \geq b$ .

Case 1:  $a = b$ . Then

$$k = \frac{2a^2-1}{a^2-1} = 2 + \frac{1}{a^2-1} \in \mathbb{Z} \Rightarrow (a^2 - 1)|1 \Rightarrow a^2 = 2, \text{ contradicting } a \in \mathbb{N}.$$

Case 2.1:  $a > b = 1$ . Then we have

$$k = \frac{a^2}{a-1} = a + 1 + \frac{1}{a-1} \in \mathbb{Z} \Rightarrow (a - 1)|1 \Rightarrow a = 2. \text{ Then } k = \frac{2^2}{2-1} = 4.$$

Case 2.2:  $a > b \geq 2$ . Then Lemma 1 applies to give  $x_2 < b$ , because

$$ab - 1 \geq 2a - 1 > 0 \text{ and } S = 2b(-1) + p(b^2 - 1) - b(-1) = \\ = b(pb - 1) - p \geq 2(2p - 1) - p = 3p - 2 \geq 1 > 0, \text{ for } a, b, p \in \mathbb{N}.$$

Then, Lemma 2 applies to give  $k = \frac{a^2+b^2-1}{ab-1} = \frac{b^2+x_2^2-1}{bx_2-1}$  because,

for  $b \geq 2, T = b^2(b^2 - 1) + 1 > 0$ ;

for  $b = 1, T = 1 > 0$ . So,  $T > 0$ , for  $b \in \mathbb{N}$ .

Finishing (2.1, 4) applies, whenever  $(ab - 1)|(a^2 + b^2 - 1)$ . ■

The first few pairs  $(a, b)$  in Ex. 7 are as follows:

2, 1; 7, 2; 26, 7; 97, 26; 362, 97; 1351, 362; 5042, 1351.

### **Remarks**

Our Lemma 1 does not immediately apply when proving the following statement:

if  $(ab - 1)|(a^2 + b^2 + 5)$  for some  $a, b \in \mathbb{N}$ , then  $\frac{a^2+b^2+5}{ab-1} = 10$ .

On the other hand, one can prove that

$k = 12$  for  $c = -14$  and  $d = 6$  ;

$k = 108$  for  $c = -14$  and  $d = 12$  ;

$k = 4$  for  $c = 2$  and  $d = 0$  ;

$k$  is a complete square for  $c = 0$  and  $d = 1$  (See [1,2]).

## **Computing Support**

### **1., Hunting for natural numbers $a, b$ with the same values of $c, d$ and $k$**

If somebody wants to hunt for some interesting integer values of  $k (= \frac{a^2+b^2+c}{ab+d})$ , then he/she can use the following jBASiC program code:

Rem "Vieta jump general input n c d output a b k.bas"

```
[start]
```

```
print chr$(13)
```

```
print "n:"
```

```
input n
```

```
print "c:"
```

```
input c
```

```
print "d:"
```

```
input d
```

```
print chr$(13)
```

```
print "c      d"
```

```
print c,d
```

```
print chr$(13)
```

```
print "a      b      k"
```

```

print chr$(13)
for b = 2 to n
for a = 1 to b
q=(a*a+b*b+c)/(a*b+d)
if q<>int(q) then goto [L1]
print a,b,q
[L1]
next a
next b
goto [start]
end

```

*Input:* n (the upper bound for natural numbers  $a, b$ ), c and d.

*Output:*  $a, b, k$  (obeying the relations  $a, b \in N$  and  $\frac{a^2+b^2+c}{ab+d} = k \in Z$ ).

Some examples are listed below [with  $0 < a < b$ ]:

<b>c</b>	<b>d</b>		
2	4		
<b>a</b>	<b>b</b>	<b>k</b>	
1	15	12	
3	37	12	
15	179	12	
37	441	12	
179	2133	12	
441	5255	12	

<b>c</b>	<b>d</b>	
1	3	
<b>a</b>	<b>b</b>	<b>k</b>
1	8	6
2	13	6
8	47	6
13	76	6
47	274	6
76	443	6
274	1597	6
443	2582	6

<b>c</b>	<b>d</b>	
-6	4	
<b>a</b>	<b>b</b>	<b>k</b>
1	7	4
3	13	4
7	27	4
13	49	4
27	101	4
49	183	4

101	377	4
183	683	4
377	1407	4
683	2549	4
1407	5251	4

<b>c</b>	<b>d</b>	
-11	3	
<b>a</b>	<b>b</b>	<b>k</b>
2	8	3
8	22	3
22	58	3
58	152	3
152	398	3
398	1042	3
1042	2728	3
2728	7142	3

<b>c</b>	<b>d</b>	
-6	8	
<b>a</b>	<b>b</b>	<b>k</b>
1	51	44

7	309	44
51	2243	44
309	13589	44

<b>c</b>	<b>d</b>	
0	-1	
<b>a</b>	<b>b</b>	<b>k</b>
1	2	5
1	3	5
2	9	5
3	14	5
9	43	5
14	67	5
43	206	5
67	321	5
206	987	5
321	1538	5
987	4729	5
1538	7369	5

<b>c</b>	<b>d</b>	
-1	-1	
<b>a</b>	<b>b</b>	<b>k</b>
1	2	4
2	7	4
7	26	4
26	97	4
97	362	4
362	1351	4
1351	5042	4

<b>c</b>	<b>d</b>	
-14	6	
<b>a</b>	<b>b</b>	<b>k</b>
1	17	12
5	61	12
17	203	12
61	727	12
203	2419	12

<b>c</b>	<b>d</b>	
-14	12	
<b>a</b>	<b>b</b>	<b>k</b>
1	119	108
11	1189	108
119	12851	108

<b>c</b>	<b>d</b>	
2	0	
<b>a</b>	<b>b</b>	<b>k</b>
1	3	4
3	11	4
11	41	4
41	153	4
153	571	4
571	2131	4
2131	7953	4

<b>c</b>	<b>d</b>	
1	0	
<b>a</b>	<b>b</b>	<b>k</b>
1	2	3
2	5	3
5	13	3
13	34	3
34	89	3
89	233	3
233	610	3
610	1597	3
1597	4181	3

<b>c</b>	<b>d</b>	
5	-1	
<b>a</b>	<b>b</b>	<b>k</b>
1	2	10
1	8	10
2	19	10
8	79	10
19	188	10
79	782	10

188	1861	10
782	7741	10

In our last example, **k** is a square number:

<b>c</b>	<b>d</b>	
0	1	
<b>a</b>	<b>b</b>	<b>k(= <math>m^2</math>)</b>
2	8	4
3	27	9
8	30	4
4	64	16
30	112	4
5	125	25
6	216	36
27	240	9
7	343	49
112	418	4
8	512	64
9	729	81
10	1000	100

64	1020	16
11	1331	121
418	1560	4
12	1728	144
240	2133	9
13	2197	169
14	2744	196
125	3120	25
15	3375	225
16	4096	256
17	4913	289
1560	5822	4
18	5832	324
19	6859	361
216	7770	36
20	8000	400

{In some cases here, we have  $b = a^3$   
and  $a^6 + a^2 = a^2(a^4 + 1)$ , so  $k = a^2$  .}

## 2., Exploiting recursivity

Equation (3) gives us a recursive formula with the help of which we can list the first few terms of an increasing sequence  $a_1, b_1 = a_2, b_2 = a_3, \dots$  obeying (2), with the same value of  $k$ . Numerical calculations can be performed by using, e.g., the following `jbasic` program code:

Rem "Vieta's descend recursive formula"

[start]

print "c="

input c

print "d="

input d

print "k="

input k

print chr\$(13)

[L1]

print "a="

input a

[L2]

$b = (a * k + \text{sqr}(a^2 * (k^2 - 4) - 4 * c + 4 * d * k)) / 2$

print b

a=b

if a < 10^9 then goto [L2]

goto [L1]

end

*Input:*  $c, d, k$  and  $a_1$ .

*Output:*  $b_1 = a_2, b_2 = a_3, b_3 = a_4, \dots$  (obeying the relations  $a, b \in N$  and  $\frac{a^2+b^2+c}{ab+d} = k \in Z$ ).

Find some numerical results as follows.

**$c=2, d=4, k=12$**

1, 15, 179, 2133, 25417, 302871, 3609035, 43005549, 5.12457553e8;

3, 37, 441, 5255, 62619, 746173, 8891457, 1.05951311e8;

**$c=1, d=3, k=6$**

1, 8, 47, 274, 1597, 9308, 54251, 316198, 1842937, 10741424, 62605607,  
3.64892218e8;

2, 13, 76, 443, 2582, 15049, 87712, 511223, 2979626, 17366533, 1.01219572e8;

**$c=-6, d=4, k=4$**

1, 7, 27, 101, 377, 1407, 5251, 19597, 73137, 272951, 1018667, 3801717,  
14188201, 52951087, 1.97616147e8;

3, 13, 49, 183, 683, 2549, 9513, 35503, 132499, 494493, 1845473, 6887399,  
25704123, 95929093, 3.58012249e8;

**$c=-11, d=3, k=3$**

2, 8, 22, 58, 152, 398, 1042, 2728, 7142, 18698, 48952, 128158, 335522, 878408,  
2299702, 6020698, 15762392, 41266478, 1.08037042e8;

**$c=-6, d=8, k=44$**

1, 51, 2243, 98641, 4337961, 1.90771643e8;

7, 309, 13589, 597607, 26281119, 1.15577163e9;

**c=0, d=-1, k=5**

2, 9, 43, 206, 987, 4729, 22658, 108561, 520147, 2492174, 11940723, 57211441,  
2.74116482e8;

3, 14, 67, 321, 1538, 7369, 35307, 169166, 810523, 3883449, 18606722,  
89150161, 4.27144083e8;

**c=-1, d=-1, k=4**

1, 2, 7, 26, 97, 362, 1351, 5042, 18817, 70226, 262087, 978122, 3650401,  
13623482, 50843527, 1.89750626e8;

**c=-14, d=6, k=12**

1, 17, 203, 2419, 28825, 343481, 4092947, 48771883, 5.81169649e8;

5, 61, 727,8663,103229, 1230085, 14657791, 1.74663407e8;

**c=-14, d=12, k=108**

1, 119, 12851, 1387789, 1.49868361e8;

11, 1189, 128401, 13866119, 1.49741245e9;

**c=2, d=0, k=4**

1, 3, 11, 41, 153, 571, 2131, 7953, 29681, 110771, 413403,1542841, 5757961,  
21489003, 80198051, 2.99303201e8;

**c=1, d=0, k=3**

1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10946, 28657, 75025, 196418, 514229,  
1346269, 3524578, 9227465, 24157817, 63245986, 1.65580141e8;

**c=5, d=-1, k=10**

2, 19, 188, 1861, 18422, 182359, 1805168, 17869321, 1.76888042e8;

8, 79, 782, 7741, 76628, 758539, 7508762, 74329081, 7.35782048e8 .

## **Conclusion**

Let  $c$  and  $d$  be fixed integer numbers. Assume that  $(a^2 + b^2 + c)$  is divisible by  $(ab + d)$  for some natural numbers  $a$  and  $b$ . Then the value of the fraction  $k (= \frac{a^2+b^2+c}{ab+d})$  remains the same. Statement of this kind, that belongs to the theory of quadratic Diophantine equations, has been proved and then illustrated on some examples. The general method of proofs has been unified and simplified.

Computing support has been provided: simple program code was described with the help of which one can “hunt” for natural numbers  $a, b$  with given and fixed integer values of  $c, d$  and  $k$ . A number of examples have been provided.

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